An Adaptive Space-Time Boundary Element Method for the Wave Equation

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Laplace transform method [Bamberger, Ha Duong 1986]

$$-\Delta u + u_{tt} = 0 \text{ in } \Omega \times \mathbb{R}_+, \quad u|_{\Gamma\times\mathbb{R}_+} = g \quad \text{on } \Gamma \times \mathbb{R}_+, \quad \text{i.c.}$$

1. Apply Laplace transform to the wave equation and end up with the Helmholtz equation in the frequency domain for $\omega \in \mathbb{C}_0$:

$$-\Delta U(\omega) - \omega^2 U(\omega) = 0 \text{ in } \Omega, \quad \gamma_0^{\mathrm{int}} U(\omega) = G(\omega) \text{ on } \Gamma$$

2. Reformulate the Helmholtz equation via potentials for $\omega \in \mathbb{C}_0$:

$$\tilde{V}(\omega)\colon\ H^{-1/2}(\Gamma) \to H^1(\Omega), \quad U(\omega) = \tilde{V}(\omega)\Lambda(\omega)$$

3. Analyse the bie explicitly in terms of $\omega \in \mathbb{C}_{\sigma_0}$ for fixed $\sigma_0 > 0$:

$$\|V(\omega)\| \le C(\sigma_0)\,|\omega|\,, \quad \left\|V(\omega)^{-1}\right\| \le C(\sigma_0)\,|\omega|^2$$

4. Apply Paley-Wiener theorem and convolution theorem for the inverse transform to the time domain:

$$u = \tilde{\mathcal{V}} * \lambda := \mathcal{L}^{-1}(\omega \mapsto \tilde{V}(\omega)\Lambda(\omega))$$

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Bilinear form and loss of time regularity

Bilinear form in the frequency domain for $\omega \in \{\mathrm{Im}\,\omega \geq \sigma_0 > 0\}$:

$$\langle \psi, -\imath\omega V(\omega)\varphi \rangle \quad \forall \varphi, \psi \in H^{-1/2}(\Gamma)$$

Define bilinear form for $\sigma \geq \sigma_0 > 0$

$$a_\sigma(\lambda, \tau) := \int_0^\infty \mathrm{e}^{-2\sigma t} \left\langle \tau(t), \left(\mathcal{V} * \frac{\mathrm{d}\lambda}{\mathrm{d}t}\right)(t) \right\rangle_\Gamma \mathrm{d}t.$$

Loss of regularity in time for $\sigma \geq \sigma_0 > 0$

$$\mathcal{V}^{-1} * \mathcal{V}* : \mathcal{H}^{3}_\sigma(\mathbb{R}, H^{-1/2}(\Gamma)) \to \mathcal{H}^{0}_\sigma(\mathbb{R}, H^{-1/2}(\Gamma)),$$

$$\mathcal{V}^{-1} * \mathcal{V}* : H^{2;-1/2}_{\sigma,\Gamma} \to H^{0;-1/2}_{\sigma,\Gamma}.$$

An energy approach [Aimi et al. 2008]

For any solution u of the wave equation

$$\begin{aligned} \partial_{tt} u(x,t) - \Delta u(x,t) &= 0 \quad \text{for } x \in \mathbb{R}^d \setminus \Gamma, \quad t \in (0,T), \\ \partial_t u(x,0) = u(x,0) &= 0 \quad \text{for } x \in \mathbb{R}^d \setminus \Gamma, \\ u(x,t) &= g(x,t) \quad \text{for } (x,t) \in \Sigma := \Gamma \times [0,T] \end{aligned}$$

it holds

$$\begin{aligned} 0 &= (\partial_{tt} u - \Delta u)\partial_t u \\ &= \partial_t \left(\frac{1}{2}(\partial_t u)^2 + \frac{1}{2}\left|\nabla u\right|^2 \right) - \nabla \cdot (\partial_t u \nabla u). \end{aligned}$$

With Green's formula it follows

$$0 = \frac{1}{2} \int_{\mathbb{R}^d \setminus \Gamma} \left\{ (\partial_t u(x,T))^2 + |\nabla u(x,T)|^2 \right\} \mathrm{d}x - \int_0^T \langle \partial_t \gamma_0^{\mathrm{int}} u(\cdot,t), [\![\gamma_1 u]\!](\cdot,t) \rangle_\Gamma \mathrm{d}t.$$

