Adaptive Mesh Refinement for Multiple Goal Functionals Applied to Elliptic Problems

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Doctoral Program
Computational Mathematics
Numerical Analysis and Symbolic Computation

- Adaptive Mesh Refinement for elliptic Problems
    1. Literature
    2. Adaptive Mesh Refinement
    3. Multiple Goal Approach
    4. Numerical Results

- Recent work for nonlinear Problems
    1. Literature
    2. Goal Oriented Mesh Refinement
    3. Numerical Examples

Overview

- Adaptive Mesh Refinement for elliptic Problems
  1. Literature
  2. Adaptive Mesh Refinement
  3. Multiple Goal Approach
  4. Numerical Results
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  1. Literature
  2. Goal Oriented Mesh Refinement
  3. Numerical Examples

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- Adaptive Mesh Refinement for elliptic Problems
  1. Literature
  2. Adaptive Mesh Refinement
  3. Multiple Goal Approach
  4. Numerical Results
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  1. Literature
  2. Goal Oriented Mesh Refinement
  3. Numerical Examples

Main Literature

R. Hartmann and P. Houston(2003),
Goal-Oriented A Posteriori Error Estimation for Multiple Target Functionals
Hyperbolic Problems: Theory, Numerics, Applications, pages 579-588. Springer Berlin Heidelberg,2003.

T. Richter and T. Wick (2015)
Variational localizations of the dual weighted residual estimator.
Journal of Computational and Applied Mathematics, 279(0): pages 192 - 208

B. Endtmayer and T. Wick. (2017)
A partition-of-unity dual-weighted residual approach for multi-objective goal functional error estimation applied to elliptic problems.
Computational Methods in Applied Mathematics, February 2017. accepted for publication, also available as NuMa Report 2016-07 at http://www.numa.uni-linz.ac.at/Publications/List/2016/2016-07.pdf.

The first reference is denoted by [R. Hartmann & P. Houston (2003)]. The second reference is denoted by [T. Richter & T. Wick (2015)]. The last one with [B. Endtmayer & T. Wick (2017)].

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- Adaptive Mesh Refinement for elliptic Problems
  1. Literature
  2. Adaptive Mesh Refinement
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  1. Literature
  2. Goal Oriented Mesh Refinement
  3. Numerical Examples

Variational Problem

Let V be a Hilbert space, $a(.,.)$ an elliptic bilinear form $a : V \times V \to \mathbb{R}$, $F \in V^*$ and we want to solve

Variational Problem (Primal Problem)

Find $u \in V$ such that

$$a(u,v) = F(v)$$

for all $v \in V$.

Furthermore let us denote the solution of the finite element discretization by u_h. Example:

$$a(u,v) = \int_\Omega \nabla u(x).\nabla v(x)dx; \quad F(v) = \int_\Omega f(x)v(x)dx;$$

Goal Functional

- not interested in the whole solution u or in norm error estimates
- but in one specific linear functional evaluation $J(u)$
- examples:
  - Point evaluation
    $J(u) = u(x_0)$ for some x_0 in the domain.
  - Mean value evaluation
    $J(u) = \frac{1}{|\Omega|} \int_\Omega u(x)dx$ for some domain Ω.
  - Flux evaluation
    $\int_\Gamma \nabla u(x).n(x)ds_x$ for some surface Γ in the domain.

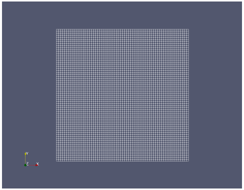
Goals and Solution

Goals:

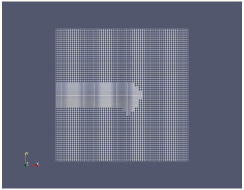
- high accuracy in our functional evaluation $J(u)$
- low computational cost

Solution:

- adaptive mesh refinement for our goal functional J



$\longrightarrow$



Local Error Estimators

But to get the information where the mesh should be refined we will need a localization of the error estimator. We are looking for η_i such that

$$|J(u) - J(u_h)| \approx \sum_i |\eta_i|$$

where $|\eta_i|$ describes the local error contribution.

