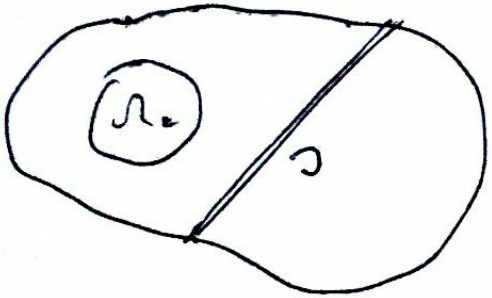
On a degenerate eddy current problem – revisited.

with Dirk Pauly

Vast literature! For example: R.C. MacCamy, E.P. Stephan (1984), H. Ammari, A. Buffa, J.-C. Nedelec (2000), M. Dauge, M. Costabel, S. Nicaise (2003), T. Pepperl (Dissertation, 2006), Ana A. Rodriguez, Alberto Valli (monograph, 2010), M. Kolmbauer, U. Langer (2011), E. Creusé, S. Nicaise (2012–2016), X. Jiang, W. Zheng (2012–14)



$\Omega_c \subseteq \Omega$

electric b.c.

$$\cancel{\partial_0 \varepsilon E} + \sigma E - \operatorname{curl} H = -J$$

$$\partial_0 \mu H + \mathring{\operatorname{curl}}\, E = 0$$

$$H = \mu^{-1} \mathring{\operatorname{curl}}\, \partial_0^{-1} E$$

$$\sigma E + \underbrace{\operatorname{curl} \mu^{-1} \mathring{\operatorname{curl}}}\, \underbrace{\partial_0^{-1} E} = -J$$

$\sigma \geq 0$

$\sigma E = \partial_0 u$, $\quad \partial_0^{-1} E = u$

$$(\operatorname{curl} \mu^{-1/2})(\mu^{-1/2} \mathring{\operatorname{curl}}) = |\mu^{-1/2} \mathring{\operatorname{curl}}|^2$$

pre-Maxwell: $N(\sigma) \neq \{0\}$ — {elliptic / parabolic}

$$\begin{tikzcd} L^2(\Omega_c) \arrow[r, "\tilde\sigma"] \arrow[d, phantom] & L^2(\Omega_c) \arrow[d, "\iota_{\Omega_c}"] \\ L^2(\Omega) \arrow[u, "\iota_{\Omega_c}^*"] \arrow[r, "\sigma"] & L^2(\Omega) \end{tikzcd}$$

$\tilde\sigma : L^2(\Omega_c) \to L^2(\Omega_c)$ strictly positive definite

$\sigma : L^2(\Omega) \to L^2(\Omega)$

Evo-Systems: A closed, densely defined lin. operator in a real Hilbert space H

M_0 selfadjoint, continuous, M_1 continuous lin mapping in H

unified approach

$$\partial_0 M_0 + M_1 + A \geq c_0 > 0 \quad \text{(numerical range condition)}$$

$$\partial_0^* M_0 + M_1^* + A^* \geq c_0 > 0$$

$$\left(\partial_0 M_0 + M_1 + A\right)^* = \overline{\partial_0^* M_0 + M_1^* + A^*}$$

$$\circledast \quad \boxed{\rho M_0 + \operatorname{sym}(M_1) + \begin{matrix} A \\ A^* \end{matrix} \geq c_0 > 0}$$

$\partial_0 \geq \rho_0 > 0$

for $\rho \geq \rho_0 > 0$

Theorem: The equation

$$\left(\overline{\partial_0 M_0 + M_1 + A}\right) u = f$$

has a unique solution for any $f \in H_{\rho,0}(\mathbb{R}, H) = L^2(\mathbb{R}, \exp(-2\rho t)\,dt)$. Moreover

$$\left\| \left(\overline{\partial_0 M_0 + M_1 + A}\right)^{-1} \right\| \leq \frac{1}{c_0} \quad \text{(i.e. continuous dependence)}$$

and

$$\chi_{]-\infty,0]} \left(\overline{\partial_0 M_0 + M_1 + A}\right)^{-1} \left(1 - \chi_{]-\infty,0]}\right) = 0 \quad \text{(i.e. causality)}$$

Here: $A = A^*, \quad M_1 = 0, \quad H = L^2(\Omega)$

$\partial_0 \sigma + \operatorname{curl} \mu^{-1} \mathring{\operatorname{curl}} \geq c_0 > 0$?

$\varrho \sigma + \operatorname{curl} \mu^{-1} \mathring{\operatorname{curl}} \geq c_0 > 0$?

$$\langle u | \partial_0 \sigma u \rangle_{\varrho,0,H} + \langle u | \operatorname{curl} \mu^{-1} \mathring{\operatorname{curl}} u \rangle_{\varrho,0,H} \geq c_0 \langle u | u \rangle_{\varrho,0,H}$$

$$\varrho |\sigma_{\varrho} u|^2_{\varrho,0,H} + |\mu^{-1/2} \mathring{\operatorname{curl}} u|^2_{\varrho,0,H} \geq c_0 |u|^2_{\varrho,0,H}$$

o.B.d.A. $\varrho \geq 1$.

