Superconvergent graded meshes for Dirichlet control problems

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Plan of the talk

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The elliptic Dirichlet control problem

- bounded polygonal domain Ω with boundary Γ
- state variable $y \in Y := L^2(\Omega)$
- control variable $u \in U_{ad} := \{u \in L^2(\Gamma) : \ a \leq u(x) \leq b \text{ for a.a. } x \in \Gamma\}$

Dirichlet control problem

$$\min_{(y,u)\in Y\times U_{ad}} J(y,u) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Gamma)}$$

$$\text{subject to} \quad -\Delta y = 0 \quad \text{in } \Omega, \quad y = u \quad \text{on } \Gamma, \quad \text{in very weak sense}$$

$$\Leftrightarrow \quad (y, \Delta v)_{L^2(\Omega)} = (u, \partial_n v)_{L^2(\Gamma)} \quad \forall v \in H^1_0(\Omega) \cap H^1_\Delta(\Omega)$$

- desired state $y_d \in H^s(\Omega)$ with some $s \geq 0$
- small parameter ν

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First order optimality conditions

- adjoint state $\bar{p} \in V := \{v \in H_0^1(\Omega) : \Delta v \in L^2(\Omega)\}$
- projection operator $\Pi_{[a,b]}(c) := \min\{b, \max\{a, c\}\}$

First order optimality system

$$
\begin{aligned}
-\Delta \bar{y} &= f && \text{in } \Omega, & \bar{y} &= \bar{u} && \text{on } \partial\Omega && \text{in very weak sense} \\
-\Delta \bar{p} &= \bar{y} - y_d && \text{in } \Omega, & \bar{p} &= 0 && \text{on } \partial\Omega && \text{in weak sense} \\
& && & \bar{u} &= \Pi_{[a,b]}\left(\tfrac{1}{\nu}\partial_n \bar{p}\right) && \text{on } \partial\Omega &&
\end{aligned}
$$

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Discretization

Let $\mathcal{T}_h$ be a conforming finite element mesh. Define

$$Y_h = \{v_h \in H^1(\Omega) : v_h|_T \in \mathcal{P}_1 \ \forall T \in \mathcal{T}_h\},\ Y_{0h} = Y_h \cap H_0^1(\Omega),\ U_{ad}^h = Y_h|_\Gamma \cap U_{ad}$$

Discrete Dirichlet control problem

$$\min_{(y_h,u_h)\in Y_h \times U_{ad}^h} J(y_h, u_h) := \frac{1}{2}\|y_h - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u_h\|^2_{L^2(\Gamma)}$$

subject to $(\nabla y_h, \nabla v_h)_{L^2(\Omega)} = (f, v_h)_{L^2(\Omega)}$ for all $v_h \in Y_{0h}$ and $y_h|_\Gamma = u_h$

Discrete optimality system

$$\begin{aligned}
(\nabla \bar{y}_h, \nabla v_h)_{L^2(\Omega)} &= 0 && \forall v_h \in Y_{0h} \text{ and } \bar{y}_h|_\Gamma = \bar{u}_h,\\
(\nabla \bar{p}_h, \nabla v_h)_{L^2(\Omega)} &= (\bar{y}_h - y_d, v_h)_{L^2(\Omega)} && \forall v_h \in Y_{0h},\\
(\nu \bar{u}_h - \partial_n^h \bar{p}_h, u_h - \bar{u}_h)_{L^2(\Gamma)} &\geq 0 && \forall u_h \in U_{ad}^h,
\end{aligned}$$

where the discrete normal derivative $\partial_n^h \bar{p}_h \in Y_h|_\Gamma$ is defined by

$$(\partial_n^h \bar{p}_h, v_h)_{L^2(\Gamma)} = -(\bar{y}_h - y_d, v_h)_{L^2(\Omega)} + (\nabla \bar{p}_h, \nabla v_h)_{L^2(\Omega)} \quad \text{for all } v_h \in Y_h \setminus Y_{0h}$$

Approximation results

Contributions, among others:

- E. Casas, J.-P. Raymond: Error estimates for the numerical approximation of Dirichlet boundary control for semilinear elliptic equations. SICON 45(2006), 1586–1611.
- K. Deckelnick, A. Günther, M. Hinze: Finite element approximation of Dirichlet boundary control for elliptic PDEs on two- and three-dimensional curved domains. SICON 48(2009), 2798–2819.
- S. May, R. Rannacher, B. Vexler: Error analysis for a finite element approximation of elliptic Dirichlet boundary control problems. SICON 51(2013), 2585–2611.
- Th. Apel, M. Mateos, J. Pfefferer, A. Rösch: Error estimates for Dirichlet control problems in polygonal domains: Quasi-uniform meshes. arXiv:1704.08843

General error estimate

- $S_h : U_h \to Y_h$ is the discrete harmonic extension
- $Q_h : L^2(\Gamma) \to U_h$ is the $L^2(\Gamma)$-projection
- $u_h^* \in U_{ad}^h$ such that $(\nu \bar{u} - \partial_n \bar{p}, u_h^* - \bar{u})_{L^2(\Gamma)} = 0$.

General error estimate

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \le c \left(\|\bar{u} - u_h^*\|_{L^2(\Gamma)} + \|\bar{y} - S_h Q_h \bar{u}\|_{L^2(\Omega)} + \sup_{\psi_h \in U_h} \frac{\left|(\nabla \bar{p}, \nabla S_h \psi_h)_{L^2(\Omega)}\right|}{\|\psi_h\|_{L^2(\Gamma)}} \right)$$

- first term: quasi-interpolation error
- second term: contains approximation of non-smooth boundary condition, in general $y \notin H^1(\Omega)$
- third term: corresponds to error estimate of normal derivative, determines the overall convergence order, numerator equals: $\left|(\nabla(\bar{p} - I_h \bar{p}), \nabla S_h \psi_h)_{L^2(\Omega)}\right|$ note the $L^2(\Gamma)$-norm in the denominator

Plan of the talk

Superconvergence meshes: contributions

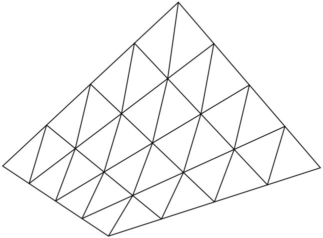
Many contributions, among them:

- M. Křížek, P. Neittaanmäki. On superconvergence techniques. Acta Applicandae Mathematicae, 9(3):175–198, 1987.
- J.Z. Zhu, O.C. Zienkiewicz. Superconvergence recovery technique and a posteriori error estimators. IJNME, 30:1321–1339, 1990.
- L.B. Wahlbin. Superconvergence in Galerkin Finite Element Methods. Springer, Berlin, 1995.
- A.M. Lakhany, I. Marek, J.R. Whiteman. Superconvergence results on mildly structured triangulations. CMAME, 189(1):1–75, 2000.
- J. Brandts, M. Křížek. History and future of superconvergence in three-dimensional finite element methods. In Finite element methods (Jyväskylä, 2000), pages 22–33. Gakkotosho, Tokyo, 2001.
- R.E. Bank, J. Xu. Asymptotically exact a posteriori error estimators, part I: Grids with superconvergence. SINUM, 41(6):2294–2312, 2003.

All for quasi-uniform meshes.

Superconvergence meshes: definition

[Bank/Xu 03]: meshes with $O(h^2)$ approximate parallelogram property



Mesh produced by Max Winkler

The lengths of any two opposite edges differ by $O(h^2)$

except in a region of size $O(h^{2\sigma})$

for some applications (Neumann boundary conditions) there is another condition for boundary edges

Result for any $u \in W^{3,\infty}(\Omega)$ and any $v_h \in V_h$ (p.w. linears):

$$\left| \int_\Omega \nabla(u - I_h u) \cdot \nabla v_h \right| \leq c h^{1+\min\{1,\sigma\}} |\log h|^{1/2} \|u\|_{W^{3,\infty}(\Omega)} |v_h|_{H^1(\Omega)}$$

Note that a piecewise $O(h^2)$ approximate parallelogram property is sufficient.

Superconvergence meshes: applications

Applications of the formula

$$\left|\int_\Omega \nabla(u - I_h u) \cdot \nabla v_h\right| \le ch^{1+\min\{1,\sigma\}} |\log h|^{1/2} \|u\|_{W^{3,\infty}(\Omega)} |v_h|_{H^1(\Omega)}$$

1. Supercloseness of interpolant:

$$c_1 \|u_h - I_h u\|_{H^1(\Omega)} \le \sup_{v_h \in V_h} \frac{a(u_h - I_h u, v_h)}{\|v_h\|_{H^1(\Omega)}} = \sup_{v_h \in V_h} \frac{a(u - I_h u, v_h)}{\|v_h\|_{H^1(\Omega)}}$$

$$\le ch^{1+\min\{1,\sigma\}} |\log h|^{1/2} \|u\|_{W^{3,\infty}(\Omega)}$$

Note the $H^1(\Omega)$-norm in the denominator.

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Superconvergence meshes: applications

Applications of the formula

$$\left|\int_\Omega \nabla(u - I_h u)\cdot\nabla v_h\right| \le ch^{1+\min\{1,\sigma\}}|\log h|^{1/2}\|u\|_{W^{3,\infty}(\Omega)}|v_h|_{H^1(\Omega)}$$

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$$c_1\|u_h - I_h u\|_{H^1(\Omega)} \le \sup_{v_h\in V_h}\frac{a(u_h - I_h u, v_h)}{\|v_h\|_{H^1(\Omega)}} = \sup_{v_h\in V_h}\frac{a(u - I_h u, v_h)}{\|v_h\|_{H^1(\Omega)}}$$
$$\le ch^{1+\min\{1,\sigma\}}|\log h|^{1/2}\|u\|_{W^{3,\infty}(\Omega)}$$

Corollary: properties of gradient recovery [Bank/Xu 03, Thm. 4.2]:

$$\|\nabla u - Q_h\nabla u_h\|_{L^2(\Omega)} \le ch^{1+\min\{1,\sigma\}}|\log h|^{1/2}\|u\|_{W^{3,\infty}(\Omega)}$$

$Q_h : L^2(\Omega) \to V_h$ is the $L^2(\Omega)$-projection operator

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Superconvergence meshes: applications

Applications of the formula

$$\left|\int_\Omega \nabla(u - I_h u)\cdot\nabla v_h\right| \le c h^{1+\min\{1,\sigma\}}|\log h|^{1/2}\|u\|_{W^{3,\infty}(\Omega)}|v_h|_{H^1(\Omega)}$$

2. Approximation of normal derivatives: By the definition of the variational normal derivative $\partial_n^h u_h$ we have

$$(\partial_n u - \partial_n^h u_h, z_h)_\Gamma = (\nabla(u - u_h), \nabla z_h)_\Omega \quad \forall z_h \in V_h$$

Moreover, we have

$$\begin{aligned}\|\partial_n u - \partial_n^h u_h\|^2_{L^2(\Gamma)} &= (\partial_n u - \partial_n^h u_h, \partial_n u - \partial_n^h u_h)_\Gamma \\ &= (\partial_n u - \partial_n^h u_h, \partial_n u - Q_h\partial_n u)_\Gamma + (\partial_n u - \partial_n^h u_h, Q_h\partial_n u - \partial_n^h u_h)_\Gamma\end{aligned}$$

With $e_h := Q_h\partial_n u - \partial_n^h u_h$ and the discrete harmonic extension operator S_h we get

$$(\partial_n u - \partial_n^h u_h, e_h)_\Gamma = (\nabla(u - u_h), \nabla S_h e_h)_\Omega = (\nabla(u - I_h u), \nabla S_h e_h)_\Omega$$

which can be estimated by the Bank/Xu-formula

Superconvergence meshes: applications

3. Deckelnick/Günther/Hinzeconsidered the approximation of smooth domains and modified the estimate with $r > 2$ to

$$\left| \int_{\Omega_h} \nabla(u - I_h u) \cdot \nabla v_h \right| \leq c \|u\|_{W^{3,r}(\Omega_h)} \left(h^{1+\min\{1,\sigma\}} \|v_h\|_{H^1(\Omega_h)} + h^{3/2} \|v_h\|_{L^2(\Gamma_h)} \right)$$

and used it in the analysis of Dirichlet control problems with L^2-regularization. The approximation order $\frac{3}{2}$ for control and state is optimal due to regularity issues.

