On a Degenerate Eddy Current Problem-revisited.

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Rainer Picard, TU Dresden
(joint work with Dirk Pauly, U Duisburg-Essen)

What are Evo-Systems?

Consider systems of the form

$$\left(\partial_0 \mathscr{M} + A\right) U = F,$$

where A, A^* are accretive in a real Hilbert space H and $\mathscr{M}$ is a so-called material law operator. Solutions are discussed in a weighted real L^2-space $H_\rho\left(\mathbb{R}, H\right)$, constructed by completion of the space $\mathring{C}_1\left(\mathbb{R}, H\right)$ of differentiable H-valued functions with compact support w.r.t. $\langle \cdot | \cdot \rangle_{\rho,H}$ (norm: $|\cdot|_{\rho,H}$)

$$(\varphi, \psi) \mapsto \int_{\mathbb{R}} \langle \varphi(t) \,|\, \psi(t) \rangle_H \exp\left(-2\rho t\right) dt.$$

Time-differentiation ∂_0 as a closed operator in $H_\rho\left(\mathbb{R}, H\right)$ induced by

$$\begin{aligned} \mathring{C}_1\left(\mathbb{R}, H\right) \subseteq H_\rho\left(\mathbb{R}, H\right) &\to H_\rho\left(\mathbb{R}, H\right), \\ \varphi &\mapsto \varphi'. \end{aligned}$$

What are Evo-Systems?

Time-differentiation ∂_0 is a normal operator in $H_\rho(\mathbb{R}, H)$. For $\rho_0 \in]0,\infty[$, $\rho \in]\rho_0,\infty[$, we have

$$\mathrm{sym}\,\partial_0 = \rho \geq \rho_0 > 0,$$

i.e. ∂_0 is a strictly (and uniformly w.r.t. $\rho \in [\rho_0,\infty[$) positive definite operator

with respect to the real inner product

$$(\phi,\psi) \mapsto \mathfrak{Re}\,\langle\phi|\psi\rangle_{\rho,H}\,.$$

Allows the discussion of the problem class:

$$\overline{\left(\partial_0 M\left(\partial_0^{-1}\right) + A\right)}\,U = F \qquad \text{(Evo-Sys)}$$

Basic Solution Theory of Evo-Systems in $H_\rho(\mathbb{R}, H)$

$$\overline{\left(\partial_0 M\left(\partial_0^{-1}\right) + A\right)} U = F \qquad \text{(Evo-Sys)}$$

Theorem

Let $z \mapsto M(z)$ be a rational $\mathscr{L}(H,H)$-valued function in a neighborhood of 0 such that

$$\rho M(0) + \operatorname{sym} M'(0) + A,\ \rho M(0) + \operatorname{sym} M'(0) + A^* \geq c_0 > 0$$

for some $c_0 \in \mathbb{R}$ and all $\rho \in]0,\infty[$ sufficiently large. Then well-posedness of (Evo-Sys) follows for all $\rho \in]\rho_0,\infty[$. Moreover, the solution operator $\overline{\left(\partial_0 M\left(\partial_0^{-1}\right) + A\right)}^{-1}$ is causal in the sense that

$$\chi_{]-\infty,0]} \overline{\left(\partial_0 M\left(\partial_0^{-1}\right) + A\right)}^{-1} = \chi_{]-\infty,0]} \overline{\left(\partial_0 M\left(\partial_0^{-1}\right) + A\right)}^{-1} \chi_{]-\infty,0]}.$$

The Pre-Maxwell Equations.