<u>To show:</u> $|\chi_{\Omega_c} u|^2_H + |\mathring{\operatorname{curl}} u|^2_H \geq c_0 |u|^2_H$

False!

$$N(\chi_{\Omega_c}) \cap N(\mathring{\operatorname{curl}}) = L^2(\Omega \setminus \overline{\Omega_c}) \cap N(\mathring{\operatorname{curl}})$$
$$= N(\mathring{\operatorname{curl}}_{\Omega \setminus \overline{\Omega_c}})$$

e.g. (segment property!) $D(\mathring{\operatorname{curl}}_{\Omega\setminus\overline{\Omega_c}}) \subseteq D(\mathring{\operatorname{curl}})$

We obviously need to restrict our investigation to

$$H_0 = \underbrace{(N(\chi_{\Omega_c}) \cap N(\mathring{\operatorname{curl}}))^{\perp}}_{R(\operatorname{curl}_{\Omega\setminus\overline{\Omega_c}}) \oplus L^2(\Omega_c)} \qquad \langle \cdot | \cdot \rangle_{\varrho,0,0}, \ |\cdot|_{\varrho,0,0}$$

Assumption: ~~…~~ Ω ~~…~~ such that ~~…~~ $R(\operatorname{curl})$ ~~…~~ is closed

e.g. compact embedding

$D(\mathring{\operatorname{curl}}) \cap D(\operatorname{div}) \hookrightarrow L^2$

Witsch, Weck & π 2001

$$H_0 = R(\mathrm{curl}) \oplus \underbrace{H_1 \oplus H_2}_{N(\mathring{\mathrm{curl}})}$$

$$H_1 = \underset{L^2(\Omega_c)}{\underset{\shortparallel}{R(\sigma)}} \cap N(\mathring{\mathrm{curl}}) = N(\mathring{\mathrm{curl}}_{\Omega_c})$$

$$H_2 = \big(N(\mathring{\mathrm{curl}}) \cap H_0 \big) \ominus H_1 = N(\mathring{\mathrm{curl}}) \cap \left(\overline{R(\mathring{\mathrm{curl}}_{\Omega_c})} \cup \overline{R(\mathring{\mathrm{curl}}_{\Omega \setminus \Omega_c})} \right)$$

$$u = \overset{\overset{R(\mathrm{curl})}{\Downarrow}}{u_0} + u_1 + u_2$$

$$u_1 = \chi_{\Omega_c} u_1 \perp u_2 \qquad \Longrightarrow \qquad \chi_{\Omega_c} u_1 \perp \chi_{\Omega_c} u_2$$

$$0 = \langle \chi_{\Omega_c} u_1 \mid u_2 \rangle_0 = \langle \chi_{\Omega_c} u_1 \mid \chi_{\Omega_c} u_2 \rangle_0$$

$U_2 = \mathrm{grad}\,\varphi$ and $\mathrm{div}\,\mathrm{grad}\,\varphi = 0$ in $\Omega \setminus \dot{\Omega}_c$

Lemma 1:

$$|\chi_{\Omega_c}(u_1+u_2)|_0^2 = |\chi_{\Omega_c} u_1|_0^2 + |\chi_{\Omega_c} u_2|_0^2$$
$$= |u_1|_0^2 + |\chi_{\Omega_c} u_2|_0^2$$

Lemma 2:

$$|u_0|_0^2 \le k_0 \, |\operatorname{curl} \mathring{u}_0|_0^2 \quad , \; k_0 \ge 1$$

Proof ~~[illegible] embedding [illegible]~~ closed range! Here we used the compact embedding assumption!

Lemma 3:

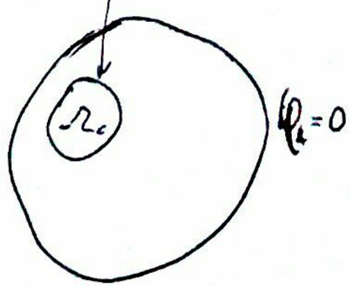
$$|u_2|_0^2 \le k_2 \, |\chi_{\Omega_c} u_2|_0^2 \quad , \; k_2^* \ge 1 ,$$

Proof:

$$u_2 = \operatorname{grad} \varphi$$

$$-\operatorname{div} \operatorname{grad} \varphi = 0 \quad \text{in } \Omega \setminus \mathring{\Omega}_c$$

φ on $\dot\Omega_c$ coinciding b.c. in Ω_c & $\Omega \setminus \bar\Omega_c$



Lemma 4: $$|\chi_{\Omega_c} u|_0^2 + |\operatorname{curl} u|_0^2 \geq c_1 |u|_0^2$$

Proof:
$$\begin{aligned}
|u|_0^2 &= |u_0|_0^2 + |u_1|_0^2 + |u_2|_0^2 \quad \text{Lemma 3} \\
&\leq |u_0|_0^2 + |\chi_{\Omega_c} u_1|_0^2 + k_2^2 |\chi_{\Omega_c} u_2|_0^2 \quad \text{Lemma 1} \\
&\leq |u_0|_0^2 + k_2^2 |\chi_{\Omega_c} (u_1 + u_2)|_0^2 \\
&\leq |u_0|_0^2 (1 + 2k_2^2) + 2k^2 |\chi_{\Omega_c} u|_0^2 \quad \text{Lemma 2} \\
&\leq (1 + 2k_2) k_0 |\operatorname{curl} u_0|_0^2 + 2k^2 |\chi_{\Omega_c} u|_0^2 \\
&\leq 3 k_2 k_0 \left(|\chi_{\Omega_c} u|_0^2 + |\operatorname{curl} u|_0^2 \right).
\end{aligned}$$

Theorem: The degenerate eddy current problem is a well-posed ~~evolutionary~~ problem.