A new approach to mixed methods for biharmonic problems in 2D and 3D

Walter Zulehner

Institute of Computational Mathematics
JKU Linz, Austria

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Wolfgang Krendl, Katharina Rafetseder, Martin Neumüller (Linz),
Dirk Pauly (Essen)

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Outline

Biharmonic problems

domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, bounded and simply connected

boundary $\partial\Omega = \Gamma$ connected, polygonal/polyhedral

find u such that

$$\Delta^2 u = f \quad \text{in } \Omega, \qquad u = \partial_n u = 0 \quad \text{on } \Gamma$$

observe that

$$\Delta^2 u = \operatorname{div} \operatorname{Div} (\nabla^2 u)$$

primal variational formulation

find $u \in W = H_0^2(\Omega)$ such that

$$\int_\Omega \nabla^2 u : \nabla^2 v \, dx = \langle f, v \rangle \quad \text{for all } v \in W = H_0^2(\Omega)$$

in short

$$\Delta^2 u = f \quad \text{in } W^* = H^{-2}(\Omega)$$

observe that

$$\Delta^2 u = -\operatorname{div}\operatorname{Div}\mathbf{M} \quad \text{with} \quad \mathbf{M} = -\nabla^2 u$$

mixed variational formulation

find $\mathbf{M} \in \mathbf{L}^2(\Omega, \mathbb{S})$ and $u \in H_0^2(\Omega)$ such that

$$\begin{aligned} \mathbf{M} \qquad\qquad + \nabla^2 u &= 0 \quad \text{in } \mathbf{L}^2(\Omega, \mathbb{S}) \\ \operatorname{div}\operatorname{Div}\mathbf{M} \qquad\qquad &= -f \quad \text{in } H^{-2}(\Omega) \end{aligned}$$

in detail

$$\begin{aligned} \int_\Omega \mathbf{M} : \mathbf{N}\, dx \quad + \int_\Omega (\operatorname{div}\operatorname{Div}\mathbf{N})\, u\, dx &= 0 && \text{for all } \mathbf{N} \in \mathbf{L}^2(\Omega, \mathbb{S}) \\ \int_\Omega (\operatorname{div}\operatorname{Div}\mathbf{M})\, v\, dx \qquad\qquad &= -\langle f, v \rangle && \text{for all } v \in \mathrm{H}_0^2(\Omega) \end{aligned}$$

Biharmonic problems

New variational formulation for $f \in \mathrm{H}^{-1}(\Omega)$

Find $\mathbf{M} \in \mathbf{V}$ and $u \in H_0^1(\Omega)$ such that

$$\begin{aligned} \mathbf{M} \qquad\quad + \nabla^2 u &= 0 \quad \text{in } \mathbf{V}^* \\ \operatorname{div}\operatorname{Div}\mathbf{M} \qquad &= -f \quad \text{in } \mathrm{H}^{-1}(\Omega) \end{aligned}$$

The space $\mathbf{L}^2(\Omega, \mathbb{S})$ is replaced by

$$\mathbf{V} = \{\mathbf{N} \in \mathbf{L}^2(\Omega, \mathbb{S}) \colon \operatorname{div}\operatorname{Div}\mathbf{N} \in \mathrm{H}^{-1}(\Omega)\} \equiv \mathbf{H}^{0,-1}(\operatorname{div}\operatorname{Div}; \Omega, \mathbb{S})$$

norm

$$\|\mathbf{N}\|_{\mathbf{V}} = \left(\|\mathbf{N}\|_0^2 + \|\operatorname{div}\operatorname{Div}\mathbf{N}\|_{-1}^2 \right)^{1/2}$$

Bernardi/Girault/Maday (1992), Z. (2015), Pechstein/Schöberl (2011)

Biharmonic problems

Find $\mathbf{M} \in \mathbf{V} = \mathbf{H}^{0,-1}(\operatorname{div}\operatorname{Div}; \Omega, \mathbb{S})$ and $u \in Q = H_0^1(\Omega)$ such that

$$\begin{aligned} \langle \mathbf{M}, \mathbf{N} \rangle \quad + \langle \operatorname{div}\operatorname{Div}\mathbf{N}, u \rangle &= 0 \qquad \text{for all } \mathbf{N} \in \mathbf{V} \\ \langle \operatorname{div}\operatorname{Div}\mathbf{M}, v \rangle \quad &= -\langle f, v \rangle \quad \text{for all } v \in Q \end{aligned}$$