Note: It is not obvious whether this estimate holds for meshes with only piecewise $O(h^2)$ approximate parallelogram property.

This result stimulated our treatment of superconvergence meshes within the investigation of Dirichlet control problems in non-smooth domains.

K. Deckelnick, A. Günther, M. Hinze: Finite element approximation of Dirichlet boundary control for elliptic PDEs on two- and three-dimensional curved domains, SIAM J. Control Optim. 48(2009), 2798–2819.

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Graded meshes: contributions

Many contributions, among them:

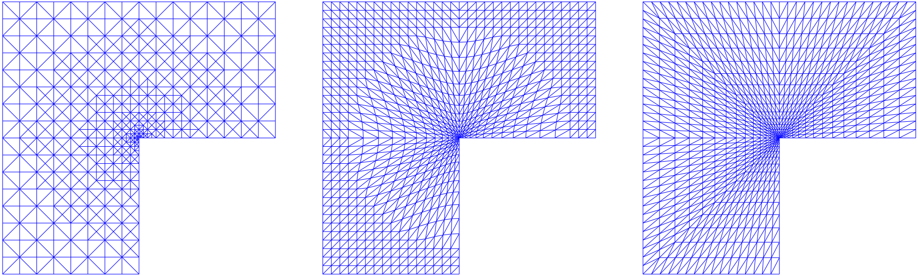
- L.A. Oganesyan, L.A. Rukhovets. Variational-difference schemes for linear second-order elliptic equations in a two-dimensional region with piecewise smooth boundary. Zh. Vychisl. Mat. Mat. Fiz., 8:97–114, 1968.
- I. Babuška. Finite element method for domains with corners. Computing, 6:264–273, 1970.
- G. Raugel. Résolution numérique par une méthode d'éléments finis du problème de Dirichlet pour le Laplacien dans un polygone. C. R. Acad. Sci. Paris, Sèr. A, 286(18):A791–A794, 1978.
- A.H. Schatz, L.B. Wahlbin. Maximum norm estimates in the finite element method on plane polygonal domains. Part 2: Refinements. Math. Comp., 33(146):465–492, 1979.
- R. Fritzsch, P. Oswald. Zur optimalen Gitterwahl bei Finite-Elemente-Approximationen. WZ TU Dresden, 37(3):155–158, 1988.
- C. Băcuţă, V. Nistor, L.T. Zikatanov. Improving the rate of convergence of high order finite elements on polygons and domains with cusps. Numer. Math., 100(2):165–184, 2005.

All without superconvergence.

Definition of graded meshes

With global mesh parameter h and grading parameter $\mu \in (0, 1]$,
let the element size $h_T := diamT$ be related to the distance r_T to the corner

$$h_T \sim \begin{cases} h^{1/\mu} & \text{for } r_T = 0 \\ h r_T^{1-\mu} & \text{for } R \geq r_T > 0 \\ h & \text{for } r_T > R \end{cases} \qquad (*)$$



Purpose of mesh grading: regularity issues

Consider polygonal domain Ω with boundary Γ and

$$-\Delta u + u = f \quad \text{in } \Omega$$

with Dirichlet or Neumann boundary conditions.

We have near corners with interior angle ω

$$u = u_r + \xi(r)\, r^{\lambda}\, \Phi(\varphi)$$

with regular part u_r, cut-off function $\xi(r)$, smooth function $\Phi(\varphi)$, and $\lambda = \pi/\omega$.
The letter λ denotes the singularity exponent in the whole talk.

- $u \in H^2(\Omega)$ for $\omega < \pi$
- $u \in W^{2,\infty}(\Omega)$ for $\omega < \pi/2$

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Purpose of mesh grading: error estimates

For the piecewise linear finite element approximation we have:

$$\|u - u_h\|_{H^1(\Omega)} \leq ch^{\min\{1, \lambda/\mu - \varepsilon\}} \|f\|_{L^2(\Omega)}$$
$$\|u - u_h\|_{L^2(\Omega)} \leq ch^{2\min\{1, \lambda/\mu - \varepsilon\}} \|f\|_{L^2(\Omega)}$$

[Oganesyan/Rukhovets 68, 74, 79], [Babuška 70], [Raugel 78], . . . , [Pfefferer 12, 15]

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[Oganesyan/Rukhovets 68, 74, 79], [Babuška 70], [Raugel 78], . . . , [Pfefferer 12, 15]

$$\|u - u_h\|_{L^\infty(\Omega)} \le ch^{\min\{2,\lambda/\mu\}-\varepsilon}\|f\|_X$$

[Schatz/Wahlbin 79] for smooth right hand sides f and $c = c(f)$
[Sirch 10] for $X = C^{0,\sigma}(\Omega)$, $h^{-\varepsilon} \widehat{=} |\log h|^{3/2}$, Dirichlet problem
[Rogovs et al. 17] for $X = C^{0,\sigma}(\Omega)$, $h^{-\varepsilon} \widehat{=} |\log h|^{1}$, Neumann and Dirichlet problems

Purpose of mesh grading: error estimates

For the piecewise linear finite element approximation we have:

$$\|u - u_h\|_{L^2(\Gamma)} \le c h^{\min\{2,(\frac{1}{2}+\lambda)/\mu\}-\varepsilon} \|f\|_X$$

[Pfefferer et al. 15] for $X = C^{0,\sigma}(\Omega)$, $h^{-\varepsilon} \widehat{=} |\log h|^{1+\delta}$, $\delta = \delta(\lambda, \mu)$, Neumann problem

$$\|\partial_n u - \Pi_h \partial_n u\|_{L^2(\Gamma)} \le c h^{\min\{1/2,(\lambda-\frac{1}{2})/\mu\}-\varepsilon} \|f\|_{L^2(\Omega)}$$

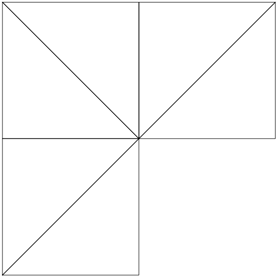
[Apel/Nicaise/Pfefferer 16] Dirichlet problem, $\Pi_h \ldots L^2(\Gamma)$-projection

Generation of graded meshes 1: Bisection

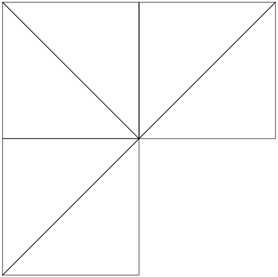
Initialize refinement algorithm with coarse start mesh. Afterwards, mark every element $T \in \mathcal{T}_h$ for refinement which satisfies

$$h_T > h \quad \text{or} \quad h_T > h \left(\frac{r_T}{R}\right)^{1-\mu}$$

until the desired mesh is reached. [Fritzsch/Oswald 88]



$h = 1.4142$, $\mu = 1.0$



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mesh hierarchy, smooth transition from one element to the next
but approximate parallelogram property is violated

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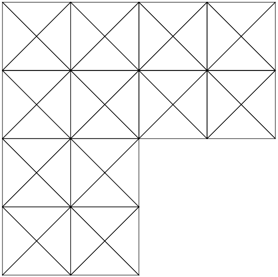
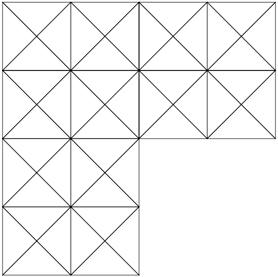


$h = 0.5$, $\mu = 1.0$



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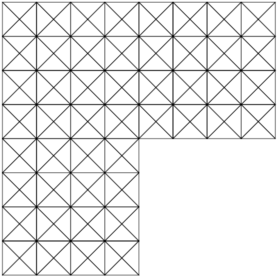
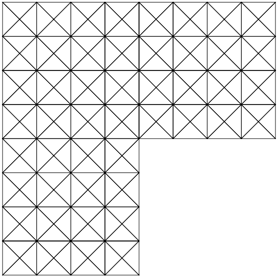


$h = 0.25$, $\mu = 1.0$



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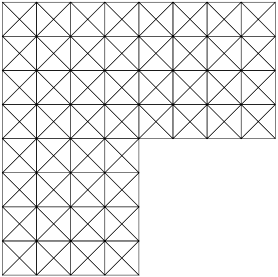

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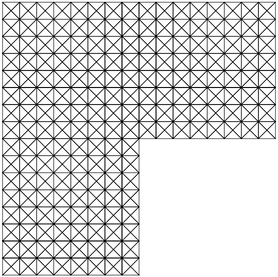

$h = 0.125$, $\mu = 1.0$

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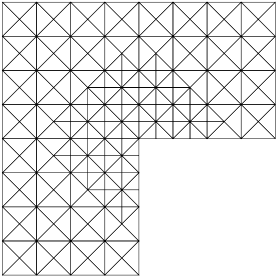
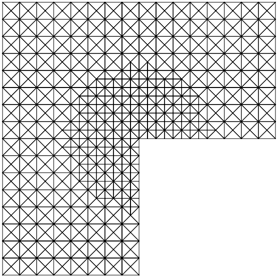


$h = 0.25$, $\mu = 0.9$



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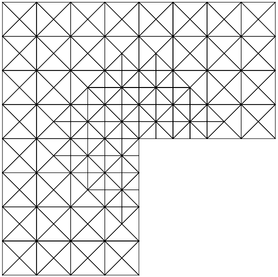
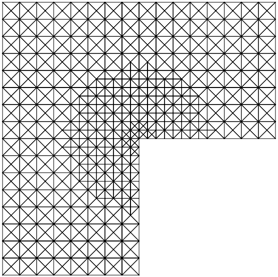


$h = 0.25$, $\mu = 0.8$



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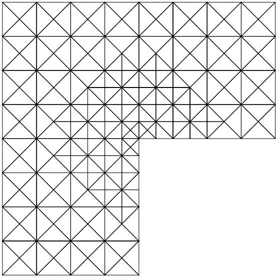
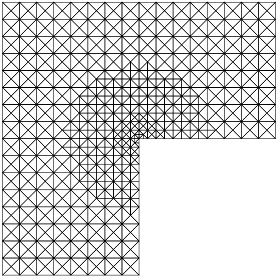


