Space-time methods for optimal control models in pedestrian dynamics

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AANMPDE-9-16, St. Wolfgang, July 4, 2016

Outline

Pedestrian motion

- Empirical studies of human crowds started about 50 years ago.
- Nowadays there is a large literature on different micro- and macroscopic approaches available.
- Challenges: microscopic interactions not clearly defined, multiscale effects, finite size effects,.....

Microscopic and macroscopic models

Microscopic models (model every particle):

- force based models: position of a particle is determined by forces acting on it
- stochastic optimal control models: each agent wants to minimize a stochastic cost functional
- lattice based models: domain divided into cells, particles may (or may not) jump from one cell to another with a certain transition probability

Macroscopic models: number of individuals goes to infinity, nonlinear transport-diffusion equation based on conservation of mass

Nonlinear diffusion transportation models

Intuitive assumption: total number of individuals is conserved in time and the speed of individuals is linked to the density of the surrounding pedestrian flow.

Conservation law:

$$\partial_t \rho(\boldsymbol{x}, t) + \mathrm{div}(F(\rho(\boldsymbol{x}, t))\boldsymbol{v}(\boldsymbol{x}, t)) = 0,$$

where $x \in \Omega \subset \mathbb{R}^d$ with $d = \{1, 2, 3\}$ is the position in space, $t \in (0, T]$ the time, $\rho(\boldsymbol{x}, t)$ the pedestrian density, $\boldsymbol{v}(\boldsymbol{x}, t)$ the velocity and $F(\rho)$ the **mobility/penalization function** for high densities such as $F(\rho) = \rho_{max} - \rho$ or $F(\rho) = \rho(\rho_{max} - \rho)^2$ with ρ_{max} being the maximal density, e.g., we will choose $\rho_{max} = 1$ later. See also

- R. L. Hughes. A continuum theory for the flow of pedestrians. *Transportation Research Part B: Methodological*, 36(6):507–535, 2002.

Hughes' model for pedestrian flow

- Pedestrians have a common sense/drive of the task described via a potential ϕ, where $-\nabla\phi$ gives the direction.
- Pedestrians try to minimize the travel time.
- Pedestrians try to avoid high densities, speed depends on the density of the surrounding pedestrian flow.

$$\rho_t - \operatorname{div}(\rho f^2(\rho)\nabla\phi) = 0,$$
$$|\nabla\phi| = \frac{1}{f(\rho)},$$

where $f(\rho)$ provides a weighting or cost wrt high densities, i.e., saturation for $\rho \to \rho_{max}$. More detailed discussion can be found in

- M. Burger, M. Di Francesco, P. Markowich and M-T. Wolfram. Mean field games with nonlinear mobilities in pedestrian dynamics,
A continuum theory for the flow of pedestrians. *DCDS B*, 19, 2014.

An optimal control approach for fast exit scenarios

Let us consider an evacuation/fast exit scenario, i.e., a room with one or several exits from which a group wants to leave as fast as possible. Let $\Omega \subset \mathbb{R}^2$, $\Gamma = \partial\Omega = \Gamma_E \cup \Gamma_N$, and $(0, T)$ be the time interval. The minimization reads as: $\min_{\rho,\boldsymbol{v}} \mathcal{J}(\rho, \boldsymbol{v})$ with

$$\mathcal{J}(\rho,\boldsymbol{v}) = \underbrace{\frac{1}{2}\int_0^T \int_\Omega \rho(\boldsymbol{x},t)|\boldsymbol{v}(\boldsymbol{x},t)|^2 d\boldsymbol{x}\,dt}_{\text{kinetic energy}} + \underbrace{\frac{\alpha}{2}\int_0^T \int_\Omega \rho(\boldsymbol{x},t)\,d\boldsymbol{x}\,dt}_{\text{exit time}},$$

s.t.

$$\partial_t \rho + \operatorname{div}(\rho\boldsymbol{v}) = \frac{\sigma^2}{2}\triangle\rho, \qquad \text{in } \Omega \times (0,T),$$

$$(-\frac{\sigma^2}{2}\nabla\rho + \rho\boldsymbol{v}) \cdot \boldsymbol{n} = 0, \qquad \text{on } \Gamma_N \times (0,T),$$