Define the energy

$$E(T) := \frac{1}{2} \int_{\mathbb{R}^d \setminus \Gamma} \left\{ (\partial_t u(x,T))^2 + |\nabla u(x,T)|^2 \right\} \mathrm{d}x$$

and represent the solution by $u = \tilde{\mathcal{V}} * \lambda$. Hence,

$$E(T) = \int_0^T \langle \partial_t (\mathcal{V} * \lambda)(t), \lambda(t) \rangle_\Gamma \mathrm{d}t = \langle \partial_t (\mathcal{V} * \lambda), \lambda \rangle_\Sigma.$$

Define the bilinear form

$$a_E(\lambda, \varphi) := \langle \partial_t (\mathcal{V} * \lambda), \varphi \rangle_\Sigma.$$

One-dimensional wave equation

$Q := (0, L) \times (0, T)$ and $\Sigma := \{0, L\} \times [0, T]$.

Single layer potential:

$$\tilde{\mathcal{V}} * w(x,t) = \frac{1}{2}\int_{-\infty}^{t-|x|} w_0(s)\,\mathrm{d}s + \frac{1}{2}\int_{-\infty}^{t-|x-L|} w_L(s)\,\mathrm{d}s$$

for $t \in \mathbb{R},\ x \in \mathbb{R} \setminus \{0, L\}$ and a density $w = \begin{pmatrix} w_0 \\ w_L \end{pmatrix}$.

Single layer operator $\mathcal{V}*\colon\ L^2(\Sigma) \to H^1_{\{0\}}(\Sigma)$

$$\mathcal{V} * w(t) = \frac{1}{2} \begin{pmatrix} \int_0^t w_0(s)\mathrm{d}s + \int_0^{t-L} w_L(s)\mathrm{d}s \\ \int_0^{t-L} w_0(s)\mathrm{d}s + \int_0^t w_L(s)\mathrm{d}s \end{pmatrix} \quad \text{for } t \in [0, T]$$

with

$$L^2(\Sigma) := \left\{ v = \begin{pmatrix} v_0 \\ v_L \end{pmatrix} : v_0, v_L \in L^2(0,T) \right\},$$

$$H^1_{\{0\}}(\Sigma) := \left\{ v = \begin{pmatrix} v_0 \\ v_L \end{pmatrix} : v_0, v_L \in H^1(0,T),\ \gamma v_0(0) = \gamma v_L(0) = 0 \right\}.$$

Theorem (Aimi et al. 2008)

The bilinear form $a_E\colon\ L^2(\Sigma)\times L^2(\Sigma)\to\mathbb{R}$,

$$a_E(w,v) := \langle \partial_t(\mathcal{V} * w), v\rangle_{L^2(\Sigma)},$$

is bounded and coercive:

$$a_E(\varphi,\varphi) \geq C(T)\|\varphi\|^2_{L^2(\Sigma)} \quad \forall \varphi \in L^2(\Sigma)$$

Find $w \in L^2(\Sigma)$ for given $g \in H^1_{\{0\}}(\Sigma)$ such that

$$a_E(w,v) = \frac{1}{2}\langle \partial_t g, v\rangle_{L^2(\Sigma)} + \langle \partial_t(\mathcal{K} * g), v\rangle_{L^2(\Sigma)} \quad \forall v \in L^2(\Sigma),$$

where the double layer operator is given by

$$\mathcal{K} * g(t) = \mathcal{K} * \begin{pmatrix} g_0 \\ g_L \end{pmatrix}(t) = -\frac{1}{2}\begin{pmatrix} g_L(t-L) \\ g_0(t-L) \end{pmatrix} \quad \text{for } t \in [0,T].$$

Decomposition

$$\Sigma = \bigcup_{i=1}^{N_0+N_L} \overline{\tau}_i.$$

A conforming ansatz space for $L^2(\Sigma)$ is

$$\begin{aligned} S^0_{\Delta t}(\Sigma) &:= S^0_{\Delta t_0}(0,T) \times S^0_{\Delta t_L}(0,T) \\ &= \operatorname{span}\left\{ \begin{pmatrix} \varphi^0_{0,i} \\ 0 \end{pmatrix} \right\}_{i=1}^{N_0} \cup \operatorname{span}\left\{ \begin{pmatrix} 0 \\ \varphi^0_{L,j} \end{pmatrix} \right\}_{j=1}^{N_L} \\ &= \operatorname{span}\left\{ \hat{\varphi}^0_i \right\}_{i=1}^{N_0+N_L}. \end{aligned}$$

The resulting linear system is

$$V_{\Delta t} \underline{w} = \underline{g}$$

with $V_{\Delta t} \in \mathbb{R}^{(N_0+N_L)\times(N_0+N_L)}$, $\underline{g} \in \mathbb{R}^{N_0+N_L}$ and $\underline{w} \in \mathbb{R}^{N_0+N_L}$.