Theorem

The mixed problem is well-posed in $\mathbf{V} \times Q$, *equipped with the norms*

$$\|\mathbf{N}\|_{\mathbf{V}} = \left(\|\mathbf{N}\|_0^2 + \|\operatorname{div}\operatorname{Div}\mathbf{N}\|_{-1}^2 \right)^{1/2}, \quad \|v\|_Q = |v|_1.$$

Its solution $(\mathbf{M}, u)$ *satisfies* $u \in W = \mathrm{H}_0^2(\Omega)$ *with*

$$\Delta^2 u = f \quad \textit{in } W^* = \mathrm{H}^{-2}(\Omega)$$

and, vice versa, ...

Krendl/Rafetseder/Z. (2014,2016)

A Helmholtz-like decomposition

Each $\mathbf{M} \in \mathbf{H}^{0,-1}(\operatorname{div}\operatorname{Div};\Omega,\mathbb{S})$ can be uniquely written as

$$\mathbf{M} = p\,\mathbf{I} + \mathbf{M}_0 \quad \text{with} \quad \mathbf{I} \quad \text{identity matrix in } \mathbb{R}^d$$

where $p \in H_0^1(\Omega)$ is determined by

$$\underbrace{\operatorname{div}\operatorname{Div}(p\,\mathbf{I})}_{=\Delta p} = \operatorname{div}\operatorname{Div}\mathbf{M}$$

and

$$\operatorname{div}\operatorname{Div}\mathbf{M}_0 = 0$$

Kernel of div Div in 2D

Let $\mathbf{M} \in \mathbf{L}^2(\Omega; \mathbb{S})$. Then

$$\operatorname{div} \operatorname{Div} \mathbf{M} = 0 \quad \Longleftrightarrow \quad \mathbf{M} = \operatorname{sym} \operatorname{Curl} \phi, \; \phi \in H^1(\Omega)^2$$

with

$$\operatorname{Curl} \phi = \begin{bmatrix} \partial_2 \phi_1 & -\partial_1 \phi_1 \\ \partial_2 \phi_2 & -\partial_1 \phi_2 \end{bmatrix}$$

Proof.

Since $\operatorname{div} (\operatorname{Div} \mathbf{M}) = 0$ and $\operatorname{Div} \mathbf{M} \in H^{-1}(\Omega)$, we have

$$\operatorname{Div} \mathbf{M} = \operatorname{curl} \rho = -\operatorname{Div} \operatorname{Spin}(\rho) \text{ with } \operatorname{Spin}(\rho) = \begin{bmatrix} 0 & -\rho \\ \rho & 0 \end{bmatrix}.$$

for some $\rho \in L^2(\Omega)$. □

Kernel of div Div in 2D

Proof.

Hence $\operatorname{Div}(\mathbf{M} + \operatorname{Spin}(\rho)) = 0$. Therefore

$$\mathbf{M} + \operatorname{Spin}(\rho) = \operatorname{Curl}\phi.$$

for some $\phi \in H^1(\Omega)^2$.

This implies

$$\mathbf{M} = \operatorname{sym}\operatorname{Curl}\phi \quad \text{and} \quad \operatorname{Spin}(\rho) = \operatorname{skw}\operatorname{Curl}\phi.$$

□

Beirão da Veiga/Niiranen/Stenberg(2007), Huang/Huang/Xu (2011)

Kernel of div Div in 3D

Let $\mathbf{M} \in \mathbf{L}^2(\Omega; \mathbb{S})$. Then

$$\operatorname{div} \operatorname{Div} \mathbf{M} = 0 \quad \Longleftrightarrow \quad \mathbf{M} = \operatorname{sym} \operatorname{Curl} \mathbf{\Phi}, \; \mathbf{\Phi} \in \mathbf{H}^1(\Omega)$$

Proof.