But how do we find $|\eta_i|$??

Adjoint Variational Problem

Find $z \in V$ such that

$$a(u, z) = J(u)$$

for all $u \in V$.

Dual Weighted Residual (DWR) with Partition of Unity(PU) [T. Richter & T. Wick (2015)]

One localization approach is Partition of Unity(PU), which is presented in [T. Richter & T. Wick (2015)]. Let ψ_i be such that

$$\sum_i \psi_i \equiv 1.$$

Then one can show

$$J(u) - J(u_h) = \sum_i \underbrace{F((z - z_h)\psi_i) - a(u_h, (z - z_h)\psi_i)}_{\eta_i^{PU}}.$$

- $|\eta_i^{PU}|$ can be computed if we "compute z" by solving the dual variational problem.

Overview

Multiple Goal Approach

- up to now we considered only one functional $J(.)$
- now we consider N linear functionals $J_i(.)$ for $i \in \{1, \ldots, N\}$
- we do refinements until for all $|J_i(u) - J_i(u_h)| < TOL_i$ holds
- but that means we have to solve **N** linear systems!!
- therefore we try to combine the functionals

Combined Functional

Idea:

- combine the functionals to a new functional $J_c(.)$ as it is done in [R. Hartmann & P. Houston (2003)], where

$$J_c(v) := \sum_{k}^{N} J_k(v) w_k$$

  for some weights w_i(not only positive ones)
- the choice of these weights is very crucial
- bad choice of weights may lead to error cancelling.

Weights

What the weights should be fulfil:

- no error cancelling
- similar relative error

How this can be solved:

- assuring that errors have same sign
- the weights can be chosen as

$$w_k := \frac{\mathrm{sign}(J_k(u) - J_k(u_h))\omega_k}{|J_k(u_h)|},$$

where ω_k is a self-chosen but positive weight (as it is done in [R. Hartmann & P. Houston (2003)] with $\omega_k = 1$).

Problem:

- we don't know $J_k(u)$ and therefore we are not able to compute the sign!

How to get the Signs?

For this we consider the

Adjoint Adjoint Variational Problem (see [R. Hartmann & P. Houston (2003)])

Find $e \in V$ such that

$$a(e, v) = R_{u_h}(v),$$

where $R_{u_h}(v) := F(v) - a(u_h, v)$ for all $v \in V$.

- solution e is $u - u_h$, i.e. the discretization error.
- allows to construct the $J_c(v)$.
- advantage of this adjoint-adjoint approach: just have to solve **2 linear systems** instead of **N**.

The discretization of the Adjoint-Adjoint Problem

Let us denote a second finite element space by $V_h^{(2)}$ and the approximate solution of the adjoint-adjoint problem by $e_h^{(2)}$.

Theorem (see [B. Endtmayer & T. Wick (2017)])

Let a fulfil the assumptions of Lax-Milgram and $u_h^{(2)}$ be the approximate solution by using the finite element space $V_h^{(2)}$. Furthermore let $V_h \subseteq V_h^{(2)}$. Then for the approximate solution of the adjoint-adjoint problem holds

$$e_h^{(2)} = u_h^{(2)} - u_h.$$

- allows us to solve the primal problem on the FE-space $V_h^{(2)}$ instead of solving the adjoint-adjoint problem.

Summary /Algorithm (see [B. Endtmayer & T. Wick (2017)])

1. compute the approximate solutions $u_h, u_h^{(2)}$
2. construct the combined functional $J_c(v)$
3. if $|J_c(u_h^{(2)}) - J_c(u_h)| = |J_c(e_h^{(2)})| < TOL$ stop
4. approximate the solution dual variational problem $a(u, z) = J_c(u)$ for the weights of the error functional
5. compute local error estimators $|\eta_i|$
6. if $|\eta_i| > \frac{1}{n_h} \sum_i |\eta_i|$ we flag the i-th element for refinement
7. refine all the flagged elements in the mesh
8. go to 1