$h = 0.25$, $\mu = 0.7$



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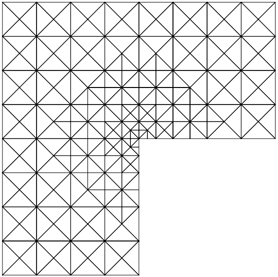
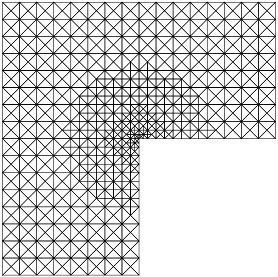


$h = 0.25$, $\mu = 0.6$



$h = 0.125$, $\mu = 0.6$

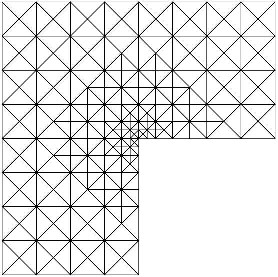
mesh hierarchy, smooth transition from one element to the next
but approximate parallelogram property is violated

Generation of graded meshes 1: Bisection

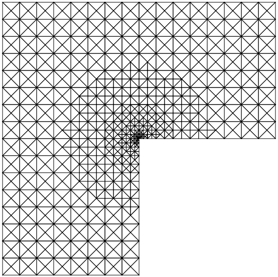
Initialize refinement algorithm with coarse start mesh. Afterwards, mark every element $T \in \mathcal{T}_h$ for refinement which satisfies

$$h_T > h \quad \text{or} \quad h_T > h \left(\frac{r_T}{R} \right)^{1-\mu}$$

until the desired mesh is reached. [Fritzsch/Oswald 88]



$h = 0.25$, $\mu = 0.5$



$h = 0.125$, $\mu = 0.5$

mesh hierarchy, smooth transition from one element to the next
but approximate parallelogram property is violated

Generation of graded meshes 1: Bisection

Initialize refinement algorithm with coarse start mesh. Afterwards, mark every element $T \in \mathcal{T}_h$ for refinement which satisfies

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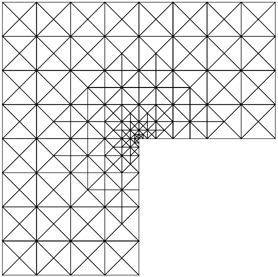
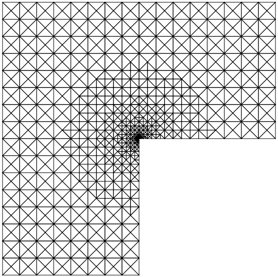
until the desired mesh is reached. [Fritzsch/Oswald 88]



$h = 0.25$, $\mu = 0.4$



$h = 0.125$, $\mu = 0.4$

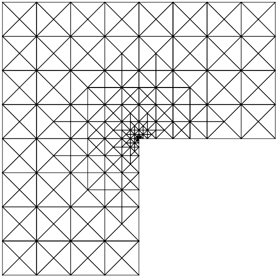
mesh hierarchy, smooth transition from one element to the next
but approximate parallelogram property is violated

Generation of graded meshes 1: Bisection

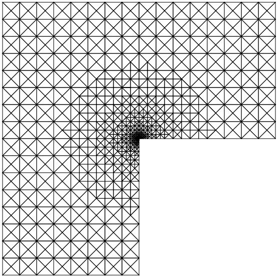
Initialize refinement algorithm with coarse start mesh. Afterwards, mark every element $T \in \mathcal{T}_h$ for refinement which satisfies

$$h_T > h \quad \text{or} \quad h_T > h \left(\frac{r_T}{R} \right)^{1-\mu}$$

until the desired mesh is reached. [Fritzsch/Oswald 88]



$h = 0.25$, $\mu = 0.3$



$h = 0.125$, $\mu = 0.3$

mesh hierarchy, smooth transition from one element to the next
but approximate parallelogram property is violated

Generation of graded meshes 1: Bisection

Initialize refinement algorithm with coarse start mesh. Afterwards, mark every element $T \in \mathcal{T}_h$ for refinement which satisfies

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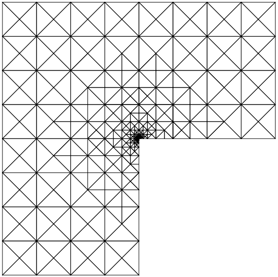
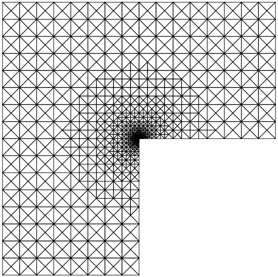
until the desired mesh is reached. [Fritzsch/Oswald 88]



$h = 0.25$, $\mu = 0.2$



$h = 0.125$, $\mu = 0.2$

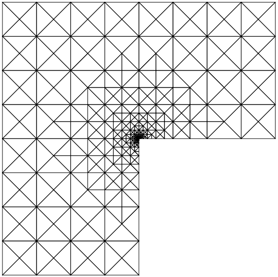
mesh hierarchy, smooth transition from one element to the next
but approximate parallelogram property is violated

Generation of graded meshes 1: Bisection

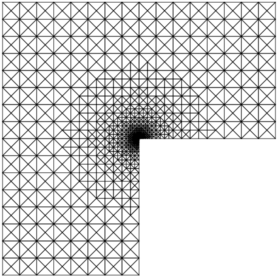
Initialize refinement algorithm with coarse start mesh. Afterwards, mark every element $T \in \mathcal{T}_h$ for refinement which satisfies

$$h_T > h \quad \text{or} \quad h_T > h \left(\frac{r_T}{R} \right)^{1-\mu}$$

until the desired mesh is reached. [Fritzsch/Oswald 88]



$h = 0.25$, $\mu = 0.1$

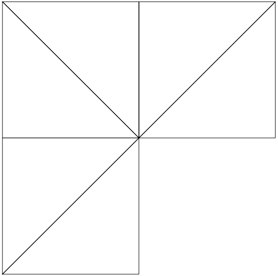


$h = 0.125$, $\mu = 0.1$

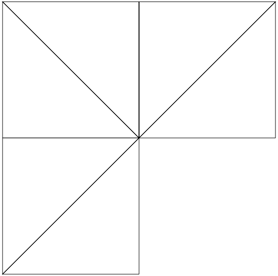
mesh hierarchy, smooth transition from one element to the next
but approximate parallelogram property is violated

Generation of graded meshes 2: Relocation

1. Refine a coarse start mesh uniformly until $h_T \sim h \, \forall T \in \mathcal{T}_h$ with desired mesh size h.
2. Transform the nodes $X^{(i)} \in \Omega \cap S_R$ according to $X_{new}^{(i)} = X^{(i)} \left(\frac{r(X^{(i)})}{R} \right)^{1/\mu - 1}$ [Oganesyan/Rukhovets 68, 74, 79]



$h = 1.4142$, $\mu = 1.0$

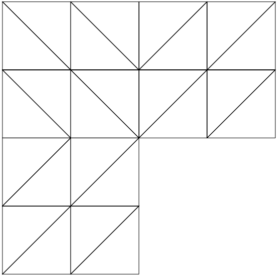


$h = 1.4142$, $\mu = 1.0$

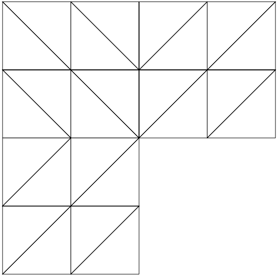
no mesh hierarchy but approximate parallelogram property can be achieved

Generation of graded meshes 2: Relocation

1. Refine a coarse start mesh uniformly until $h_T \sim h \, \forall T \in \mathcal{T}_h$ with desired mesh size h.
2. Transform the nodes $X^{(i)} \in \Omega \cap S_R$ according to $X^{(i)}_{new} = X^{(i)} \left(\frac{r(X^{(i)})}{R} \right)^{1/\mu - 1}$ [Oganesyan/Rukhovets 68, 74, 79]



$h = 0.7071$, $\mu = 1.0$

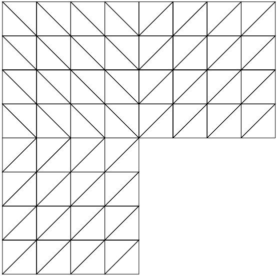


$h = 0.7071$, $\mu = 1.0$

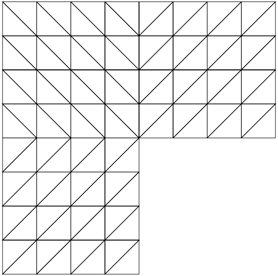
no mesh hierarchy but approximate parallelogram property can be achieved

Generation of graded meshes 2: Relocation

1. Refine a coarse start mesh uniformly until $h_T \sim h \, \forall T \in \mathcal{T}_h$ with desired mesh size h.
2. Transform the nodes $X^{(i)} \in \Omega \cap S_R$ according to $X^{(i)}_{new} = X^{(i)} \left(\frac{r(X^{(i)})}{R} \right)^{1/\mu - 1}$ [Oganesyan/Rukhovets 68, 74, 79]



$h = 0.3536$, $\mu = 1.0$

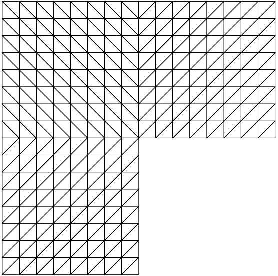


$h = 0.3536$, $\mu = 1.0$

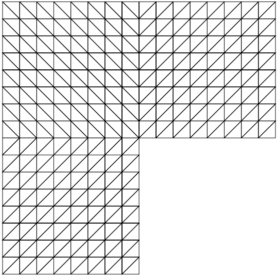
no mesh hierarchy but approximate parallelogram property can be achieved

Generation of graded meshes 2: Relocation

1. Refine a coarse start mesh uniformly until $h_T \sim h \, \forall T \in \mathcal{T}_h$ with desired mesh size h.
2. Transform the nodes $X^{(i)} \in \Omega \cap S_R$ according to $X^{(i)}_{new} = X^{(i)} \left(\frac{r(X^{(i)})}{R} \right)^{1/\mu - 1}$ [Oganesyan/Rukhovets 68, 74, 79]



$h = 0.1768$, $\mu = 1.0$

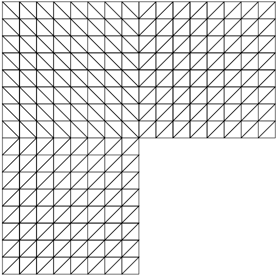


$h = 0.1768$, $\mu = 1.0$

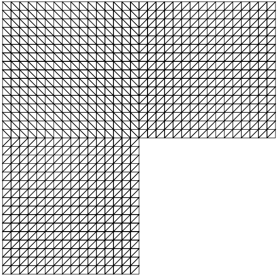
no mesh hierarchy but approximate parallelogram property can be achieved

Generation of graded meshes 2: Relocation

1. Refine a coarse start mesh uniformly until $h_T \sim h \, \forall T \in \mathcal{T}_h$ with desired mesh size h.
2. Transform the nodes $X^{(i)} \in \Omega \cap S_R$ according to $X^{(i)}_{new} = X^{(i)} \left(\frac{r(X^{(i)})}{R} \right)^{1/\mu - 1}$ [Oganesyan/Rukhovets 68, 74, 79]