Since $\operatorname{div} (\operatorname{Div} \mathbf{M}) = 0$ and $\operatorname{Div} \mathbf{M} \in H^{-1}(\Omega)$, we have

$$\operatorname{Div} \mathbf{M} = \operatorname{curl} \vec{\rho} = -\operatorname{Div} \operatorname{Spin}(\vec{\rho}) \text{ with } \operatorname{Spin}(\vec{\rho}) = \begin{bmatrix} 0 & -\rho_3 & \rho_2 \\ \rho_3 & 0 & -\rho_1 \\ -\rho_2 & \rho_1 & 0 \end{bmatrix}.$$

for some $\vec{\rho} \in L^2(\Omega)$. □

Kernel of div Div in 3D

Proof.

Hence $\operatorname{Div}(\mathbf{M} + \operatorname{Spin}(\vec{\rho})) = 0$. Therefore

$$\mathbf{M} + \operatorname{Spin}(\vec{\rho}) = \operatorname{Curl} \mathbf{\Phi}.$$

for some $\mathbf{\Phi} \in \mathbf{H}^1(\Omega)$.

This implies

$$\mathbf{M} = \operatorname{sym} \operatorname{Curl} \mathbf{\Phi} \quad \text{and} \quad \operatorname{Spin}(\vec{\rho}) = \operatorname{skw} \operatorname{Curl} \mathbf{\Phi}.$$

□

Since $\operatorname{sym} \operatorname{Curl}(q\,\mathbf{I}) = 0$ for all $q \in L^2(\Omega)$, we also have

$$\operatorname{div} \operatorname{Div} \mathbf{M} = 0 \quad \Longleftrightarrow \quad \mathbf{M} = \operatorname{sym} \operatorname{Curl} \mathbf{\Phi}, \ \mathbf{\Phi} \in \mathbf{H}(\operatorname{sym} \operatorname{Curl}; \Omega, \mathbb{T})$$

Decomposition of the biharmonic problem

Find $\mathbf{M} \in \mathbf{V} = \mathbf{H}^{0,-1}(\operatorname{div}\operatorname{Div};\Omega,\mathbb{S})$ and $u \in Q = H_0^1(\Omega)$ such that

$$\begin{aligned} \langle \mathbf{M}, \mathbf{N} \rangle \quad &+ \langle \operatorname{div}\operatorname{Div}\mathbf{N}, u \rangle = 0 && \text{for all } \mathbf{N} \in \mathbf{V} \\ \langle \operatorname{div}\operatorname{Div}\mathbf{M}, v \rangle \quad &= -\langle f, v \rangle && \text{for all } v \in Q \end{aligned}$$

in 2D

$$\mathbf{M} = p\,\mathbf{I} + \operatorname{sym}\operatorname{Curl}\phi, \quad \mathbf{N} = q\,\mathbf{I} + \operatorname{sym}\operatorname{Curl}\psi$$

in 3D

$$\mathbf{M} = p\,\mathbf{I} + \operatorname{sym}\operatorname{Curl}\mathbf{\Phi}, \quad \mathbf{N} = q\,\mathbf{I} + \operatorname{sym}\operatorname{Curl}\mathbf{\Psi}$$

Decomposition of the biharmonic problem

- $$\langle \mathbf{M}, \mathbf{N}\rangle + \langle \operatorname{div}\operatorname{Div}\mathbf{N}, u\rangle = 0 \quad \text{for} \quad \mathbf{N} = q\,\mathbf{I}$$
  $$\iff \quad \langle p\,\mathbf{I}, q\,\mathbf{I}\rangle + \langle \operatorname{sym}\operatorname{Curl}\phi, q\,\mathbf{I}\rangle + \langle \Delta q, p\rangle = 0$$
- $$\langle \mathbf{M}, \mathbf{N}\rangle + \langle \operatorname{div}\operatorname{Div}\mathbf{N}, u\rangle = 0 \quad \text{for} \quad \mathbf{N} = \operatorname{sym}\operatorname{Curl}\psi$$
  $$\iff \quad \langle \mathbf{M}, \operatorname{sym}\operatorname{Curl}\psi\rangle = 0$$
  $$\iff \quad \langle p\,\mathbf{I}, \operatorname{sym}\operatorname{Curl}\psi\rangle + \langle \operatorname{sym}\operatorname{Curl}\phi, \operatorname{sym}\operatorname{Curl}\psi\rangle = 0$$
- $$\langle \operatorname{div}\operatorname{Div}\mathbf{M}, v\rangle = -\langle f, v\rangle \quad \iff \quad \langle \Delta p, v\rangle = -\langle f, v\rangle$$