We want to inspect

$$\left(\partial_0 \sigma + \operatorname{curl} \mu^{-1} \mathring{\operatorname{curl}}\right) E = -\partial_0 J =: f,$$

where σ, μ are continuous selfadjoint mappings with $\sigma \geq 0$ and μ strictly positive definite. We read off

$$\rho \left\langle E | \sigma E \right\rangle_{\rho,0,0} + \left\langle \mathring{\operatorname{curl}} E | \mu^{-1} \mathring{\operatorname{curl}} E \right\rangle_{\rho,0,0} = \left\langle E | f \right\rangle_{\rho,0,0}$$

showing that

$$E \in N(\sigma) \cap N\left(\mathring{\operatorname{curl}}\right)$$

will cause difficulties, unless

$$f \in H_0 := \left(N(\sigma) \cap N\left(\mathring{\operatorname{curl}}\right)\right)^{\perp}.$$

Moreover,

$$E \in H_0$$

will enforce uniqueness.

The Pre-Maxwell Equations.

History: vast literature!
For example: R.C. MacCamy, E. P. Stephan (1984), H. Ammari, A. Buffa, J.-C. Nedelec (2000), T. Pepperl (Dissertation, 2006), M. Costabel, M. Dauge, S. Nicaise (2003), M. Kolmbauer, U. Langer (2011) , E. Creusé, S. Nicaise (2012-2016), Ana A. Rodriguez, Alberto Valli (monograph, 2010), X. Jiang, W. Zheng (2012-14)

- $\Omega = \mathbb{R}^3$
  - non-degenerate
    - time-harmonic
    - time-dependent
  - degenerate
    - time-harmonic
    - time-dependent
- $\Omega \subseteq \mathbb{R}^3$ bounded domain
  - degenerate

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The Pre-Maxwell Equations.

What is missing?
Solution theory with general operator coefficients for 'rough' domains.

$$\mathrm{Op}\, u = f.$$

Well-posedness = "The linear operator $\mathrm{Op} : \mathrm{D}(\mathrm{Op}) \subseteq \mathrm{X} \to \mathrm{Y}$ is injective and surjective, X, Y Banach spaces".
Frequently, Y is not complete, sometimes Op is replaced by $\widetilde{\mathrm{Op}}$.
Always smooth (at least Lipschitz) domains and interfaces.

The Pre-Maxwell Equations.

Evo-System: $M\left(\partial_0^{-1}\right) = \sigma,\ A = A^* = \operatorname{curl}\mu^{-1}\mathring{\operatorname{curl}}.$

For the underlying domain Ω we assume $\mathscr{H}_{D,\Omega}=\{0\}$ and the compact embedding property

$$D\left(\mathring{\operatorname{curl}}\right)\cap D(\operatorname{div})\hookrightarrow\hookrightarrow L^2(\Omega).$$

Since for $E\in D\left(\operatorname{curl}\mu^{-1}\mathring{\operatorname{curl}}\right)$

$$\left\langle E\middle|\operatorname{curl}\mu^{-1}\mathring{\operatorname{curl}}E\right\rangle_{\rho,0,0} = \left\langle\mathring{\operatorname{curl}}E\middle|\mu^{-1}\mathring{\operatorname{curl}}E\right\rangle_{\rho,0,0}$$

it suffices to show

$$\langle E|\sigma E\rangle_{\rho,0,0} + \left\langle\mathring{\operatorname{curl}}E\middle|\mu^{-1}\mathring{\operatorname{curl}}E\right\rangle_{\rho,0,0} \geq c_0\langle E|E\rangle_{\rho,0,0}$$

for all $E\in D\left(\mathring{\operatorname{curl}}\right)\cap H_0$.

The Pre-Maxwell Equations.

σ is degenerate in the sense that

$$\sigma = \iota_{\Omega_c} \widetilde{\sigma}\, \iota_{\Omega_c}^{*}$$

for

$$\widetilde{\sigma} : L^2\left(\Omega_c, \mathbb{R}^3\right) \to L^2\left(\Omega_c, \mathbb{R}^3\right)$$

such that $\widetilde{\sigma}$ is strictly positive definite, where the open set $\Omega_c \subseteq \Omega$ is assumed to be such that $\overline{\Omega_c} \subseteq \Omega$ and that Ω_c has a Lebesgue null set as topological boundary.

$$H_0^{\perp} = N\left(\mathring{\operatorname{curl}}\right) \cap L^2\left(\Omega \setminus \overline{\Omega_c}, \mathbb{R}^3\right).$$

As an assumption on the boundary quality we require

$$H_0^{\perp} = N\left(\mathring{\operatorname{curl}}_{\Omega \setminus \overline{\Omega_c}}\right)$$

for which for example segment property for Ω_c would be sufficient.