A posteriori error estimator [Steinbach, Schulz 2000]

$$\eta_i := \left\| \gamma_1^{\text{int}} u - w_{\Delta t} \right\|_{L^2(\tau_i)} \quad \text{for } i = 1, \ldots, N_0 + N_L$$

Approximate solution in Q :

$$\tilde{u}_{\Delta t} := \tilde{\mathcal{V}} * w_{\Delta t} - \mathcal{W} * g$$

Neumann trace of the approximate solution:

$$\tilde{w}_{\Delta t} := \gamma_1^{\text{int}} \tilde{u}_{\Delta t} = \frac{1}{2} w_{\Delta t} + \mathcal{K}' * w_{\Delta t} + \mathcal{D} * g$$

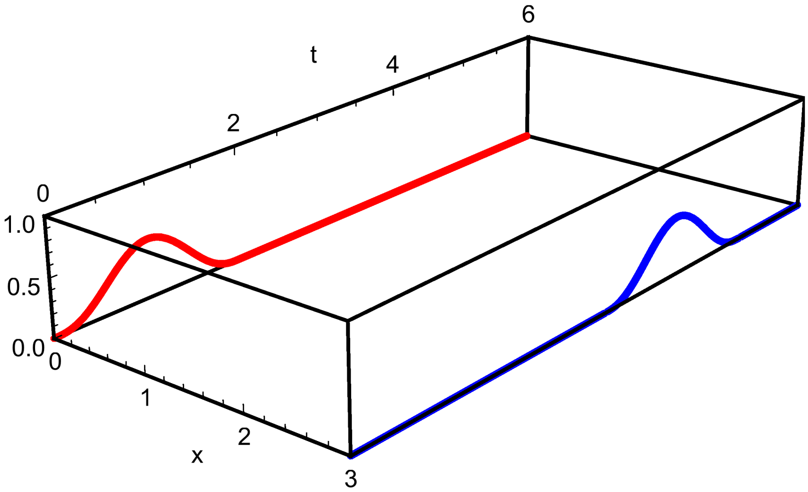
Local error estimator:

$$\tilde{\eta}_i := \| \tilde{w}_{\Delta t} - w_{\Delta t} \|_{L^2(\tau_i)} \quad \text{for } i = 1, \ldots, N_0 + N_L$$

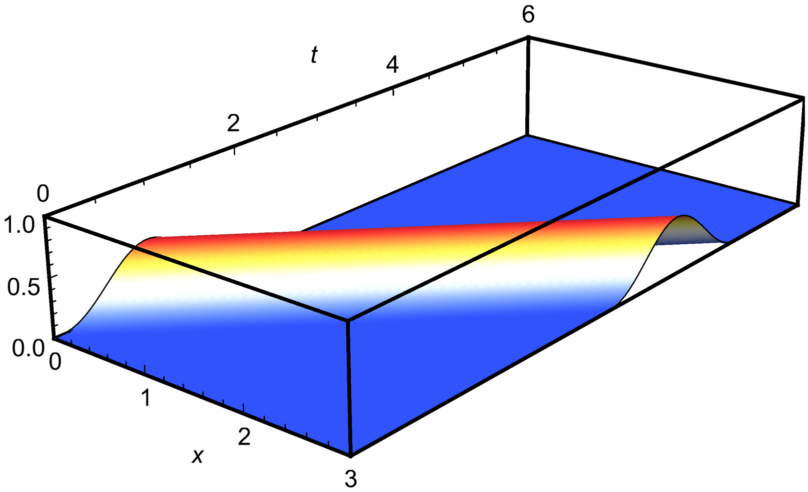
For a parameter $\theta \in [0, 1]$ refine all elements τ_i where