Decomposition of the biharmonic problem

Find $p \in H_0^1(\Omega)$, $\phi \in \mathrm{H}^1(\Omega)^2$, and $u \in H_0^1(\Omega)$ such that

$$
\begin{aligned}
\langle p\,\mathbf{I}, q\,\mathbf{I}\rangle \quad &+ \langle \operatorname{sym}\operatorname{Curl}\phi, q\,\mathbf{I}\rangle &+ \boxed{\langle \Delta q, u\rangle} &= 0 \\
\langle p\,\mathbf{I}, \operatorname{sym}\operatorname{Curl}\psi\rangle &+ \boxed{\langle \operatorname{sym}\operatorname{Curl}\phi, \operatorname{sym}\operatorname{Curl}\psi\rangle} & &= 0 \\
\boxed{\langle \Delta p, v\rangle} & & &= -\langle f, v\rangle
\end{aligned}
$$

for all $q \in H_0^1(\Omega)$, $\psi \in \mathrm{H}^1(\Omega)^2$, and $v \in H_0^1(\Omega)$.

Decomposition of the biharmonic problem

Find $p \in H_0^1(\Omega)$, $\boldsymbol{\Phi} \in \mathbf{H}(\text{sym Curl}, \Omega, \mathbb{T})$, and $u \in H_0^1(\Omega)$ such that

$$\begin{aligned}
&\langle p\,\mathbf{I}, q\,\mathbf{I}\rangle && + \langle \text{sym Curl}\,\boldsymbol{\Phi}), q\,\mathbf{I}\rangle && + \boxed{\langle \Delta q, u\rangle} &&= 0\\
&\langle p\,\mathbf{I}, \text{sym Curl}\,\boldsymbol{\Psi}\rangle && + \boxed{\langle \text{sym Curl}\,\boldsymbol{\Phi}, \text{sym Curl}\,\boldsymbol{\Psi}\rangle} && && = 0\\
&\boxed{\langle \Delta p, v\rangle} && && && = -\langle f, v\rangle
\end{aligned}$$

for all $q \in H_0^1(\Omega)$, $\boldsymbol{\Psi} \in \mathbf{H}(\text{sym Curl}, \Omega, \mathbb{T})$, and $v \in H_0^1(\Omega)$.

Kernel of sym Curl in 2D

Let $\phi \in L^2(\Omega)^2$. Then

$$\operatorname{sym}\operatorname{Curl}\phi = 0 \quad\Longleftrightarrow\quad \phi \in RT_0 = \{ax + b\colon\ a \in \mathbb{R},\ b \in \mathbb{R}^2\}$$

Proof.

$$\operatorname{sym}\operatorname{Curl}\phi = 0 \implies \operatorname{Curl}\phi = \operatorname{skw}\operatorname{Curl}\phi = \operatorname{Spin}(\rho)$$

$$\implies \quad 0 = \operatorname{Div}(\operatorname{Spin}(\rho)) = -\operatorname{curl}\rho$$

$$\operatorname{curl}\rho = 0 \implies \rho(x) = a \quad \text{for some } a \in \mathbb{R}$$

$$\operatorname{Curl}(\phi(x) - a\,x) = \operatorname{Spin}\rho - \operatorname{Spin}a = 0$$

$$\implies \quad \phi(x) - a\,x = b \quad \text{some } b \in \mathbb{R}^2.$$

□

Kernel of sym Curl in 3D

Let $\boldsymbol{\Phi} \in \mathbf{L}^2(\Omega)$. Then

$$\operatorname{sym}\operatorname{Curl}\boldsymbol{\Phi} = 0 \quad \Longleftrightarrow \quad \boldsymbol{\Phi} = q\,\mathbf{I} + \nabla v,\ q \in L^2(\Omega),\ v \in H^1(\Omega)^3$$

Proof.

$$\operatorname{sym}\operatorname{Curl}\boldsymbol{\Phi} = 0 \implies \operatorname{Curl}\boldsymbol{\Phi} = \operatorname{skw}\operatorname{Curl}\boldsymbol{\Phi} = \operatorname{Spin}(\vec{\rho})$$
$$\implies \quad 0 = \operatorname{Div}(\operatorname{Spin}(\vec{\rho})) = -\operatorname{curl}\vec{\rho}$$

$$\operatorname{curl}\vec{\rho} = 0 \implies \vec{\rho} = \nabla q \quad \text{for some } q \in L^2(\Omega)$$

$$\operatorname{Curl}(\boldsymbol{\Phi} - q\,\mathbf{I}) = \operatorname{Spin}\rho - \operatorname{Spin}(\nabla q) = 0$$
$$\implies \quad \boldsymbol{\Phi} - q\,\mathbf{I} = \nabla v \quad \text{some } v \in H^1(\Omega)^3.$$