$h = 0.1768$, $\mu = 1.0$

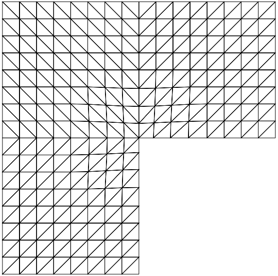


$h = 0.0884$, $\mu = 1.0$

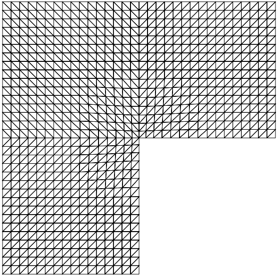
no mesh hierarchy but approximate parallelogram property can be achieved

Generation of graded meshes 2: Relocation

1. Refine a coarse start mesh uniformly until $h_T \sim h \, \forall T \in \mathcal{T}_h$ with desired mesh size h.
2. Transform the nodes $X^{(i)} \in \Omega \cap S_R$ according to $X^{(i)}_{new} = X^{(i)} \left(\frac{r(X^{(i)})}{R} \right)^{1/\mu - 1}$ [Oganesyan/Rukhovets 68, 74, 79]



$h = 0.1768$, $\mu = 0.9$

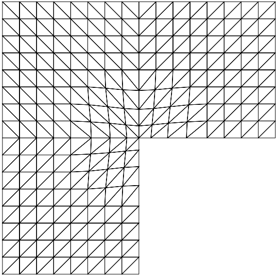


$h = 0.0884$, $\mu = 0.9$

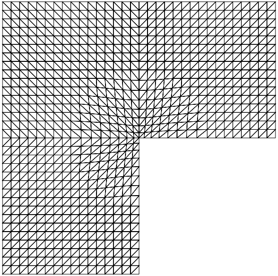
no mesh hierarchy but approximate parallelogram property can be achieved

Generation of graded meshes 2: Relocation

1. Refine a coarse start mesh uniformly until $h_T \sim h \, \forall T \in \mathcal{T}_h$ with desired mesh size h.
2. Transform the nodes $X^{(i)} \in \Omega \cap S_R$ according to $X^{(i)}_{new} = X^{(i)} \left(\frac{r(X^{(i)})}{R}\right)^{1/\mu-1}$ [Oganesyan/Rukhovets 68, 74, 79]



$h = 0.1768$, $\mu = 0.8$

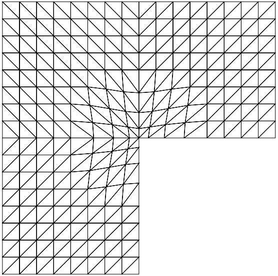


$h = 0.0884$, $\mu = 0.8$

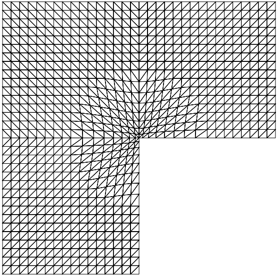
no mesh hierarchy but approximate parallelogram property can be achieved

Generation of graded meshes 2: Relocation

1. Refine a coarse start mesh uniformly until $h_T \sim h \, \forall T \in \mathcal{T}_h$ with desired mesh size h.
2. Transform the nodes $X^{(i)} \in \Omega \cap S_R$ according to $X^{(i)}_{new} = X^{(i)} \left(\frac{r(X^{(i)})}{R} \right)^{1/\mu - 1}$ [Oganesyan/Rukhovets 68, 74, 79]



$h = 0.1768$, $\mu = 0.7$

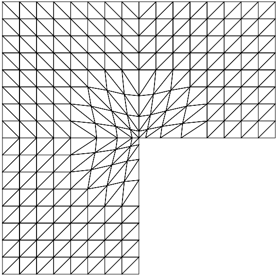


$h = 0.0884$, $\mu = 0.7$

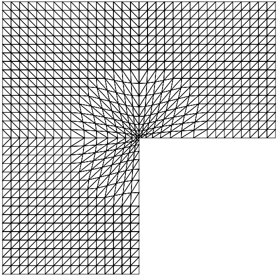
no mesh hierarchy but approximate parallelogram property can be achieved

Generation of graded meshes 2: Relocation

1. Refine a coarse start mesh uniformly until $h_T \sim h \, \forall T \in \mathcal{T}_h$ with desired mesh size h.
2. Transform the nodes $X^{(i)} \in \Omega \cap S_R$ according to $X^{(i)}_{new} = X^{(i)} \left(\frac{r(X^{(i)})}{R} \right)^{1/\mu - 1}$ [Oganesyan/Rukhovets 68, 74, 79]



$h = 0.1768$, $\mu = 0.6$



$h = 0.0884$, $\mu = 0.6$

no mesh hierarchy but approximate parallelogram property can be achieved

Generation of graded meshes 2: Relocation

1. Refine a coarse start mesh uniformly until $h_T \sim h \, \forall T \in \mathcal{T}_h$ with desired mesh size h.
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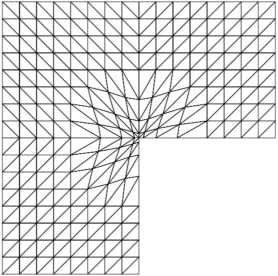
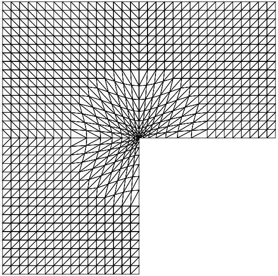


$h = 0.1768$, $\mu = 0.5$



$h = 0.0884$, $\mu = 0.5$

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Generation of graded meshes 2: Relocation

1. Refine a coarse start mesh uniformly until $h_T \sim h \, \forall T \in \mathcal{T}_h$ with desired mesh size h.
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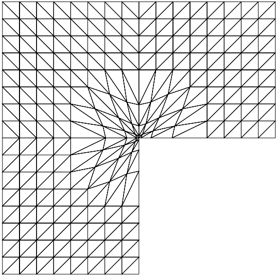
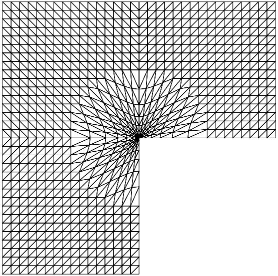


$h = 0.1768$, $\mu = 0.4$



$h = 0.0884$, $\mu = 0.4$

no mesh hierarchy but approximate parallelogram property can be achieved

Generation of graded meshes 2: Relocation

1. Refine a coarse start mesh uniformly until $h_T \sim h \, \forall T \in \mathcal{T}_h$ with desired mesh size h.
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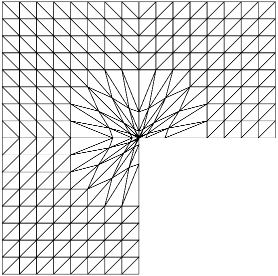
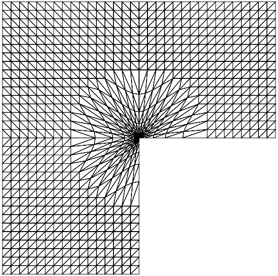


$h = 0.1768$, $\mu = 0.3$

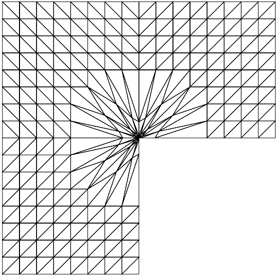


$h = 0.0884$, $\mu = 0.3$

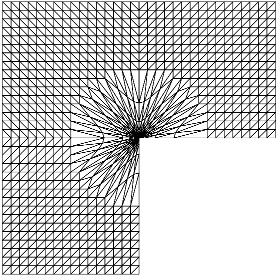
no mesh hierarchy but approximate parallelogram property can be achieved

Generation of graded meshes 2: Relocation

1. Refine a coarse start mesh uniformly until $h_T \sim h \; \forall T \in \mathcal{T}_h$ with desired mesh size h.
2. Transform the nodes $X^{(i)} \in \Omega \cap S_R$ according to $X^{(i)}_{new} = X^{(i)} \left(\frac{r(X^{(i)})}{R} \right)^{1/\mu - 1}$ [Oganesyan/Rukhovets 68, 74, 79]



$h = 0.1768$, $\mu = 0.2$



$h = 0.0884$, $\mu = 0.2$

no mesh hierarchy but approximate parallelogram property can be achieved

Generation of graded meshes 2: Relocation

1. Refine a coarse start mesh uniformly until $h_T \sim h \, \forall T \in \mathcal{T}_h$ with desired mesh size h.
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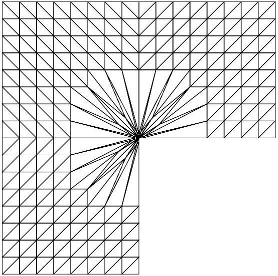
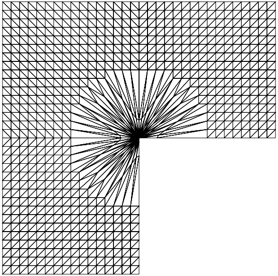


$h = 0.1768$, $\mu = 0.1$

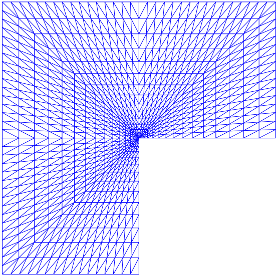


$h = 0.0884$, $\mu = 0.1$

no mesh hierarchy but approximate parallelogram property can be achieved

Generation of graded meshes 2: Relocation

Other metric can be used for relocation [Raugel]:



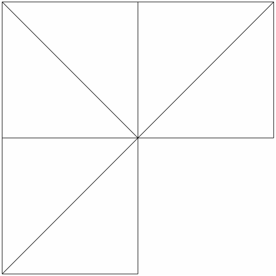
Generation of graded meshes 3: hierarchic with relocation

Contributions, among others:

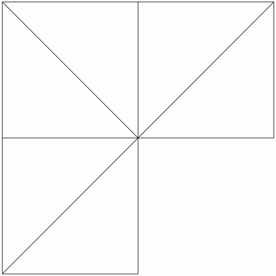
- Th. Apel, J. Schöberl. Multigrid methods for anisotropic edge refinement. SINUM, 40(5):1993–2006, 2002.
- C. Băcuţă, V. Nistor, L.T. Zikatanov. Improving the rate of convergence of high order finite elements on polygons and domains with cusps. Numer. Math., 100(2):165–184, 2005.
- L. Chen, H. Li. Superconvergence of gradient recovery schemes on graded meshes for corner singularities. J. Comp. Math., 28(1):11–31, 2010.

Generation of graded meshes 3: hierarchic with relocation

Start with a coarse mesh and split the triangles recursively into four subtriangles by introducing new nodes at each edge. The new nodes are usually the midpoints of the edges except if an edge is adjacent to a singular corner. In this case the edge is split in the ratio $2^{-1/\mu} : 1$.