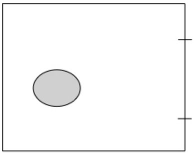
$$(-\frac{\sigma^2}{2}\nabla\rho + \rho\boldsymbol{v}) \cdot \boldsymbol{n} = \beta\rho, \text{ on } \Gamma_E \times (0,T), \qquad \rho(\cdot,0) = \rho_0(\cdot), \text{ in } \Omega.$$

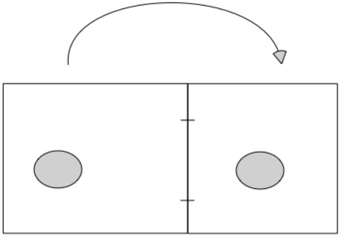
Two different approaches to solve the problem

Target: evacuate the group of people as fast as possible

Approach 1: group has to leave the room via the exit(s)

Approach 2: group has to get from one place to the other
$\rightarrow$ transport problem





Space-time approach to optimal mass transport

- J.-D. Benamou and Y. Brenier. A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem, *Numerische Mathematik*, 84(3):375–393, 2000.

proposed to reset the L^2 Monge-Kantorovich mass transfer problem in a fluid mechanics framework: $\min_{\rho,\boldsymbol{v}} \mathcal{J}(\rho,\boldsymbol{v})$ with

$$\mathcal{J}(\rho,\boldsymbol{v}) = \int_0^T \int_\Omega \rho(\boldsymbol{x},t)|\boldsymbol{v}(\boldsymbol{x},t)|^2 d\boldsymbol{x}\, dt,$$

subject to

$$\partial_t \rho + \operatorname{div}(\rho\boldsymbol{v}) = 0, \qquad \rho(\boldsymbol{x},0) = \rho_0(\boldsymbol{x}), \qquad \rho(\boldsymbol{x},T) = \rho_T(\boldsymbol{x}).$$

More efficient numerical treatment proposed in

- E. Haber and R. Horesh. A multilevel method for the solution of time dependent optimal transport. *Numerical Mathematics: Theory, Methods and Applications*, 8(1):97–111, 2015.

by setting the momentum $\boldsymbol{m} = \rho\boldsymbol{v}$.

Generalization of the optimal control problem

Introducing the function $F(\rho)$ describing the nonlinear mobilities, we consider the following optimization problem:

$$\min_{\rho,\boldsymbol{v}} \mathcal{J}(\rho,\boldsymbol{v}) = \min_{\rho,\boldsymbol{v}} \frac{1}{2}\int_{Q_T} F(\rho(\boldsymbol{x},t))|\boldsymbol{v}(\boldsymbol{x},t)|^2 d\boldsymbol{x}\,dt + \frac{\alpha}{2}\int_{Q_T} \rho(\boldsymbol{x},t)\,d\boldsymbol{x}\,dt,$$

such that

$$\begin{cases} \partial_t\rho(\boldsymbol{x},t) + \operatorname{div}(F(\rho(\boldsymbol{x},t))\boldsymbol{v}(\boldsymbol{x},t)) = \dfrac{\sigma^2}{2}\triangle\rho(\boldsymbol{x},t), & \text{in } \Omega\times(0,T), \\ \left(F(\rho)\boldsymbol{v} - \dfrac{\sigma^2}{2}\nabla\rho\right)\cdot\boldsymbol{n} = 0, & \text{on } \Gamma_N\times(0,T), \\ \left(F(\rho)\boldsymbol{v} - \dfrac{\sigma^2}{2}\nabla\rho\right)\cdot\boldsymbol{n} = \beta\rho, & \text{on } \Gamma_E\times(0,T), \\ \rho(\boldsymbol{x},0) = \rho_0(\boldsymbol{x}), & \text{in } \Omega. \end{cases}$$