□

Hilbert complexes

Let $\boldsymbol{\Phi} \in \mathbf{L}^2(\Omega, , \mathbb{T})$. Then

$$\operatorname{sym} \operatorname{Curl} \boldsymbol{\Phi} = 0 \quad \Longleftrightarrow \quad \boldsymbol{\Phi} = \operatorname{dev} \nabla v, \;\; v \in H^1(\Omega)^3$$

exact Hilbert complex

$$H^1(\Omega)^2 \xrightarrow{\operatorname{sym} \operatorname{Curl}} \mathbf{H}^{0,-1}(\operatorname{div} \operatorname{Div}; \Omega, \mathbb{S}) \xrightarrow{\operatorname{div} \operatorname{Div}} H^{-1}(\Omega)$$

exact Hilbert complex

$$H^1(\Omega)^3 \xrightarrow{\operatorname{dev} \nabla} \mathbf{H}(\operatorname{sym} \operatorname{Curl}; \Omega, \mathbb{T}) \xrightarrow{\operatorname{sym} \operatorname{Curl}} \mathbf{H}^{0,-1}(\operatorname{div} \operatorname{Div}; \Omega, \mathbb{S}) \xrightarrow{\operatorname{div} \operatorname{Div}} H^{-1}(\Omega)$$

Quenneville-Bélair (2015)

$$\operatorname{dev} \nabla v = 0 \quad \Longleftrightarrow \quad v \in RT_0 = \{ax + b : \;\; a \in \mathbb{R}, \;\; b \in \mathbb{R}^3\}$$

Gauging

$$\operatorname{div}\operatorname{Div}\mathbf{M} = 0 \quad \Longleftrightarrow \quad \mathbf{M} = \operatorname{sym}\operatorname{Curl}\boldsymbol{\Phi}, \; \boldsymbol{\Phi} \in \mathbf{H}(\operatorname{sym}\operatorname{Curl}; \Omega, \mathbb{T})$$

Observe that $\boldsymbol{\Phi}$ is uniquely determined in

$$\mathbf{H}(\operatorname{sym}\operatorname{Curl}, \Omega, \mathbb{T}) \cap \mathbf{H}_0(\operatorname{Div}, \Omega)$$

with

$$\operatorname{Div}\boldsymbol{\Phi} = 0 \quad \text{in } \Omega, \qquad \boldsymbol{\Phi} n = 0 \quad \text{on } \Gamma.$$

Decomposition of biharmonic problems in 2D

Since

$$\text{sym}\,\text{Curl}\,\psi = (\text{div}\,\psi^\perp)\,\mathbf{I} - \text{sym}\,\nabla\psi^\perp \quad \text{with} \quad \psi^\perp = \begin{bmatrix} -\psi_2 \\ \psi_1 \end{bmatrix}$$

it follows that

$$\boxed{\langle \text{sym}\,\text{Curl}\,\phi, \text{sym}\,\text{Curl}\,\psi \rangle} = \int_\Omega \text{sym}\,\nabla\phi^\perp : \text{sym}\,\nabla\psi^\perp \, dx$$

Decomposition of biharmonic problems in 2D

decomposition

1. **Poisson problem** with Dirichlet boundary conditions for p

$$\Delta p = f \quad \text{in } \Omega, \qquad p = 0 \quad \text{on } \Gamma$$

2. **pure traction problem** with Poisson ratio 0 for ϕ

$$-\operatorname{Div}\left(\operatorname{sym}\nabla\phi^{\perp}\right) = \nabla p \quad \text{in } \Omega, \qquad \left(\operatorname{sym}\nabla\phi^{\perp}\right) n = 0 \quad \text{on } \Gamma$$

3. **Poisson problem** with Dirichlet boundary conditions for u

$$\Delta u = 2\,p + \operatorname{div}\phi^{\perp} \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma$$