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  2. Adaptive Mesh Refinement
  3. Multiple Goal Approach
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  1. Literature
  2. Goal Oriented Mesh Refinement
  3. Numerical Examples

Numerical Results

Example 1a:

Let us have a look at the problem (L-shaped domain):

$$\Omega = (-1,1) \times (-1,1) \setminus (-1,0) \times (-1,0)$$

$$-\Delta u(x,y) = f(x,y) \quad \forall (x,y) \in \Omega,$$
$$u(x,y) = 0 \quad \forall (x,y) \in \partial\Omega.$$

The data f is choosen such that the exact solution u is given by

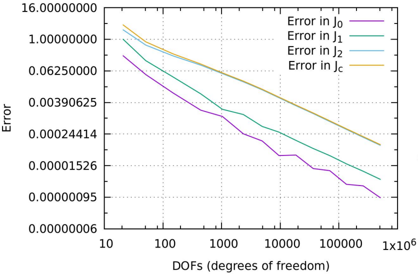
$$u(x,y) = x(y^2-1)(x^2-1)(e^{3y}-1).$$

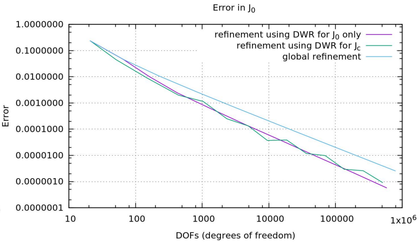
We consider the following three goal functionals:

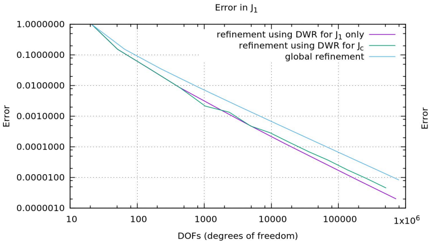
$$J_0(u) := u(0.5, 0.5),$$
$$J_1(u) := \int_{\Omega_1} u(x,y) d(x,y),$$
$$J_2(u) := \int_{\Gamma_1} \nabla u(x,y).n \; d(x,y),$$

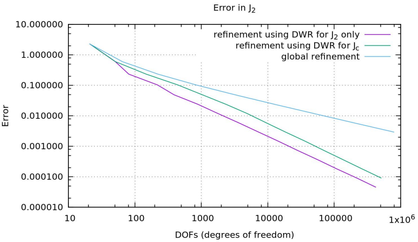
where $\Omega_1 = (-0.5, 0) \times (0.5, 1)$ and $\Gamma_1 = \{1\} \times (0, 1)$.

Errors with respect to the DOF's

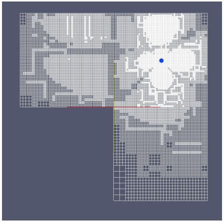




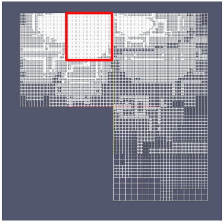




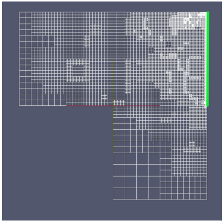
Meshes



Example 1a: Mesh for DWR for J_0.



Example 1a: Mesh for DWR for J_1.



Example 1a: Mesh for DWR for J_2.

Example 1a: Mesh for DWR for J_c.

Example 4b:

We consider the Laplace equation on a slit domain (see J. Andersson & H. Mikayelyan 2015):

$$
\begin{aligned}
-\Delta u(x,y) &= 0 \quad \forall (x,y) \in \Omega,\\
u(x,y) &= g(x,y) \quad \forall (x,y) \in \Gamma_D,\\
\nabla u(x,y) \cdot n(x,y) &= 0 \quad \forall (x,y) \in \Gamma_N,
\end{aligned}
$$

where

$$\Omega = (-1,1) \times (-1,1) \setminus \{(x,0) | -1 \leq x \leq 0\}.$$

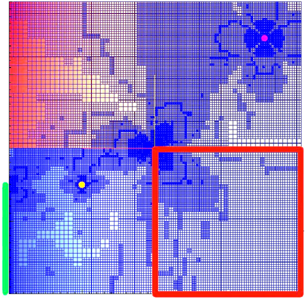
The data is give by:

$$
\begin{aligned}
\Gamma_N &= \{(x,0) | -1 \leq x \leq 0\},\\
\Gamma_D &= \partial\Omega \setminus \Gamma_N,\\
g(x,y) &:= sign(y) \frac{1}{\sqrt{2}} \sqrt{\sqrt{x^2+y^2} - x}.
\end{aligned}
$$