The Pre-Maxwell Equations.

Here $\mathring{\operatorname{curl}}_{\Omega\setminus\overline{\Omega_c}}$ denotes the operator $\mathring{\operatorname{curl}}$ constructed with Ω replaced by $\Omega\setminus\overline{\Omega_c}$. In this case

$$\begin{aligned} H_0 &= \overline{R\left(\operatorname{curl}_{\Omega\setminus\overline{\Omega_c}}\right)} \oplus L^2\left(\Omega_c,\mathbb{R}^3\right) \\ &= \left(R\left(\mathring{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}}\right) \oplus \mathscr{H}_{D,\Omega\setminus\overline{\Omega_c}}\right)^{\perp} \oplus L^2\left(\Omega_c,\mathbb{R}^3\right) \\ &= \left(N\left(\operatorname{div}_{\Omega\setminus\overline{\Omega_c}}\right) \cap \mathscr{H}^{\perp}_{D,\Omega\setminus\overline{\Omega_c}}\right) \oplus L^2\left(\Omega_c,\mathbb{R}^3\right), \end{aligned}$$

where $\mathring{\operatorname{grad}}_{\Omega\setminus\overline{\Omega_c}}$ denotes the operator $\mathring{\operatorname{grad}}$ constructed with Ω replaced by $\Omega\setminus\overline{\Omega_c}$, $\operatorname{div}_{\Omega\setminus\overline{\Omega_c}} := -\mathring{\operatorname{grad}}^{*}_{\Omega\setminus\overline{\Omega_c}}$ with adjoint taken in $L^2\left(\Omega\setminus\overline{\Omega_c},\mathbb{R}^3\right)$ and

$$\mathscr{H}_{D,\Omega\setminus\overline{\Omega_c}} := N\left(\operatorname{div}_{\Omega\setminus\overline{\Omega_c}}\right) \cap N\left(\mathring{\operatorname{curl}}_{\Omega\setminus\overline{\Omega_c}}\right)$$

denotes the space of harmonic Dirichlet fields in $L^2\left(\Omega\setminus\overline{\Omega_c},\mathbb{R}^3\right)$.

The Pre-Maxwell Equations.

We will use the orthogonal decomposition

$$H_0 = R(\operatorname{curl}) \oplus H_1 \oplus H_2. \tag{1}$$

where

$$H_1 := N\left(\mathring{\operatorname{curl}}\right) \cap R(\sigma) = N\left(\mathring{\operatorname{curl}}_{\Omega_c}\right) \subseteq N\left(\mathring{\operatorname{curl}}\right)$$

$$H_2 := N\left(\mathring{\operatorname{curl}}\right) \cap \left(\left(N(\operatorname{div}_{\Omega_c}) \cap \mathscr{H}^{\perp}_{D,\Omega_c}\right) \oplus \left(N\left(\operatorname{div}_{\Omega\setminus\overline{\Omega_c}}\right) \cap \mathscr{H}^{\perp}_{D,\Omega\setminus\overline{\Omega_c}}\right)\right) \subseteq N\left(\mathring{\operatorname{curl}}\right).$$

The Pre-Maxwell Equations.

Lemma : We have for $U_k \in H_k,\ k = 1, 2,$

$$\left|\chi_{\Omega_c}(U_1 + U_2)\right|^2 = |U_1|^2 + \left|\chi_{\Omega_c} U_2\right|^2 .$$

Proof : (Idea) $U_1 = \chi_{\Omega_c} U_1 = \mathring{\operatorname{grad}}_{\Omega_c} \psi \perp N(\operatorname{div}_{\Omega_c})$ and $\chi_{\Omega_c} U_2 \in N(\operatorname{div}_{\Omega_c})$.