$h = 1.4142$, $\mu = 0.5$

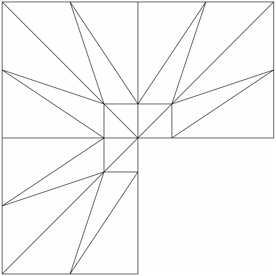


$h = 1.4142$, $\mu = 0.3$

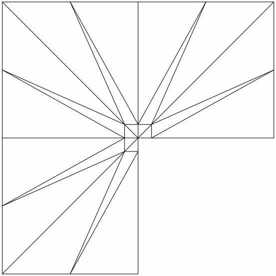
mesh hierarchy and approximate parallelogram property is satisfied up to region of size $O(h)$ (edges of coarse grid and edges of non-bisecting nodes)

Generation of graded meshes 3: hierarchic with relocation

Start with a coarse mesh and split the triangles recursively into four subtriangles by introducing new nodes at each edge. The new nodes are usually the midpoints of the edges except if an edge is adjacent to a singular corner. In this case the edge is split in the ratio $2^{-1/\mu} : 1$.



$h = 1.0607$, $\mu = 0.5$

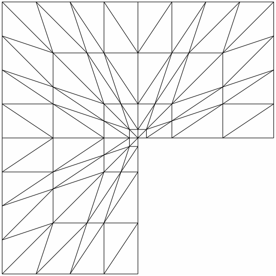


$h = 1.2739$, $\mu = 0.3$

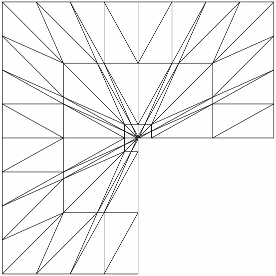
mesh hierarchy and approximate parallelogram property is satisfied up to region of size $O(h)$ (edges of coarse grid and edges of non-bisecting nodes)

Generation of graded meshes 3: hierarchic with relocation

Start with a coarse mesh and split the triangles recursively into four subtriangles by introducing new nodes at each edge. The new nodes are usually the midpoints of the edges except if an edge is adjacent to a singular corner. In this case the edge is split in the ratio $2^{-1/\mu} : 1$.



$h = 0.5303$, $\mu = 0.5$

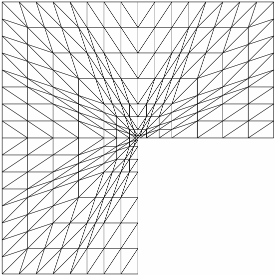


$h = 0.6370$, $\mu = 0.3$

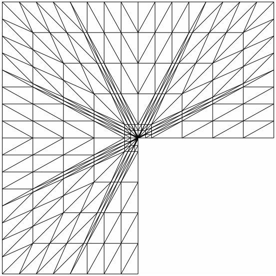
mesh hierarchy and approximate parallelogram property is satisfied up to region of size $O(h)$ (edges of coarse grid and edges of non-bisecting nodes)

Generation of graded meshes 3: hierarchic with relocation

Start with a coarse mesh and split the triangles recursively into four subtriangles by introducing new nodes at each edge. The new nodes are usually the midpoints of the edges except if an edge is adjacent to a singular corner. In this case the edge is split in the ratio $2^{-1/\mu}:1$.



$h = 0.2652$, $\mu = 0.5$

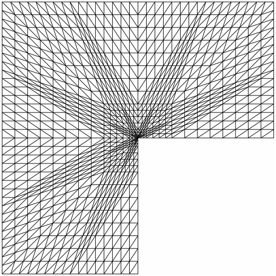


$h = 0.3185$, $\mu = 0.3$

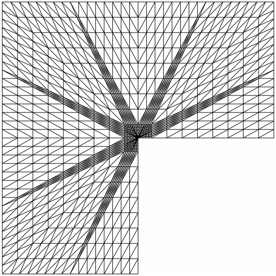
mesh hierarchy and approximate parallelogram property is satisfied up to region of size $O(h)$ (edges of coarse grid and edges of non-bisecting nodes)

Generation of graded meshes 3: hierarchic with relocation

Start with a coarse mesh and split the triangles recursively into four subtriangles by introducing new nodes at each edge. The new nodes are usually the midpoints of the edges except if an edge is adjacent to a singular corner. In this case the edge is split in the ratio $2^{-1/\mu} : 1$.



$h = 0.1326$, $\mu = 0.5$

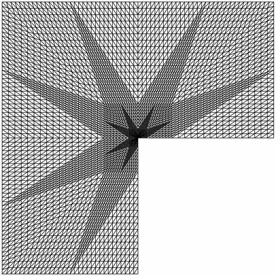


$h = 0.1592$, $\mu = 0.3$

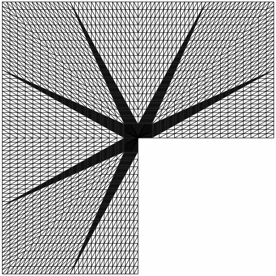
mesh hierarchy and approximate parallelogram property is satisfied up to region of size $O(h)$ (edges of coarse grid and edges of non-bisecting nodes)

Generation of graded meshes 3: hierarchic with relocation

Start with a coarse mesh and split the triangles recursively into four subtriangles by introducing new nodes at each edge. The new nodes are usually the midpoints of the edges except if an edge is adjacent to a singular corner. In this case the edge is split in the ratio $2^{-1/\mu} : 1$.



$h = 0.0663$, $\mu = 0.5$



$h = 0.0796$, $\mu = 0.3$

mesh hierarchy and approximate parallelogram property is satisfied up to region of size $O(h)$ (edges of coarse grid and edges of non-bisecting nodes)

Plan of the talk

Superconvergent graded meshes: contributions

Contributions, among others:

- Y.-Q. Huang. The superconvergence of finite element methods on domains with reentrant corners. In Finite element methods (Jyväskylä, 1997), pages 169–182. Dekker, New York, 1998.

for Raugel-type meshes

- L. Chen, H. Li. Superconvergence of gradient recovery schemes on graded meshes for corner singularities. J. Comp. Math., 28(1):11–31, 2010.

for hierarchic meshes with relocation

Results:

- estimates for $\|\nabla(u_h - I_h u)\|_{L^2(\Omega)}$ and for recovered gradient with $\mu < \min\{1, \frac{1}{2}\lambda\}$,
- not the right estimate for our purposes

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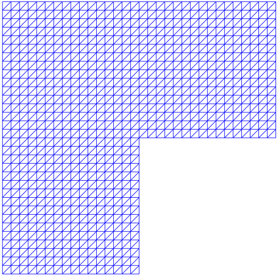
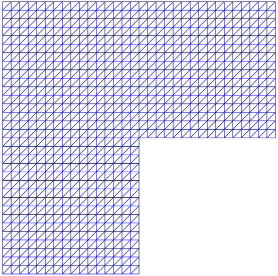
Superconvergent graded meshes

We modify the relocation strategy and are able to generate superconvergent graded meshes:



old, $\mu = 1.0$

$\Rightarrow$



new, $\mu = 1.0$

approximate parallelogram property is satisfied

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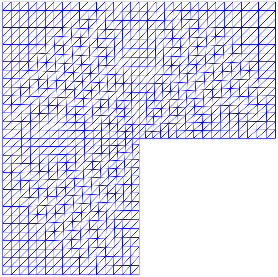
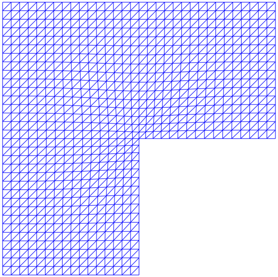
Superconvergent graded meshes

We modify the relocation strategy and are able to generate superconvergent graded meshes:



old, $\mu = 0.9$

$\Rightarrow$



new, $\mu = 0.9$

approximate parallelogram property is satisfied

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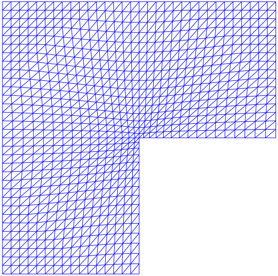
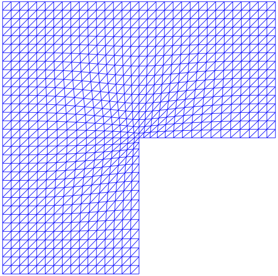
Superconvergent graded meshes

We modify the relocation strategy and are able to generate superconvergent graded meshes:



old, $\mu = 0.8$

$\Rightarrow$



new, $\mu = 0.8$

approximate parallelogram property is satisfied

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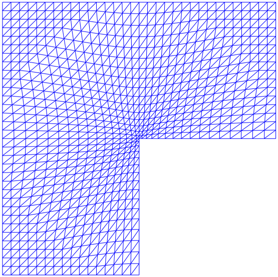
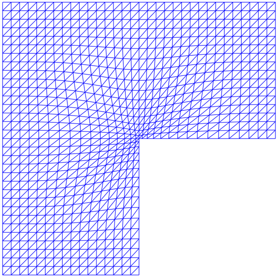
Superconvergent graded meshes

We modify the relocation strategy and are able to generate superconvergent graded meshes:



old, $\mu = 0.7$

$\Rightarrow$



new, $\mu = 0.7$

approximate parallelogram property is satisfied

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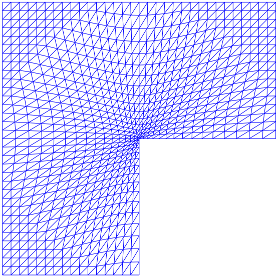
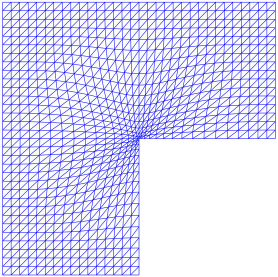
Superconvergent graded meshes

We modify the relocation strategy and are able to generate superconvergent graded meshes:



old, $\mu = 0.6$

$\Rightarrow$



new, $\mu = 0.6$

approximate parallelogram property is satisfied

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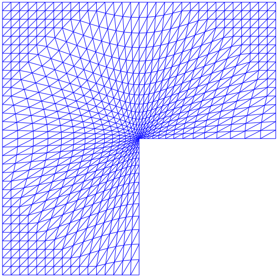
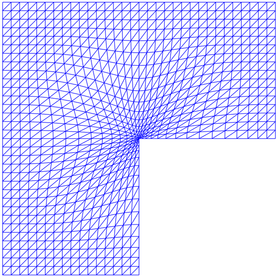
Superconvergent graded meshes

We modify the relocation strategy and are able to generate superconvergent graded meshes:



old, $\mu = 0.5$

$\Rightarrow$



new, $\mu = 0.5$

approximate parallelogram property is satisfied

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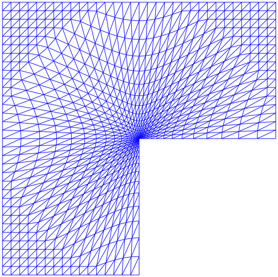
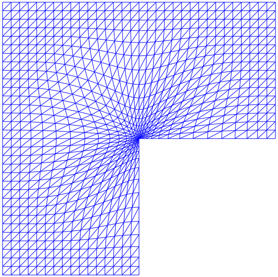
Superconvergent graded meshes