Momentum formulation

Denoting the momemtum $\boldsymbol{m} = F(\rho)\boldsymbol{v}$, we can rewrite the minimization functional as

$$\mathcal{J}(\rho, \boldsymbol{m}) = \frac{1}{2}\int_0^T \int_\Omega \frac{|\boldsymbol{m}(\boldsymbol{x},t)|^2}{F(\rho(\boldsymbol{x},t))} d\boldsymbol{x}\, dt + \frac{\alpha}{2}\int_0^T \int_\Omega \rho(\boldsymbol{x},t)\, d\boldsymbol{x}\, dt,$$

s.t.

$$\begin{aligned}
\partial_t \rho + \operatorname{div}(\boldsymbol{m}) &= \frac{\sigma^2}{2}\triangle\rho, && \text{in } \Omega \times (0,T),\\
(\boldsymbol{m} - \frac{\sigma^2}{2}\nabla\rho)\cdot \boldsymbol{n} &= 0, && \text{on } \Gamma_N \times (0,T),\\
(\boldsymbol{m} - \frac{\sigma^2}{2}\nabla\rho)\cdot \boldsymbol{n} &= \beta\rho, && \text{on } \Gamma_E \times (0,T),\\
\rho(\boldsymbol{x},0) &= \rho_0(\boldsymbol{x}), && \text{in } \Omega.
\end{aligned}$$

Optimality system

Find the solution $(\rho, \boldsymbol{m}, \lambda)$, where λ is the Lagrange multiplier, of

$$
\begin{aligned}
&\partial_t \rho + \operatorname{div}(\boldsymbol{m}) - \frac{\sigma^2}{2}\triangle \rho = 0, && \text{in } \Omega \times (0,T),\\
&\frac{\boldsymbol{m}}{F(\rho)} - \nabla \lambda = 0, && \text{in } \Omega \times (0,T),\\
&\partial_t \lambda + \frac{F'(\rho)|\boldsymbol{m}|^2}{2F^2(\rho)} + \frac{\sigma^2}{2}\triangle \lambda = \frac{\alpha}{2}, && \text{in } \Omega \times (0,T),\\
&(\boldsymbol{m} - \frac{\sigma^2}{2}\nabla\rho)\cdot \boldsymbol{n} = 0, \quad \frac{\sigma^2}{2}\nabla\lambda \cdot \boldsymbol{n} = 0, && \text{on } \Gamma_N \times (0,T),\\
&(\boldsymbol{m} - \frac{\sigma^2}{2}\nabla\rho)\cdot \boldsymbol{n} = \beta\rho, \quad \frac{\sigma^2}{2}\nabla\lambda \cdot \boldsymbol{n} + \beta\lambda = 0, && \text{on } \Gamma_E \times (0,T),\\
&\rho(\boldsymbol{x},0) = \rho_0(\boldsymbol{x}), \quad \lambda(\boldsymbol{x},T) = 0, && \text{in } \Omega.
\end{aligned}
$$

A priori estimate for the forward problem

$$V = L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)),\ Q = [L^2(\Omega \times (0,T))]^d$$

Lemma

Let $\rho_0 \in L^2(\Omega)$. *Let* $F(\rho) \in C^1(\mathbb{R})$ *be bounded and non-negative for* $0 \leq \rho \leq \rho_{max}$ *and let* $\sigma > 0$, $\beta \geq 0$. *Let* $\boldsymbol{v} \in Q$ *and let* $\rho \in V$ *be a weak solution of*

$$\langle \partial_t \rho, \psi \rangle_{H^{-1},H^1} + \int_\Omega \left(\frac{\sigma^2}{2} \nabla \rho - F(\rho)\boldsymbol{v} \right) \cdot \nabla \psi \, d\boldsymbol{x} = - \int_{\Gamma_E} \beta \rho \psi \, ds,$$

for all $\psi \in H^1(\Omega)$. *Then there exist constants* $C_1, C_2 > 0$ *depending on* F, σ, Ω *and* T *only, such that*

$$\|\rho\|_V \leq C_1 \|\boldsymbol{v}\|_Q + C_2.$$

Existence and uniqueness

Theorem

Under the assumptions of the Lemma the variational problem for $\min_{(\rho,\mathbf{v})\in V\times Q} \mathcal{J}(\rho,\mathbf{v})$ *subject to* $\partial_t \rho + div(F(\rho)\mathbf{v}) = \frac{\sigma^2}{2}\triangle\rho$ *has at least a weak solution in* $V \times Q$ *with given* $\rho_0 \in L^2(\Omega)$.