The exact solution is given by

$$u(x,y) = g(x,y).$$

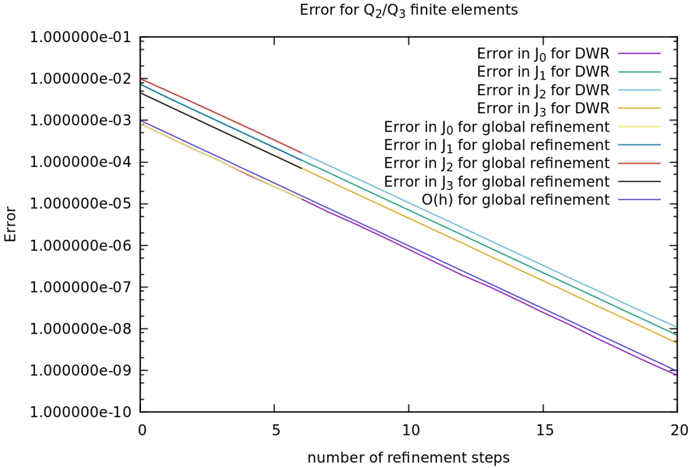
We are interested in the functional evaluations



$$J_0(u) := u(0.75, 0.75),$$
$$J_1(u) := u(-0.5, -0.25),$$
$$J_2(u) := \int_{\Gamma_7} \nabla u(x,y).n \; d(x,y),$$
$$J_3(u) := \int_{\Omega_7} u(x,y) \; d(x,y),$$

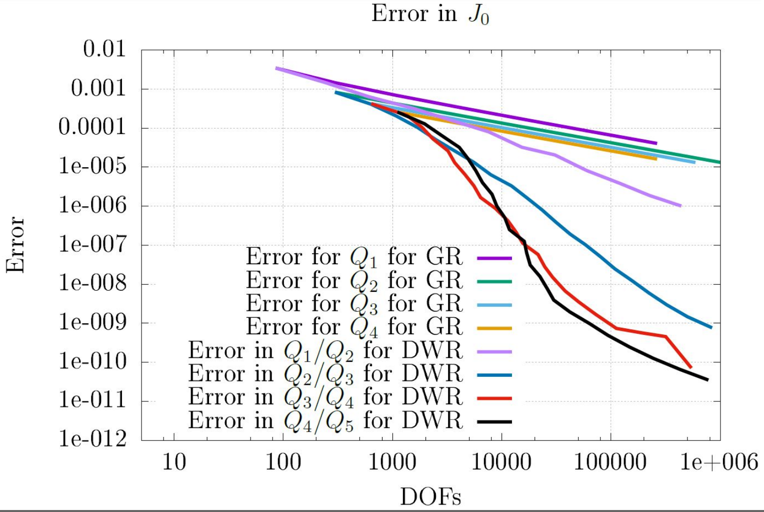
where and $\Gamma_7 = \{-1\} \times (-1, -0.25)$ and $\Omega_7 = (0,1) \times (-1,0)$.

Error for Q_2/Q_3 finite elements



Error versus $DOFs$

Here we compare the error with the number of $DOFs$.



Overview

- Adaptive Mesh Refinement for elliptic Problems
  1. Literature
  2. Adaptive Mesh Refinement
  3. Multiple Goal Approach
  4. Numerical Results
- Recent work for nonlinear Problems
  1. Literature
  2. Goal Oriented Mesh Refinement
  3. Numerical Examples

Overview

- Adaptive Mesh Refinement for elliptic Problems
  1. Literature
  2. Adaptive Mesh Refinement
  3. Multiple Goal Approach
  4. Numerical Results
- Recent work for nonlinear Problems
  1. Literature
  2. Goal Oriented Mesh Refinement
  3. Numerical Examples

Main Literature

R. Rannacher and J. Vihharev(2013),
Adaptive finite element analysis of nonlinear problems: balancing of discretization and iteration errors
J. Numer. Math. 21 (2013), no. 1, 23–61. MR3043432.