We modify the relocation strategy and are able to generate superconvergent graded meshes:



old, $\mu = 0.4$

$\Rightarrow$



new, $\mu = 0.4$

approximate parallelogram property is satisfied

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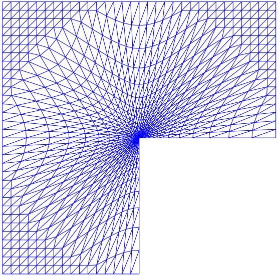
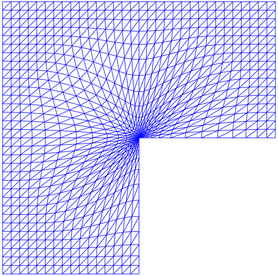
Superconvergent graded meshes

We modify the relocation strategy and are able to generate superconvergent graded meshes:



old, $\mu = 0.3$

$\Rightarrow$

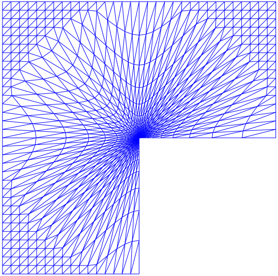


new, $\mu = 0.3$

approximate parallelogram property is satisfied

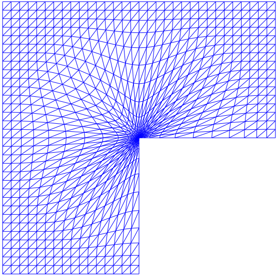
Superconvergent graded meshes

We modify the relocation strategy and are able to generate superconvergent graded meshes:



old, $\mu = 0.2$

$\Rightarrow$



new, $\mu = 0.2$

approximate parallelogram property is satisfied

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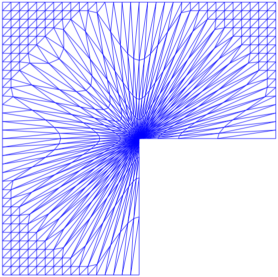
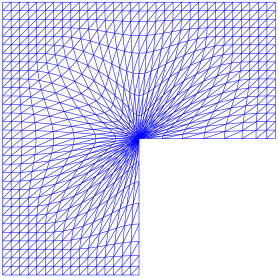
Superconvergent graded meshes

We modify the relocation strategy and are able to generate superconvergent graded meshes:



old, $\mu = 0.1$

$\Rightarrow$



new, $\mu = 0.1$

approximate parallelogram property is satisfied

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Error estimate

If

- $\mathcal{T}_h$ is a superconvergent graded mesh with grading parameter μ, and
- the approximate parallelogram property holds for all edges (no exceptions)

we have for any $v_h \in V_h$

$$\left|\int_\Omega \nabla(u - I_h u) \cdot \nabla v_h\right| \leq c \left(\|r^{3(1-\mu)/2}\nabla^3 u\|_{L^2(\Omega)} + \|r^{(1-3\mu)/2}\nabla^2 u\|_{L^2(\Omega)}\right) \cdot \left(h^2\|r^{(1-\mu)/2}\nabla v_h\|_{L^2(\Omega)} + h^{3/2}\|v_h\|_{L^2(\Gamma)}\right)$$

provided that the norms are finite.

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Discussion of the result

$$\left|\int_{\Omega_h} \nabla(u - I_h u) \cdot \nabla v_h\right| \le c \left(\|r^{3(1-\mu)/2}\nabla^3 u\|_{L^2(\Omega)} + \|r^{(1-3\mu)/2}\nabla^2 u\|_{L^2(\Omega)} \right) \cdot \left(h^2 \|r^{(1-\mu)/2}\nabla v_h\|_{L^2(\Omega)} + h^{3/2}\|v_h\|_{L^2(\Gamma)} \right)$$

- If the function u is the solution of a homogenous Dirichlet problem with sufficiently smooth right hand side, then the assumptions $r^{3(1-\mu)/2}\nabla^3 u \in L^2(\Omega)$ and $r^{(1-3\mu)/2}\nabla^2 u \in L^2(\Omega)$ are satisfied for $\mu < \frac{2}{3}(\lambda - \frac{1}{2})$.
- In the investigation of a Dirichlet control problem we use the estimate for the adjoint state.
- In improvement of the proofs in [Bank/Xu 03] and [Deckelnick/Günther/Hinze 09] we avoided (weighted) L^r-norms with $r > 2$ of second and third derivatives of u.

Discussion of the result

$$\left|\int_{\Omega_h} \nabla(u - I_h u)\cdot\nabla v_h\right| \le c\left(\|r^{3(1-\mu)/2}\nabla^3 u\|_{L^2(\Omega)} + \|r^{(1-3\mu)/2}\nabla^2 u\|_{L^2(\Omega)}\right) \cdot \left(h^2\|r^{(1-\mu)/2}\nabla v_h\|_{L^2(\Omega)} + h^{3/2}\|v_h\|_{L^2(\Gamma)}\right)$$

- If the function v_h is discrete harmonic then

$$\|r^{(1-\mu)/2}\nabla v_h\|_{L^2(\Omega)} \le ch^{-1/2}\|v_h\|_{L^2(\Gamma)}$$

  and the second factor on the right hand side is just $h^{3/2}\|v_h\|_{L^2(\Gamma)}$. Therefore we use $\|r^{(1-\mu)/2}\nabla v_h\|_{L^2(\Omega)}$ and not just $\|\nabla v_h\|_{L^2(\Omega)}$.
- For the application to Dirichlet control problems we get for $\mu < \frac{2}{3}(\lambda - \frac{1}{2})$

$$\left|\int_{\Omega_h} \nabla(u - I_h u)\cdot\nabla v_h\right| \le ch^{3/2}\|v_h\|_{L^2(\Gamma)}$$

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Discussion of the result

- So far: For globally superconvergent meshes we get for $\mu < \frac{2}{3}(\lambda - \frac{1}{2})$ and discrete harmonic v_h

$$\left| \int_{\Omega_h} \nabla(u - I_h u) \cdot \nabla v_h \right| \leq c h^{3/2} \|v_h\|_{L^2(\Gamma)} \qquad (*)$$

What about piecewise superconvergent graded meshes?

- If $\Omega = \bigcup_{j=1}^{n} \Omega_j$ and the meshes are superconvergent in each polygon Ω_j, we get

$$\left| \int_{\Omega_h} \nabla(u - I_h u) \cdot \nabla v_h \right| \leq c h^{3/2} \sum_j \|v_h\|_{L^2(\partial\Omega_j)}$$

But we were able to show for discrete harmonic v_h and $\mu < 2\lambda - 1$

$$\sum_{j=1}^{n} \|v_h\|_{L^2(\partial\Omega_j)} \leq c_n \|v_h\|_{L^2(\Gamma)}$$

such that (*) holds also for piecewise superconvergent graded meshes.

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Plan of the talk

The elliptic Dirichlet control problem

- state variable $y \in Y := L^2(\Omega)$
- control variable $u \in U_{ad} := \{u \in L^2(\Gamma) : \ a \le u(x) \le b \text{ for a.a. } x \in \Gamma\}$

Dirichlet control problem

$$\min_{(y,u)\in Y\times U_{ad}} J(y,u) := \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Gamma)}$$

subject to $\quad -\Delta y = 0 \quad \text{in } \Omega, \quad y = u \quad \text{on } \Gamma,$ in very weak sense

$$\Leftrightarrow \quad (y,\Delta v)_{L^2(\Omega)} = (u,\partial_n v)_{L^2(\Gamma)} \quad \forall v \in H^1_0(\Omega)\cap H^1_\Delta(\Omega)$$

- adjoint state $\bar p \in V := \{v \in H^1_0(\Omega) : \Delta v \in L^2(\Omega)\}$
- projection operator $\Pi_{[a,b]}(c) := \min\{b, \max\{a,c\}\}$

First order optimality system

$$
\begin{aligned}
-\Delta \bar y &= f && \text{in } \Omega, && \bar y = \bar u && \text{on } \partial\Omega && \text{in very weak sense}\\
-\Delta \bar p &= \bar y - y_d && \text{in } \Omega, && \bar p = 0 && \text{on } \partial\Omega && \text{in weak sense}\\
&&&&& \bar u = \Pi_{[a,b]}\left(\tfrac{1}{\nu}\partial_n \bar p\right) && \text{on } \partial\Omega
\end{aligned}
$$

Discretization

Let $\mathcal{T}_h$ be a conforming finite element mesh. Define

$$Y_h = \{v_h \in H^1(\Omega) : v_h|_T \in \mathcal{P}_1 \ \forall T \in \mathcal{T}_h\},\ Y_{0h} = Y_h \cap H^1_0(\Omega),\ U^h_{ad} = Y_h|_\Gamma \cap U_{ad}$$

Discrete optimality system

$$\begin{aligned}
(\nabla \bar{y}_h, \nabla v_h)_{L^2(\Omega)} &= 0 && \forall v_h \in Y_{0h} \text{ and } \bar{y}_h|_\Gamma = \bar{u}_h,\\
(\nabla \bar{p}_h, \nabla v_h)_{L^2(\Omega)} &= (\bar{y}_h - y_d, v_h)_{L^2(\Omega)} && \forall v_h \in Y_{0h},\\
(\nu \bar{u}_h - \partial^h_n \bar{p}_h, u_h - \bar{u}_h)_{L^2(\Gamma)} &\ge 0 && \forall u_h \in U^h_{ad},
\end{aligned}$$

where the discrete normal derivative $\partial^h_n \bar{p}_h \in Y_h|_\Gamma$ is defined by

$$(\partial^h_n \bar{p}_h, v_h)_{L^2(\Gamma)} = -(\bar{y}_h - y_d, v_h)_{L^2(\Omega)} + (\nabla \bar{p}_h, \nabla v_h)_{L^2(\Omega)} \quad \text{for all } v_h \in Y_h \setminus Y_{0h}$$

General error estimate

- $S_h : U_h \to Y_h$ is the discrete harmonic extension
- $Q_h : L^2(\Gamma) \to U_h$ is the $L^2(\Gamma)$-projection
- $u_h^* \in U_{ad}^h$ such that $(\nu\bar{u} - \partial_n\bar{p}, u_h^* - \bar{u})_{L^2(\Gamma)} = 0$.

General error estimate

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} + \|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \leq c\left(\|\bar{u} - u_h^*\|_{L^2(\Gamma)} + \|\bar{y} - S_hQ_h\bar{u}\|_{L^2(\Omega)} + \sup_{\psi_h\in U_h}\frac{\left|(\nabla\bar{p}, \nabla S_h\psi_h)_{L^2(\Omega)}\right|}{\|\psi_h\|_{L^2(\Gamma)}}\right)$$

- first term: quasi-interpolation error
- second term: contains approximation of non-smooth boundary condition, in general $y \notin H^1(\Omega)$
- third term: corresponds to error estimate of normal derivative, determines the overall convergence order, numerator equals $\left|(\nabla(\bar{p} - I_h\bar{p}), \nabla S_h\psi_h)_{L^2(\Omega)}\right|$

The estimates depend on the regularity of $\bar{u}$, $\bar{p}$, and $\bar{y}$.