$$\tilde{\eta}_i \geq \theta \max_j \tilde{\eta}_j .$$

Numerical examples with $L = 3$ and $T = 6$



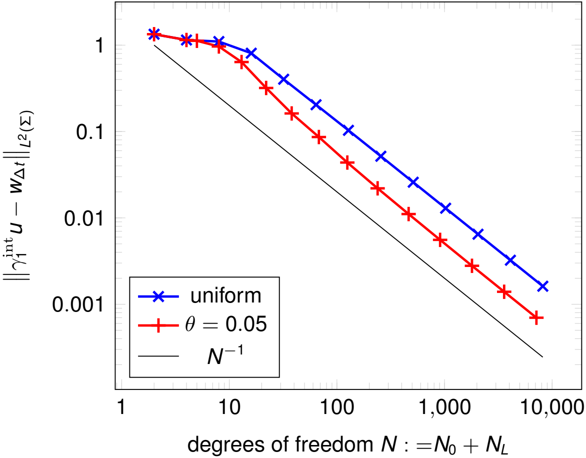
Numerical examples with $L = 3$ and $T = 6$



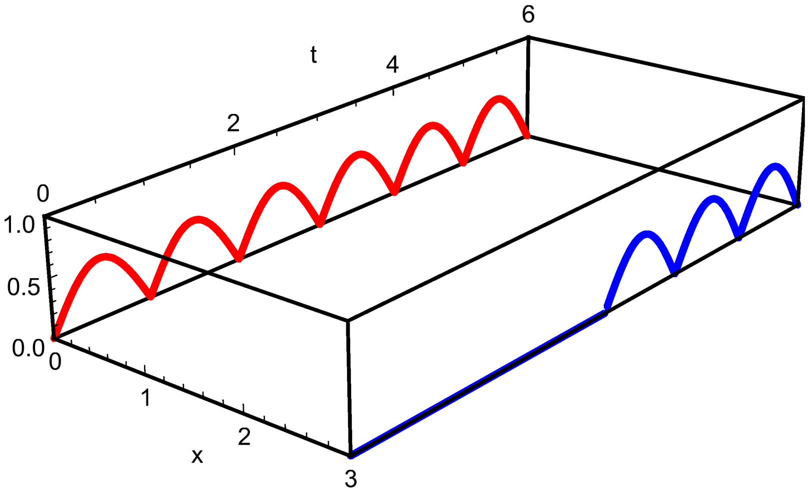
Numerical examples with $L = 3,\ T = 6$

L	$N_0 + N_L$	$\left\Vert \gamma_1^{\text{int}} u - w_{\Delta t} \right\Vert_{L^2(\Sigma)}$	eoc	$\left\Vert u - \tilde{u}_{\Delta t} \right\Vert_{L^2(Q)}$	eoc
0	2	1.33823e+00	0.00	7.26168e-01	0.00
1	4	1.14684e+00	0.22	4.49627e-01	0.69
2	8	1.10072e+00	0.06	3.24885e-01	0.47
3	16	8.06608e-01	0.45	1.46335e-01	1.15
4	32	4.02738e-01	1.00	3.75263e-02	1.96
5	64	2.04198e-01	0.98	9.55880e-03	1.97
6	128	1.03212e-01	0.98	2.49973e-03	1.94
7	256	5.17114e-02	1.00	5.95579e-04	2.07
8	512	2.58723e-02	1.00	1.56495e-04	1.93
9	1024	1.29381e-02	1.00	3.71371e-05	2.08
10	2048	6.46928e-03	1.00	9.78807e-06	1.92
11	4096	3.23467e-03	1.00	2.40515e-06	2.02
12	8192	1.61734e-03	1.00	5.99227e-07	2.00
13	16384	8.08670e-04	1.00	1.49398e-07	2.00
14	32768	4.04335e-04	1.00	3.77370e-08	1.99