We will denote this reference by
[R. Rannacher & J. Vihharev(2013)].

Overview

- Adaptive Mesh Refinement for elliptic Problems
  1. Literature
  2. Adaptive Mesh Refinement
  3. Multiple Goal Approach
  4. Numerical Results

- Recent work for nonlinear Problems
  1. Literature
  2. Goal Oriented Mesh Refinement
  3. Numerical Examples

New Primal Problem

Let V be a Banach space, $\mathcal{A}$ an operator $\mathcal{A} : V \to V^*$.

Primal Problem

Find $u \in V$ such that

$$\mathcal{A}(u)(v) = 0$$

for all $v \in V$.

We postulate:

1. the primal problem has a unique solution.
2. $\mathcal{A} \in \mathcal{C}^3(V, V^*)$.

Example:

$$\mathcal{A}(u)(v) := \int_\Omega fv - (\|\nabla u\|^2 + 1)^{\frac{p-2}{2}} \nabla u.\nabla v dx$$

New Adjoint Problem

Now we also consider a nonlinear functionals $J : V \mapsto \mathbb{R}$.

Adjojnt Problem

Find $z \in V$ such that

$$\mathcal{A}'(u)(v,z) = J'(u)(v)$$

for all $u \in V$.

- we postulate $J \in \mathcal{C}^3(V,\mathbb{R})$.

Example:

$$J(u) := \frac{1}{|\Omega|}\int_\Omega (u(x) - \frac{1}{|\Omega|}\int_\Omega u(y)dy)^2 dx$$

Error Representation

Theorem (Error Representation); see [R. Rannacher & J. Vihharev(2013)]

Let $\tilde{u}, \tilde{z} \in V$. Then it holds

$$J(u) - J(\tilde{u}) = \frac{1}{2}(\rho(\tilde{u})(z - \tilde{z}) + \rho^*(\tilde{u}, \tilde{z})(u - \tilde{u})) - \rho(\tilde{u})(\tilde{z}) + o(e^3),$$

where

$$\begin{aligned} \rho(\tilde{u})(\cdot) &:= -\mathcal{A}(\tilde{u})(\cdot), \\ \rho^*(\tilde{u}, \tilde{z})(\cdot) &:= J'(\tilde{u})(\cdot) - \mathcal{A}'(\tilde{u})(\cdot, \tilde{z}), \\ e &:= max(\|u - \tilde{u}\|_V, \|z - \tilde{z}\|_V). \end{aligned}$$

Error Estimation

For approximations $\tilde{u_h}, \tilde{z_h}$ of u, z holds:

$$J(u)-J(\tilde{u_h}) \approx \underbrace{\frac{1}{2}(\rho(\tilde{u_h})(z-\tilde{z_h})+\rho^*(\tilde{u_h},\tilde{z_h})(u-\tilde{u_h}))}_{\eta_h} - \underbrace{\rho(\tilde{u_h})(\tilde{z_h})}_{\eta_k}.$$

It turns out that

- η_h mimics the discretization error.
- η_k mimics the linearization error.
- $|\eta_k| \leq \gamma|\eta_h|$ with $\gamma \in (0,1]$ can be used as stopping rule for the nonlinear solver.

Localized discretization Error Estimation

Using again the PU localization as in [T. Richter & T. Wick (2015)] leads to

$$\eta_h := \sum_i \underbrace{\frac{1}{2}(\rho(\tilde{u_h})(\psi_i(z - \tilde{z_h})) + \rho^*(\tilde{u_h}, \tilde{z_h})(\psi_i(u - \tilde{u_h})))}_{\eta_i^{PU}},$$

where

$$\sum_i \psi_i \equiv 1.$$

This allows us adaptive refinement as in the linear case.