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Regularity in the unconstrained case $-a = b = \infty$

Optimality system

$$
\begin{array}{llll}
-\Delta \bar y = f & \text{in } \Omega, \quad \bar y = \bar u & \text{on } \partial\Omega & \text{in very weak sense} \\
-\Delta \bar p = \bar y - y_d & \text{in } \Omega, \quad \bar p = 0 & \text{on } \partial\Omega & \text{in weak sense} \\
 & \bar u = \frac{1}{\nu}\partial_n \bar p & \text{on } \partial\Omega &
\end{array}
$$

Let ω be the largest interior angle of the polygonal domain $\Omega \subset \mathbb{R}^2$, and $\lambda = \pi/\omega \in (\frac{1}{2}, 3]$ be the leading singularity exponent.

$$\bar u \in H^{\min\{\lambda - 1/2, 3/2\} - \varepsilon}(\Gamma)$$

Kinks of $\partial_n \bar p$ at corners lead to bound $\frac{3}{2} - \varepsilon$.

Corner singularities of type $c\xi r^\lambda \sin(\lambda\theta)$ in the adjoint state lead to $\lambda - \frac{1}{2} - \varepsilon$.

Th. Apel, M. Mateos, J. Pfefferer, A. Rösch: On the regularity of the solutions of Dirichlet optimal control problems in polygonal domains. SIAM J. Control Optim. 53(2015), 3620–3641.

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Approximation results – unconstrained case

$$\bar{u} \in H^{\min\{\lambda-1/2,3/2\}-\varepsilon}(\Gamma)$$

Approximation error orders for the control (up to logarithmic terms or $h^{-\varepsilon}$):

quasi-uniform meshes	
quasi-interpolation of $\bar{u}$	$\min\{\lambda - \frac{1}{2}, \frac{3}{2}\}$
literature ($\lambda > 1$)	$\min\{\frac{1}{2}\lambda, 1\}$

E. Casas, J.-P. Raymond: Error estimates for the numerical approximation of Dirichlet boundary control for semilinear elliptic equations. SICON 45(2006), 1586–1611.

S. May, R. Rannacher, and B. Vexler: Error analysis for a finite element approximation of elliptic Dirichlet boundary control problems. SICON 51(2013), 2585–2611.

Approximation results – unconstrained case

$$\bar{u} \in H^{\min\{\lambda-1/2,3/2\}-\varepsilon}(\Gamma)$$

Approximation error orders for the control (up to logarithmic terms or $h^{-\varepsilon}$):

quasi-uniform meshes	
quasi-interpolation of $\bar{u}$	$\min\{\lambda-\frac{1}{2},\frac{3}{2}\}$
literature ($\lambda > 1$)	$\min\{\frac{1}{2}\lambda,1\}$
general quasi-uniform meshes	$\min\{\lambda-\frac{1}{2},1\}$

Th. Apel, M. Mateos, J. Pfefferer, A. Rösch: Error estimates for Dirichlet control problems in polygonal domains: Quasi-uniform meshes. arXiv:1704.08843

Approximation results – unconstrained case

$$\bar{u} \in H^{\min\{\lambda-1/2,3/2\}-\varepsilon}(\Gamma)$$

Approximation error orders for the control (up to logarithmic terms or $h^{-\varepsilon}$):

quasi-uniform meshes	
quasi-interpolation of $\bar{u}$	$\min\{\lambda-\frac{1}{2},\frac{3}{2}\}$
literature ($\lambda > 1$)	$\min\{\frac{1}{2}\lambda, 1\}$
general quasi-uniform meshes	$\min\{\lambda-\frac{1}{2}, 1\}$
superconvergence meshes	$\min\{\lambda-\frac{1}{2},\frac{3}{2}\}$

Th. Apel, M. Mateos, J. Pfefferer, A. Rösch: Error estimates for Dirichlet control problems in polygonal domains: Quasi-uniform meshes. arXiv:1704.08843

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Approximation results – unconstrained case

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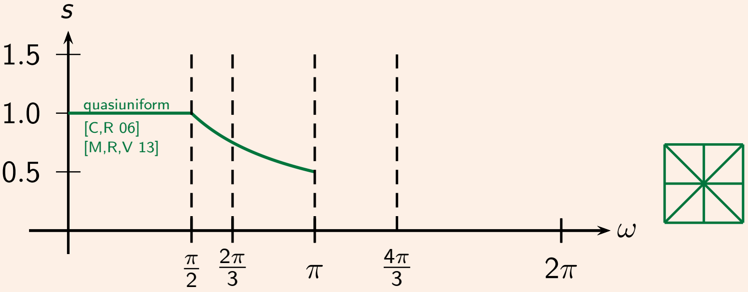
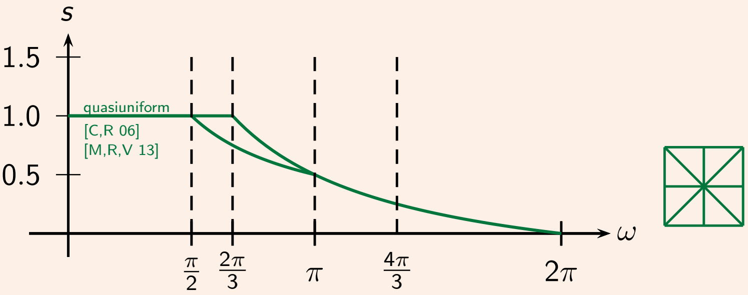


Approximation order for the control

Approximation results – unconstrained case

$$\bar{u} \in H^{\min\{\lambda-1/2,3/2\}-\varepsilon}(\Gamma)$$

Approximation error orders for the control (up to logarithmic terms or $h^{-\varepsilon}$):



Approximation order for the control

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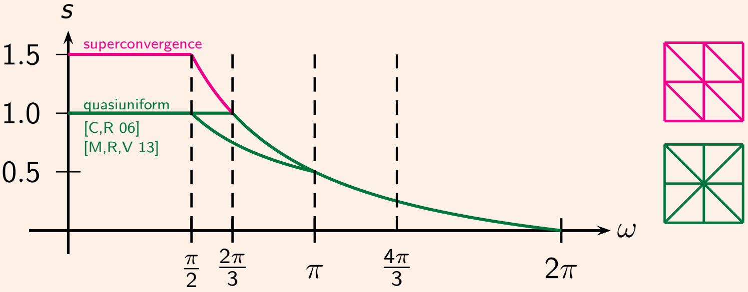
Approximation error orders for the control (up to logarithmic terms or $h^{-\varepsilon}$):



Approximation order for the control

Numerical tests show that these results are sharp.

Graded meshes – unconstrained case

General error estimate

$$\|\bar u - \bar u_h\|_{L^2(\Gamma)} + \|\bar y - \bar y_h\|_{L^2(\Omega)}$$
$$\le c\left(\|\bar u - u_h^*\|_{L^2(\Gamma)} + \|\bar y - S_h Q_h \bar u\|_{L^2(\Omega)} + \sup_{\psi_h\in U_h} \frac{\left|(\nabla \bar p, \nabla S_h\psi_h)_{L^2(\Omega)}\right|}{\|\psi_h\|_{L^2(\Gamma)}}\right)$$

- With Clément interpolant $u_h^* = C_h \bar u$:

$$\|\bar u - u_h^*\|_{L^2(\Gamma)} \le \begin{cases} ch & \text{for } \mu < \lambda - \frac{1}{2} \\ ch^{3/2-\varepsilon} & \text{for } \mu < \frac{2}{3}(\lambda - \frac{1}{2}) \end{cases}$$

- Using $\left|(\nabla \bar p, \nabla S_h\psi_h)_{L^2(\Omega)}\right| = \left|(\nabla(\bar p - I_h \bar p), \nabla S_h\psi_h)_{L^2(\Omega)}\right|$:

$$\sup_{\psi_h\in U_h} \frac{\left|(\nabla \bar p, \nabla S_h\psi_h)_{L^2(\Omega)}\right|}{\|\psi_h\|_{L^2(\Gamma)}}$$
$$\le \begin{cases} ch^{1-\varepsilon} & \text{for general graded meshes and } \mu < \lambda - \frac{1}{2} \\ ch^{3/2-\varepsilon} & \text{for superconvergent graded meshes and } \mu < \frac{2}{3}(\lambda - \frac{1}{2}) \end{cases}$$

Numerical test

- $\Omega = \{(x_1, x_2) \in (-1,1) \times (-1,1) : \ 0 < \theta < \omega\}$ with

$$\omega = \begin{cases} \frac{3}{4}\pi & \text{(convex domain)} \\ \frac{5}{4}\pi & \text{(non-convex domain)} \end{cases}$$

- $\lambda = \pi/\omega$, $\nu = 1$,

$$\bar\varphi = \begin{cases} r^\lambda \sin(\lambda\theta)(1 - x_1)(1 - x_2) & \text{if } \omega_1 = \frac{3}{4}\pi, \\ r^\lambda \sin(\lambda\theta)(1 - x_1^2)(1 - x_2) & \text{if } \omega_1 = \frac{5}{4}\pi, \end{cases}$$

$\bar u = \partial_\nu \bar\varphi$, $\bar y = S\bar u$, $y_d = \bar y + \Delta\bar\varphi$

Numerical test: bisection (general graded mesh)

$\omega = \frac{3}{4}\pi$:

μ	EOC	TOC
1.00	0.89	0.83
$0.8\overline{3}$	0.97	1.00
0.75	1.00	1.00
$0.5\overline{5}$	1.00	1.00
0.40	1.00	1.00

$\omega = \frac{5}{4}\pi$:

μ	EOC	TOC
1.00	0.30	0.30
0.75	0.37	0.40
0.50	0.61	0.60
0.30	0.97	1.00
0.25	1.00	1.00

EOC: estimated order of convergence
TOC: theoretical order of convergence $\min\{1, (\lambda - \frac{1}{2})/\mu\}$

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Numerical test: superconvergent graded meshes

S: “smooth relocation”, N: “non-smooth relocation”, H: “hierarchical”

$\omega = \frac{3}{4}\pi$:

μ	TOC	EOC for S	EOC for N	EOC for H
1.00	0.83	0.83	0.83	0.83
$0.8\overline{3}$	1.00	1.00	1.01	1.00
0.75	1.11	1.12	1.13	1.11
$0.5\overline{5}$	1.50	1.50	1.52	1.37
0.40	1.50	1.51	1.52	1.47

$\omega = \frac{5}{4}\pi$:

μ	TOC	EOC for S	EOC for N	EOC for H
1.00	0.30	0.29	0.29	0.29
0.75	0.40	0.40	0.40	0.40
0.50	0.60	0.60	0.60	0.60
0.30	1.00	1.00	1.00	1.00
0.25	1.20	1.20	1.20	1.20
0.20	1.50	1.51	1.51	1.46

$\text{TOC} = \min\{\frac{3}{2}, (\lambda - \frac{1}{2})/\mu\}$

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Regularity in the constrained case $a < 0 < b$

convex case

Optimality system

$$
\begin{aligned}
-\Delta\bar{y} &= 0 && \text{in } \Omega, \quad \bar{y} = \bar{u} \quad \text{on } \partial\Omega \quad \text{in very weak sense} \\
-\Delta\bar{p} &= \bar{y} - y_d && \text{in } \Omega, \quad \bar{p} = 0 \quad \text{on } \partial\Omega \quad \text{in weak sense} \\
\bar{u} &= \Pi_{[a,b]}\left(\tfrac{1}{\nu}\partial_n\bar{p}\right) && \text{on } \partial\Omega
\end{aligned}
$$

$\Pi_{[a,b]}$ leads to kinks but $\bar{u}$ is not more regular than $H^{3/2-\varepsilon}(\Gamma)$ anyway.