The Hellan-Herrmann-Johnson method

$k \in \mathbb{N}$, P_k polynomials of total degree $\le k$.

approximation space for $\mathbf{M}$

$$\mathbf{V}_h = \{\mathbf{N} \in \mathbf{L}^2(\Omega, \mathbb{S})\colon \mathbf{N}|_T \in P_{k-1} \text{ for all } T \in \mathcal{T}_h, \text{ and } \mathbf{N}_{nn} \text{ is continuous across all } E \in \mathcal{E}_h\}.$$

approximation space for u

$$Q_h = \mathcal{S}_{h,0} = \mathcal{S}_h \cap H_0^1(\Omega)$$

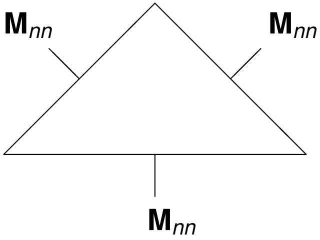
with the standard finite element spaces

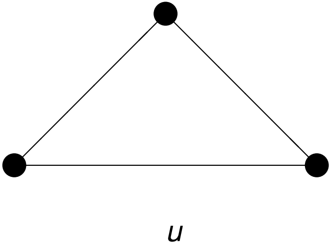
$$\mathcal{S}_h = \{v \in C(\overline{\Omega})\colon v|_T \in P_k \text{ for all } T \in \mathcal{T}_h\}$$

Discretization

Hellan-Herrmann-Johnson element of order $k = 1$

degrees of freedom

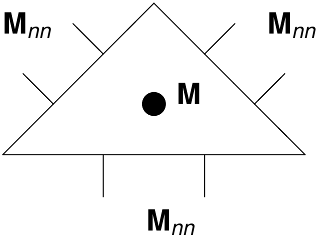


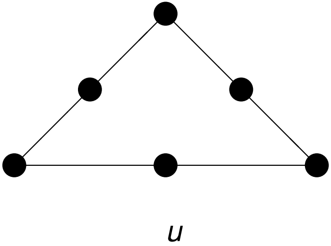


Discretization

Hellan-Herrmann-Johnson element of order $k = 2$

degrees of freedom

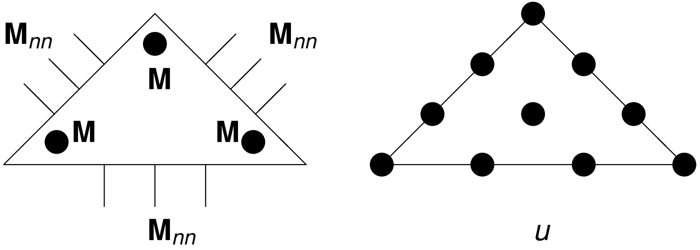




Discretization

Hellan-Herrmann-Johnson element of order $k = 3$

degrees of freedom



The HHJ method

Observe that

$$\mathbf{V}_h \not\subset \mathbf{V} = \mathbf{H}^{0,-1}(\operatorname{div}\operatorname{Div};\Omega,\mathbb{S})$$

Hellan-Herrmann-Johnson (HHJ) method:

Find $\mathbf{M}_h \in \mathbf{V}_h$ and $u_h \in Q_h$ such that

$$\begin{aligned}
\langle \mathbf{M}_h, \mathbf{N}_h \rangle \quad &+ \langle \operatorname{div}\operatorname{Div}_h \mathbf{N}_h, u_h \rangle = 0 && \text{for all } \mathbf{N}_h \in \mathbf{V}_h \\
\langle \operatorname{div}\operatorname{Div}_h \mathbf{M}_h, v_h \rangle \quad & \qquad = -\langle f, v_h \rangle && \text{for all } v_h \in Q_h
\end{aligned}$$

where

$$\langle \operatorname{div}\operatorname{Div}_h \mathbf{N}, v \rangle = \sum_{T \in \mathcal{T}_h} \left\{ \int_T \mathbf{N} : \nabla^2 v \, dx - \int_{\partial T} \mathbf{N}_{nn} \, \partial_n v \, ds \right\}$$