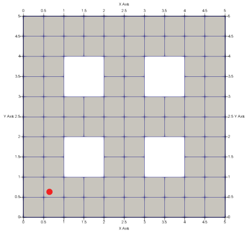
Overview

Numerical Experiment

$$-\nabla \cdot ((1 + \|\nabla u\|^2)\nabla u) = 1 \quad \text{in } \Omega,$$
$$u(x) = 0 \quad \text{on } \partial\Omega.$$

We are interested in the following functional evaluation

$$J_0(u) := u(0.6, 0.6) \approx 0.13607014401.$$



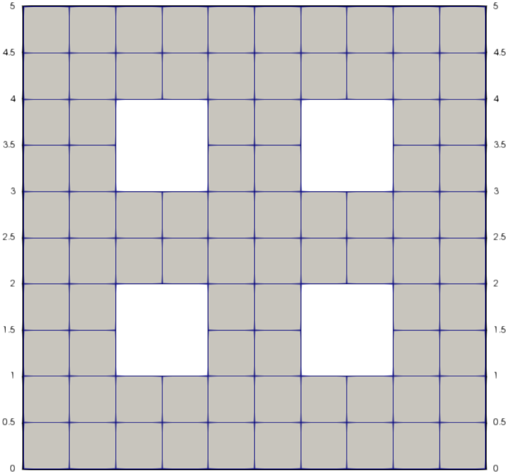
Ieff $:= |\frac{\eta_h}{J(u) - J(u_h)}|$ Ieff primal $:= |\frac{\rho(u_h)(z - z_h)}{J(u) - J(u_h)}|$ Ieff adjoint $:= |\frac{\rho^*(u_h, z_h)(u - u_h)}{J(u) - J(u_h)}|$

Adaptive Refinement

Refinementstep	DOFS primal	Exact Error	Ieff	Ieff primal	Ieff adjoint
0	117	0.01453191	0.7107785	0.4293014	0.9922556
1	161	0.008360867	0.9399308	0.795228	1.084634
2	269	0.00223129	0.945282	0.7393274	1.151237
3	629	0.000817429	0.930592	0.6851591	1.176025
4	1325	0.0003498213	0.9136854	0.6228659	1.204505
5	3057	0.0001486951	0.9121072	0.515821	1.308393
6	6985	6.594811e-05	0.9156221	0.4686931	1.362551
7	15129	3.65568e-05	0.927745	0.5250439	1.330446
8	33061	1.558321e-05	0.9261095	0.4048985	1.44732
9	68873	8.244251e-06	0.9386322	0.4350792	1.442185
10	143533	3.824049e-06	0.9457057	0.4069253	1.484486
11	300377	2.118342e-06	0.9589256	0.4776999	1.440151

- global refinement: Exact error 2.17394e-05 for 346365 DOFs (6. Refinementstep).

Mesh Evolution



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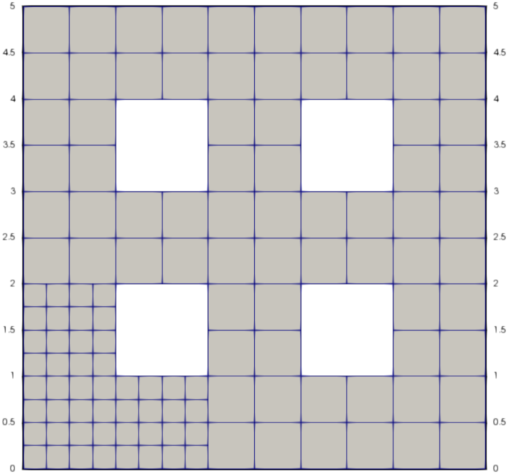