□

We also note: we have for some positive constant k_0 that

$$|U_0|^2 \leq k_0 \left|\mathring{\operatorname{curl}} U_0\right|^2$$

for all $U_0 \in D\left(\mathring{\operatorname{curl}}\right) \cap R(\operatorname{curl})$.

Proof : (Idea) compact embedding property.

□

The Pre-Maxwell Equations.

Finally we need the following more subtle lemma.

Lemma : We have a positive constant k_1 so that

$$\left|U_2\right| \leq k_2 \left|\chi_{\Omega_c} U_2\right| \tag{2}$$

for all $U_2 \in H_2$.

Proof : (Idea) $U_2 = \mathring{\operatorname{grad}} \varphi \in N\left(\mathring{\operatorname{curl}}\right)$ and

$$\operatorname{div} \operatorname{grad} \varphi = 0 \text{ in } \Omega_c \cup \Omega \setminus \overline{\Omega_c}.$$

Common boundary values on the interface! Thus, U_2 is determined by $\chi_{\Omega_c} U_2$ and the continuity estimate can be shown.

□

Lemma : There is a positive constant c_0 such that we have

$$c_0 \left|U\right|^2 \leq \left|\sigma^{1/2} U\right|^2 + \left|\mathring{\operatorname{curl}} U\right|^2 \tag{3}$$

for all $U \in D(C) \cap H_0$.

The Pre-Maxwell Equations.

Proof : We observe

$$\min\{1, c_*\}\left(\left|\chi_{\Omega_c} U\right|^2 + \left|\mathring{\operatorname{curl}} U\right|^2\right) \leq \left|\sigma^{1/2} U\right|^2 + \left|\mathring{\operatorname{curl}} U\right|^2$$

and so the estimate follows if we can show that

$$c_1 \left|U\right|^2 \leq \left|\chi_{\Omega_c} U\right|^2 + \left|\mathring{\operatorname{curl}} U\right|^2 .$$

We shall employ the above decomposition so that
$U = U_0 + U_1 + U_2$ with $U_0 \in R(\operatorname{curl})$, $U_k \in H_k$, $k = 1, 2$.

$$\begin{aligned}
\left|U\right|^2 &= \left|U_0\right|^2 + \left|U_1\right|^2 + \left|U_2\right|^2 \\
&\leq k_0^2 \left|\mathring{\operatorname{curl}} U_0\right|^2 + \left|\chi_{\Omega_c} U_1\right|^2 + k_2^2 \left|\chi_{\Omega_c} U_2\right|^2 \\
&\leq k_0^2 \left|\mathring{\operatorname{curl}} U_0\right|^2 + 2\max\left\{1, k_2^2\right\}\left|\chi_{\Omega_c}\left(U_0 + U_1 + U_2\right)\right|^2 + \\
&\quad + 2\max\left\{1, k_2^2\right\}\left|U_0\right|^2 ,
\end{aligned}$$

The Pre-Maxwell Equations.

$$
\begin{aligned}
|U|^2 &= |U_0|^2 + |U_1|^2 + |U_2|^2 \\
&\leq k_0^2 \left|\mathring{\operatorname{curl}}\, U_0\right|^2 + \left|\chi_{\Omega_c} U_1\right|^2 + k_2^2 \left|\chi_{\Omega_c} U_2\right|^2 \\
&\leq k_0^2 \left|\mathring{\operatorname{curl}}\, U_0\right|^2 + 2 \max\left\{1, k_2^2\right\} \left|\chi_{\Omega_c} (U_0 + U_1 + U_2)\right|^2 + \\
&\quad + 2 \max\left\{1, k_2^2\right\} |U_0|^2 , \\
&\leq \max\left\{1, k_0^2\right\} \left(1 + 2 \max\left\{1, k_2^2\right\}\right) \left(\left|\mathring{\operatorname{curl}}\, U\right|^2 + \left|\chi_{\Omega_c} U\right|^2\right).
\end{aligned}
$$

□