In the following we will always assume a finite number of kinks.

Convex case:

- the same regularity as in the unconstrained case
- the same approximation result (just using a different u_h^*).

The non-convex case is more interesting.

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Regularity in the constrained case $a < 0 < b$

non-convex case

Optimality system

$$
\begin{aligned}
-\Delta\bar{y} &= 0 && \text{in } \Omega, && \bar{y} = \bar{u} && \text{on } \partial\Omega && \text{in very weak sense} \\
-\Delta\bar{p} &= \bar{y} - y_d && \text{in } \Omega, && \bar{p} = 0 && \text{on } \partial\Omega && \text{in weak sense} \\
\bar{u} &= \Pi_{[a,b]}\left(\tfrac{1}{\nu}\partial_n\bar{p}\right) && \text{on } \partial\Omega
\end{aligned}
$$

Consider critical interior angle $\omega \in (\pi, 2\pi)$, i.e. $\lambda = \frac{\pi}{\omega} \in (\frac{1}{2}, 1)$.

With bootstrapping arguments follows $\bar{y} \in H^{\lambda-\epsilon}(\Omega)$, hence for $y_d \in H^{\lambda-\epsilon}(\Omega)$

$$
\begin{aligned}
\bar{p} &= p_{reg} + c_1\xi r^{\lambda}\sin(\lambda\theta) + c_2\xi r^{2\lambda}\sin(2\lambda\theta) && p_{reg} \in H^{2+\lambda-\epsilon}(\Omega) \\
\partial_n\bar{p} &= \partial_n p_{reg} - c_1 \underbrace{\xi r^{\lambda-1}}_{\to\pm\infty \text{ for } r\to 0} \mp c_2\xi r^{2\lambda-1} && \partial_n p_{reg} \in H^{1/2+\lambda-\epsilon}(\Gamma)
\end{aligned}
$$

Regularity in the constrained case $a < 0 < b$

non-convex case

Optimality system

$$
\begin{aligned}
-\Delta \bar y &= 0 && \text{in } \Omega, \quad \bar y = \bar u \quad \text{on } \partial\Omega \quad \text{in very weak sense}\\
-\Delta \bar p &= \bar y - y_d && \text{in } \Omega, \quad \bar p = 0 \quad \text{on } \partial\Omega \quad \text{in weak sense}\\
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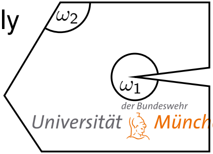
\end{aligned}
$$

- $c_1 \neq 0$: $\bar u$ is flat near the critical corner, $\bar u \in H^{3/2-\epsilon}$ locally
- i.g., convex corners determine the regularity of $\bar u$
- example: $\bar u \in H^{\min\{\Lambda-1/2,3/2\}-\epsilon}(\Gamma)$, $\Lambda = \frac{\pi}{\omega_2}$,
  i.e. $\bar u \in H^{\Lambda-1/2-\epsilon}(\Gamma)$ if $\omega_2 > \frac{\pi}{2}$.



Regularity in the constrained case $a < 0 < b$

non-convex case

Optimality system

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\begin{aligned}
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-\Delta \bar{p} &= \bar{y} - y_d && \text{in } \Omega, \quad \bar{p} = 0 \quad \text{on } \partial\Omega \quad \text{in weak sense} \\
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\end{aligned}
$$

Consider critical interior angle $\omega \in (\pi, 2\pi)$, i.e. $\lambda = \frac{\pi}{\omega} \in (\frac{1}{2}, 1)$.

With bootstrapping arguments follows $\bar{y} \in H^{\lambda-\epsilon}(\Omega)$, hence for $y_d \in H^{\lambda-\epsilon}(\Omega)$

$$
\begin{aligned}
\bar{p} &= p_{reg} + c_1 \xi r^{\lambda} \sin(\lambda\theta) + c_2 \xi r^{2\lambda} \sin(2\lambda\theta) && p_{reg} \in H^{2+\lambda-\epsilon}(\Omega) \\
\partial_n \bar{p} &= \partial_n p_{reg} - c_1 \underbrace{\xi r^{\lambda-1}}_{\to \pm\infty \text{ for } r \to 0} \mp \underbrace{c_2 \xi r^{2\lambda-1}}_{\to 0 \text{ for } r \to 0} && \partial_n p_{reg} \in H^{1/2+\lambda-\epsilon}(\Gamma)
\end{aligned}
$$

- $c_1 \neq 0$: $\bar{u}$ is flat near the critical corner, $\bar{u} \in H^{3/2-\epsilon}$ locally
- $c_1 = 0$: $\bar{u}$ is not flat locally, but $\bar{u} \in H^{2\lambda-1/2-\varepsilon}(\Gamma)$.
  This case is rare but worse than $c_1 \neq 0$.

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Regularity in the constrained case $a < 0 < b$

non-convex case

Optimality system

$$
\begin{aligned}
-\Delta\bar{y} &= 0 && \text{in } \Omega, \quad \bar{y} = \bar{u} \quad \text{on } \partial\Omega \quad \text{in very weak sense} \\
-\Delta\bar{p} &= \bar{y} - y_d && \text{in } \Omega, \quad \bar{p} = 0 \quad \text{on } \partial\Omega \quad \text{in weak sense} \\
\bar{u} &= \Pi_{[a,b]}\left(\tfrac{1}{\nu}\partial_n\bar{p}\right) && \text{on } \partial\Omega
\end{aligned}
$$

The regularity near the j-th corner is determined by

$$
\lambda_j' := \begin{cases} \lambda_j & \text{if } \lambda_j > 1, \\ 2\lambda_j & \text{if } \lambda_j < 1, \end{cases}
$$

where $\lambda_j = \pi/\omega_j$ and ω_j being the angle at j-th corner

Approximation results

$$\bar{u} \in H^{\min\{\lambda'-1/2,3/2\}-\varepsilon}(\Gamma)$$

Approximation error orders (up to logarithmic terms or $h^{-\varepsilon}$):

quasi-uniform meshes	
interpolation of $\bar{u}$	$\min\{\lambda' - \frac{1}{2}, \frac{3}{2}\}$
general quasi-uniform meshes	$\min\{\lambda' - \frac{1}{2}, 1\}$
superconvergence meshes	$\min\{\lambda' - \frac{1}{2}, \frac{3}{2}, 2\lambda\}$

Assumption: When the control bounds are active in the vicinity of some non-convex corner then they are also active for the approximate control.

The proof of this assumption is incomplete in a $O(h^{1+\varepsilon})$-neighborhood of the non-convex corners.

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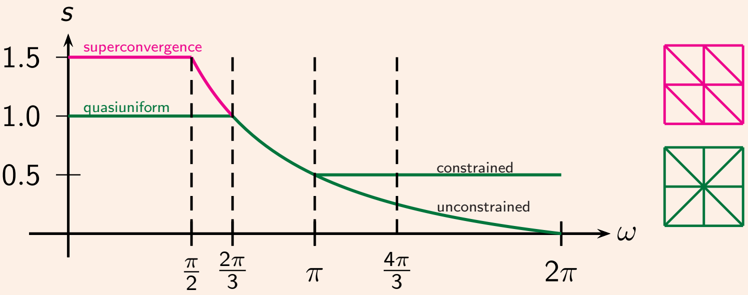
Approximation results

$$\bar{u} \in H^{\min\{\lambda'-1/2,3/2\}-\varepsilon}(\Gamma)$$

Appr

- For convex domains this is the same result as in the unconstrained case.
- For non-convex domains, the convergence order depends on the largest convex angle, this can be $\pi - \varepsilon$ leading to $\lambda' - \frac{1}{2} \approx \frac{1}{2}$.



Approximation order for the control

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Approximation results

$$\bar{u} \in H^{\min\{\lambda'-1/2, 3/2\}-\varepsilon}(\Gamma)$$

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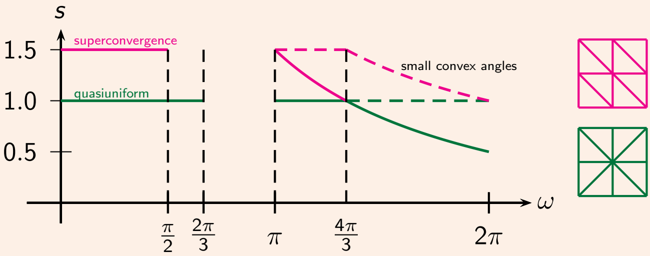
- If the largest convex angle is $\leq \frac{2\pi}{3}$ and $c_{j,1} \neq 0$, then the convergence order is 1 for general quasi-uniform meshes.
- If the largest convex angle is $\leq \frac{\pi}{2}$ and $c_{j,1} \neq 0$, then the convergence order is $\min\{\frac{3}{2}, 2\lambda\}$ for superconvergence meshes.



Approximation order for the control

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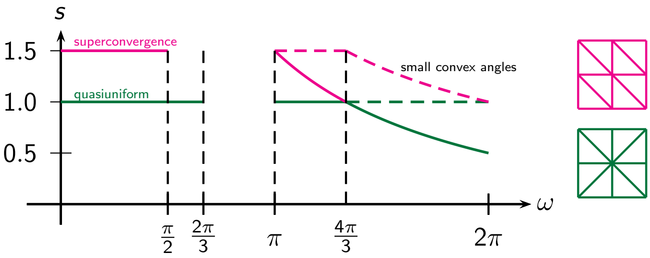
Graded meshes

What do we expect to achieve with graded meshes?



- Convergence order 1 for general graded meshes
  - $\omega \in [\frac{2}{3}\pi, \pi)$: $\mu < \lambda - \frac{1}{2}$
  - $\omega \in [\frac{4}{3}\pi, 2\pi)$: $\mu < 2\lambda - \frac{1}{2}$ (worst case)
- Convergence order $\frac{3}{2}$ for superconvergent graded meshes
  - $\omega \in [\frac{2}{3}\pi, \pi)$: $\mu < \frac{2}{3}(\lambda - \frac{1}{2})$
  - $\omega \in [\pi, 2\pi)$: $\mu < \frac{2}{3}(2\lambda - \frac{1}{2}) = \frac{4}{3}\lambda - \frac{1}{3}$ (worst case)
  - $\omega \in [\frac{4}{3}\pi, 2\pi)$: $\mu < \frac{4}{3}\lambda$ (generic case)

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Plan of the talk

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Summary

- We discussed ways of constructing graded meshes. Some of them lead to superconvergence effects.
- We observed superconvergence effects in Dirichlet control problems not only with uniform but also with graded meshes.
- In unconstrained Dirichlet control problems, we need strong mesh grading with $\mu < \frac{2}{3}(\lambda - \frac{1}{2})$ to obtain $O(h^{3/2})$ in the error of the control.
- In the constained case the necessary grading is not that strong.
- For certain families of superconvergent graded meshes we proved a core estimate.

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