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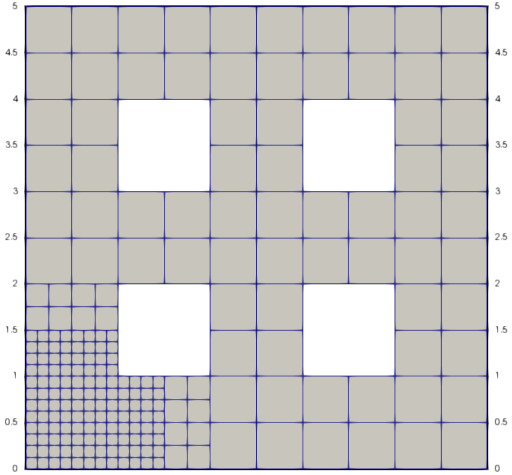

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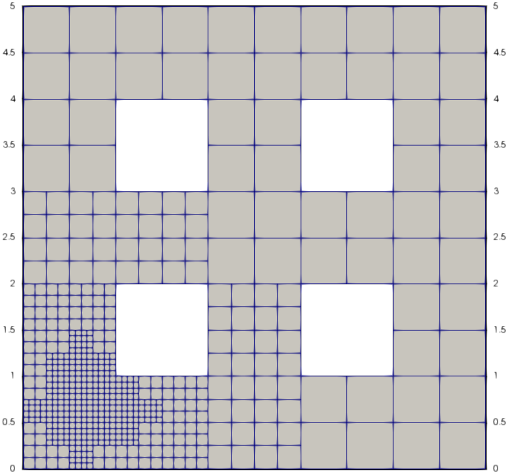

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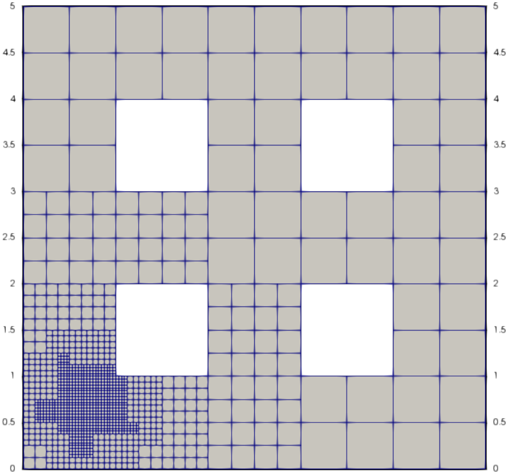

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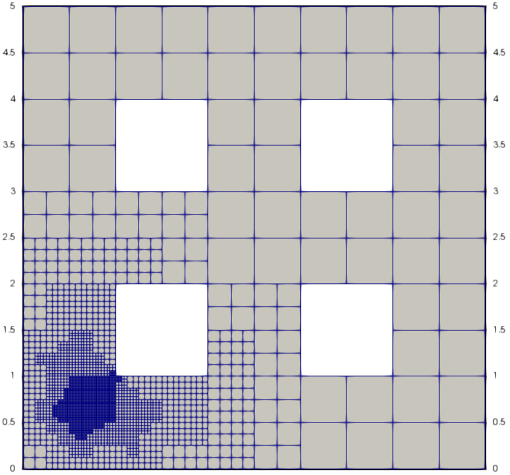

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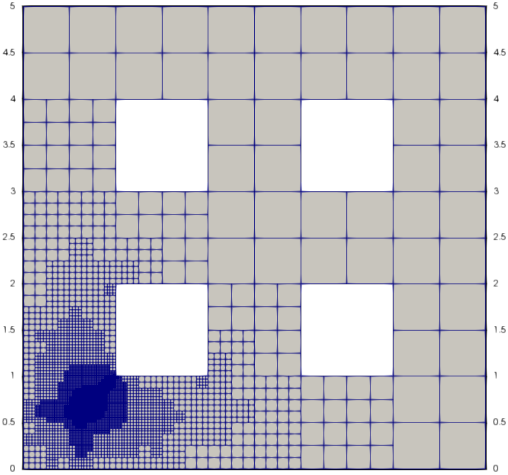