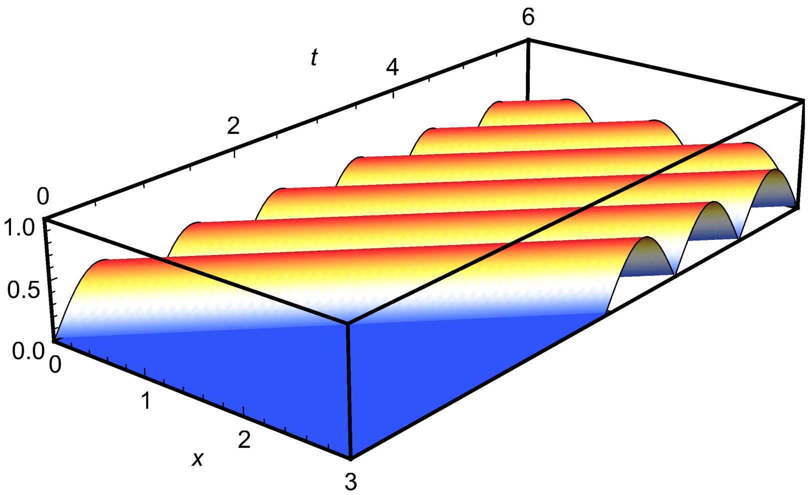
Numerical examples with $L = 3,\ T = 6$



Numerical examples with $L = 3$ and $T = 6$



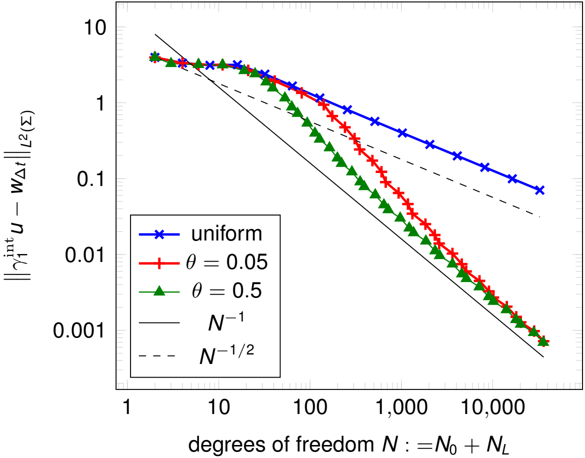
Numerical examples with $L = 3$ and $T = 6$



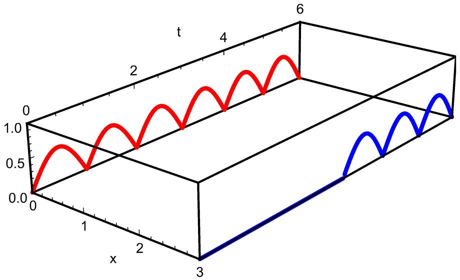
Numerical examples with $L = 3,\ T = 6$

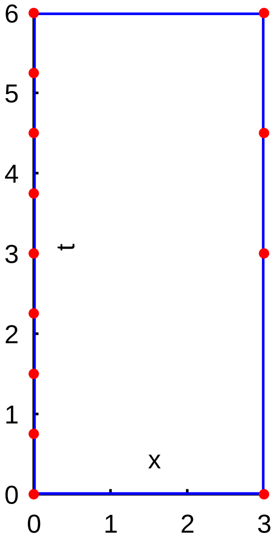
L	$N_0 + N_L$	$\left\|\gamma_1^{\mathrm{int}} u - w_{\Delta t}\right\|_{L^2(\Sigma)}$	eoc	$\left\|u - \tilde{u}_{\Delta t}\right\|_{L^2(Q)}$	eoc
0	2	3.95477e+00	0.00	2.59835e+00	0.00
1	4	3.33217e+00	0.25	5.78383e-01	2.17
2	8	3.11643e+00	0.10	4.73586e-01	0.29
3	16	3.16575e+00	-0.02	4.10036e-01	0.21
4	32	2.37997e+00	0.41	1.77812e-01	1.21
5	64	1.66423e+00	0.52	6.10341e-02	1.54
6	128	1.15613e+00	0.53	2.28464e-02	1.42
7	256	8.07589e-01	0.52	8.15019e-03	1.49
8	512	5.67073e-01	0.51	3.29593e-03	1.31
9	1024	3.99491e-01	0.51	1.33215e-03	1.31
10	2048	2.81940e-01	0.50	5.98854e-04	1.15
11	4096	1.99168e-01	0.50	2.74489e-04	1.13
12	8192	1.40764e-01	0.50	1.32773e-04	1.05
13	16384	9.95104e-02	0.50	6.47539e-05	1.04
14	32768	7.03558e-02	0.50	3.21587e-05	1.01