The HHJ method

norm on $\mathbf{V}_h$:

$$\|\mathbf{N}\|_{\mathbf{V},h} = \left(\|\mathbf{N}\|_0^2 + \|\operatorname{div} \operatorname{Div}_h \mathbf{N}\|_{-1,h}^2 \right)^{1/2}$$

with

$$\|\ell\|_{-1,h} = \sup_{0 \neq v_h \in \mathcal{S}_{h,0}} \frac{\langle \ell, v_h \rangle}{|v_h|_1}$$

norm on Q_h:

$$\|v\|_Q = |v|_1$$

Theorem

The mixed problem is uniformly well-posed in $\mathbf{V}_h \times Q_h$ *w.r.t*

$$\|\mathbf{N}\|_{\mathbf{V},h} = \left(\|\mathbf{N}\|_0^2 + \|\operatorname{div} \operatorname{Div}_h \mathbf{N}\|_{-1,h}^2 \right)^{1/2}, \quad \|v\|_Q = |v|_1.$$

Helmholtz-like decomposition

Each $\mathbf{M}_h \in \mathbf{V}_h$ can be uniquely written as

$$\mathbf{M}_h = \boldsymbol{\Pi}_h(p_h \, \mathbf{I}) + \operatorname{sym} \operatorname{Curl} \phi_h$$

with $p_h \in \mathcal{S}_{h,0}$ and $\phi_h \in (\mathcal{S}_h)^2$ and the interpolation operator

$$\boldsymbol{\Pi}_h \colon \boldsymbol{W} \subset \mathbf{V} \longrightarrow \mathbf{V}_h$$

given by the conditions

$$\int_E ((\boldsymbol{\Pi}_h \mathbf{N})_{nn} - \mathbf{N}_{nn}) \, q \, ds = 0, \quad \text{for all } q \in P_{k-1}, \ E \in \mathcal{E}_h,$$
$$\int_T (\boldsymbol{\Pi}_h \mathbf{N} - \mathbf{N}) \, q \, dx = 0, \quad \text{for all } q \in P_{k-2}, \ T \in \mathcal{T}_h$$

Brezzi/Raviart (1977)

The HHJ method

With the representation

$$\mathbf{M}_h = \boldsymbol{\Pi}_h(p_h \mathbf{I}) + \operatorname{sym} \operatorname{Curl} \phi_h \quad \text{and} \quad \mathbf{N}_h = \boldsymbol{\Pi}_h(q_h \mathbf{I}) + \operatorname{sym} \operatorname{Curl} \psi_h$$

the HHJ method reads:

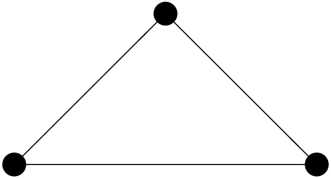
Find $p_h \in \mathcal{S}_{h,0}$, $\phi_h \in \mathcal{S}_h^2$, and $u_h \in \mathcal{S}_{h,0}$ such that

$$\begin{aligned}
&\langle \boldsymbol{\Pi}_h(p_h \mathbf{I}), \boldsymbol{\Pi}_h(q_h \mathbf{I})\rangle + \langle \operatorname{sym} \operatorname{Curl} \phi_h, \boldsymbol{\Pi}_h(q_h \mathbf{I})\rangle + \boxed{\langle \Delta u_h, q_h \rangle} &= 0 \\
&\langle \boldsymbol{\Pi}_h(p_h \mathbf{I}), \operatorname{sym} \operatorname{Curl} \psi_h\rangle + \boxed{\langle \operatorname{sym} \operatorname{Curl} \phi_h, \operatorname{sym} \operatorname{Curl} \psi_h\rangle} &= 0 \\
&\boxed{\langle \Delta p_h, v_h \rangle} &= -\langle f, v_h \rangle
\end{aligned}$$

for all $q_h \in \mathcal{S}_{h,0}$, $\psi_h \in \mathcal{S}_h^2$, and $v_h \in \mathcal{S}_{h,0}$.