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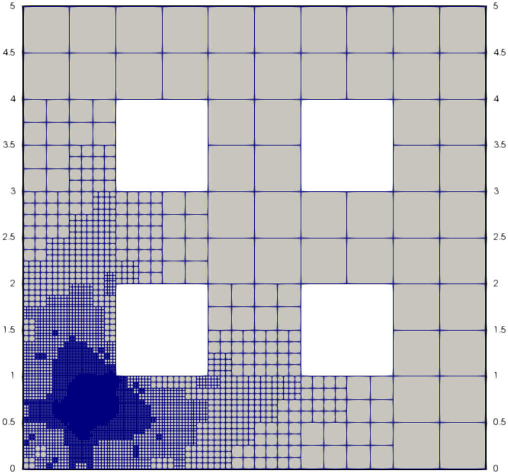

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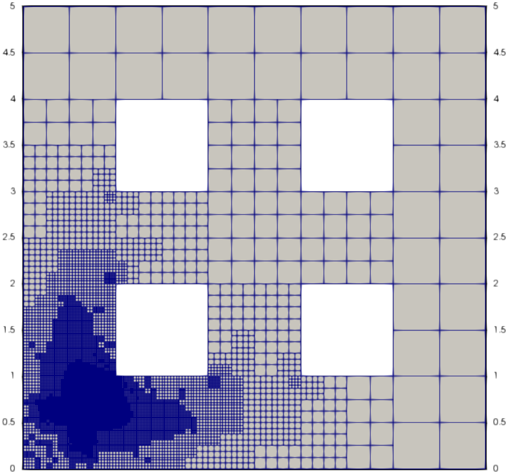

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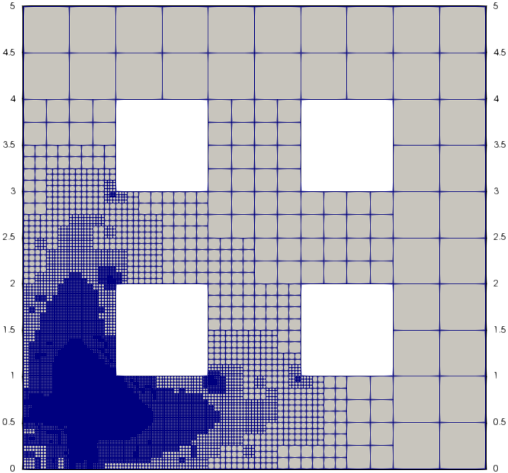

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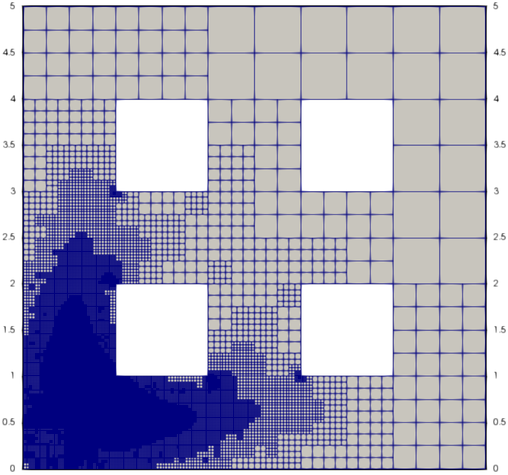

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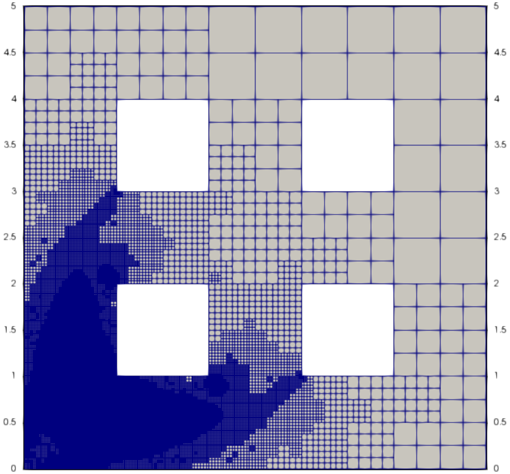

Outlook

- apply the multiple goal functional approach for nonlinear problems ,which can be done in the same way as for linear problems.
- influence of the neglected remainder therm.
- improve the computational costs of this method.

Thank you for your attention