Numerical examples with $L = 3,\ T = 6$

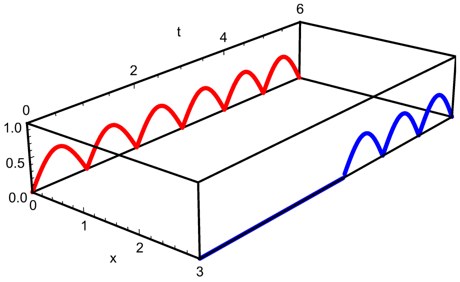


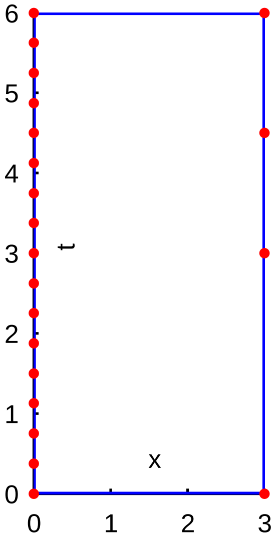
Numerical examples with $L = 3,\ T = 6, \theta = 0.5$



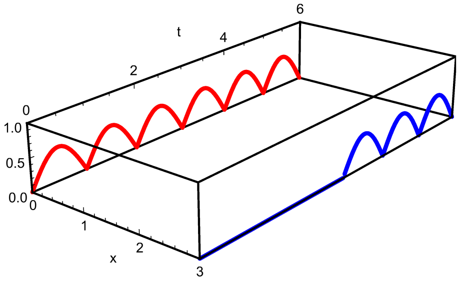


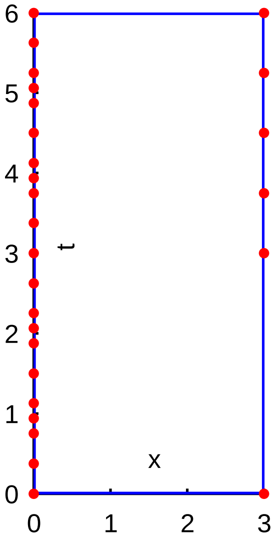
Numerical examples with $L = 3,\ T = 6,\ \theta = 0.5$



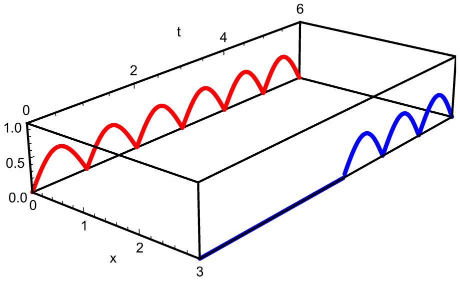


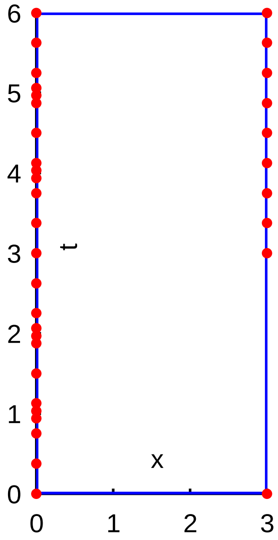
Numerical examples with $L = 3,\ T = 6, \theta = 0.5$



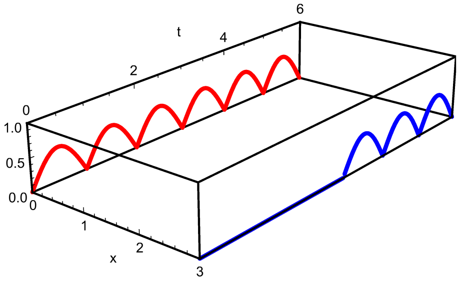


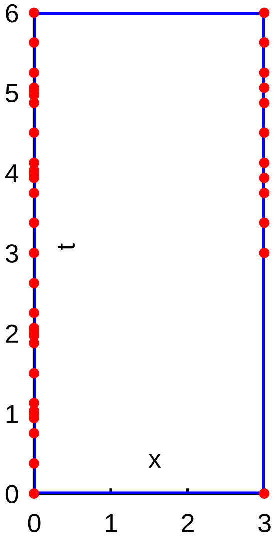
Numerical examples with $L = 3,\ T = 6,\ \theta = 0.5$





Numerical examples with $L = 3$, $T = 6$, $\theta = 0.5$





Summary and outlook

- Laplace transform method
- Loss of regularity in time
- Energy approach
- One-dimensional wave equation
- A posteriori error estimator

- Error analysis for 1D?
- Bilinear form for 2D, 3D?

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