Hellan-Herrmann-Johnson element of order $k = 1$

all degrees of freedom for $\mathbf{M}$ and u are collocated

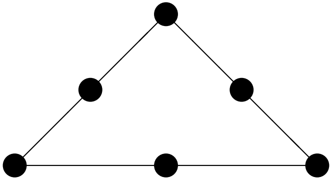


$p, \ \phi, \ u$

The HHJ method

Hellan-Herrmann-Johnson element of order $k = 2$

all degrees of freedom for $\mathbf{M}$ and u are collocated

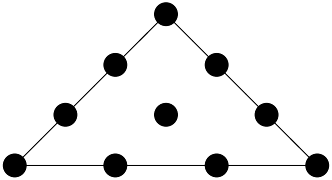


$p, \ \phi, \ u$

The HHJ method

Hellan-Herrmann-Johnson element of order $k = 3$

all degrees of freedom for **M** and u are collocated



$p, \ \phi, \ u$

Extensions

Kirchhoff-Love plate

$$-\operatorname{div}\operatorname{Div}\mathbf{M} = f \quad \text{with} \quad \mathbf{M} = -\mathcal{D}\nabla^2 u \quad \text{in } \Omega$$

with the boundary conditions

$$\begin{aligned} u = 0, &\quad \partial_n u = 0 &\quad \text{on } \Gamma_c \\ u = 0, &\quad \mathbf{M}_{nn} = 0 &\quad \text{on } \Gamma_s \\ \mathbf{M}_{nn} = 0, &\quad \partial_s \mathbf{M}_{ns} + \operatorname{Div}\mathbf{M}\cdot n = 0 &\quad \text{on } \Gamma_f \end{aligned}$$

on

$$\Gamma = \Gamma_c \cup \Gamma_s \cup \Gamma_f$$

and the corner conditions

$$[\![\mathbf{M}_{ns}]\!]_x = 0 \quad \text{for all } x \in \mathcal{V}_f,$$

Extension: Kirchhoff-Love plate

primal formulation

$$u \in W = \{v \in \mathrm{H}^2(\Omega) : v = 0 \text{ on } \Gamma_c \cup \Gamma_s,\ \partial_n v = 0 \text{ on } \Gamma_c\}$$

mixed formulation

$$u \in Q = \{v \in \mathrm{H}^1(\Omega) : v = 0 \text{ on } \Gamma_c \cup \Gamma_s\}$$

$$\mathbf{M} \in \mathbf{V} = \{\mathbf{N} \in \mathbf{L}^2(\Omega, \mathbb{S}) \colon \operatorname{div} \operatorname{Div}_W \mathbf{N} \in Q^*\}$$

with

$$\langle \operatorname{div} \operatorname{Div}_W \mathbf{N}, v \rangle = \int_\Omega \mathbf{N} : \nabla^2 v \, dx \quad \text{for all } v \in W$$

Helmholtz-like decomposition

For each $\mathbf{M} \in \mathbf{V}$ there exists a unique decomposition

$$\mathbf{M} = p\,\mathbf{I} + \operatorname{sym} \operatorname{Curl} \phi,$$

with $p \in Q$ and $\phi \in \mathrm{H}^1(\Omega)^2$ subject to

$$\langle (\operatorname{Curl} \phi) n, \nabla v \rangle_\Gamma = - \int_\Gamma p \, \partial_n v \, ds \quad \text{for all } v \in W$$

on Γ_c: void

on Γ_s: $n \cdot \partial_s \phi = 0$

on Γ_f: $n \cdot \partial_s \phi = p$ and $s \cdot \partial_s \phi = 0$

Extensions

General self-adjoint fourth-order problems

$$\operatorname{div}\operatorname{Div}\left(\mathcal{D}\nabla^2 u\right) - \operatorname{div}(K\nabla u) + c\,u = f \quad \text{in } \Omega,$$

mixed formulation

$$\begin{aligned} &\mathbf{M} & &+ \nabla^2 u = 0 & &\text{in } \mathbf{V} \\ &\operatorname{div}\operatorname{Div}\mathbf{M} & &- (-\operatorname{div}(K\nabla u) + c\,u) = -f & &\text{in } Q \end{aligned}$$

decomposition $\mathbf{M} = p\,\mathbf{I} + \operatorname{sym}\operatorname{Curl}\phi$

$$\begin{aligned} &2\,p & &+ \operatorname{curl}^*\phi & &+ \Delta u = 0 & &\text{in } Q \\ &\operatorname{curl} p & &+ \operatorname{Curl}\operatorname{sym}\operatorname{Curl}\phi & &= 0 & &\text{in } \mathbf{V} \\ &\Delta p & & & &- (-\operatorname{div}(K\nabla u) + c\,u) = -f & &\text{in } Q \end{aligned}$$