Theorem

For fixed $\rho_0 \in L^2(\Omega)$, *there exists a unique weak solution*

$$(\rho,\lambda) \in L^2(0,T;H^1(\Omega)) \times L^2(0,T;H^1(\Omega))$$

to the (reduced) optimality system.

Numerical results

The code was implemented in the **programming language Julia**, see http://julialang.org.

Let $F(\rho) = \rho(1-\rho)^2$ as in Hughes' model. The optimality system is discretized by a finite volume method in space-time.

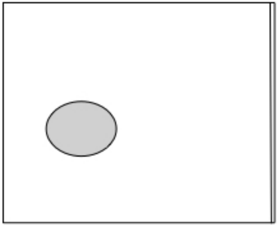
Octree mesh with 32^3 space-time cubes (32 time slices).

In order to solve the constrained optimization problem, we apply a version of the line search sequential quadratic programming (SQP) method (Newton-type scheme).

We have two types of numerical experiments:

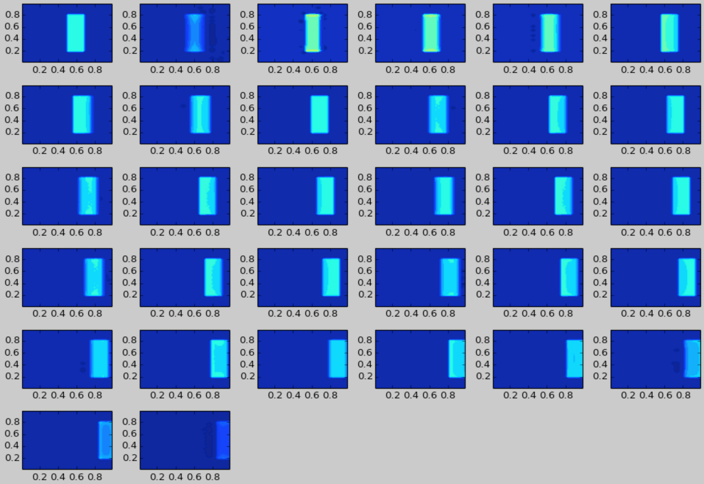
- with BCs, without final time condition, with $\sigma \neq 0$,
- with final time condition, without BCs, with $\sigma = 0$.

Numerical results - with exits

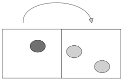


Initial distribution: $\rho_0(\boldsymbol{x}) = 0.4$ for $\boldsymbol{x} \in [0.2, 0.8] \times [0.5, 0.7]$ and $\rho_0(\boldsymbol{x}) = 0$ elsewhere; $\beta = 10$, $\sigma = 1$, $\alpha = 20$;
22 SQP iterations needed

Numerical results



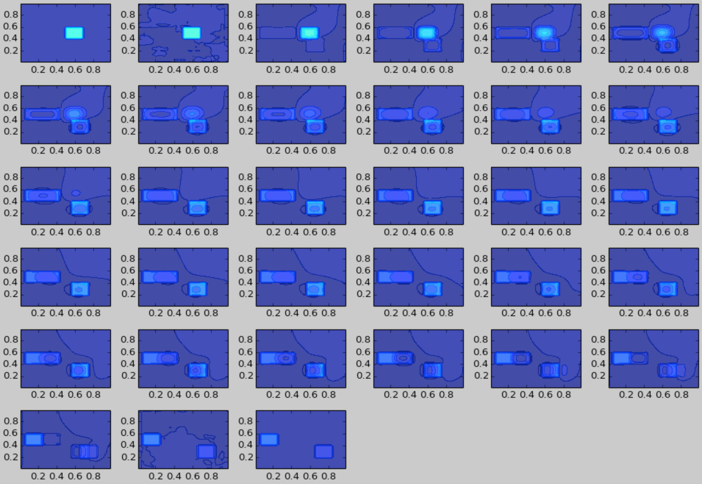
Numerical results - mass transport



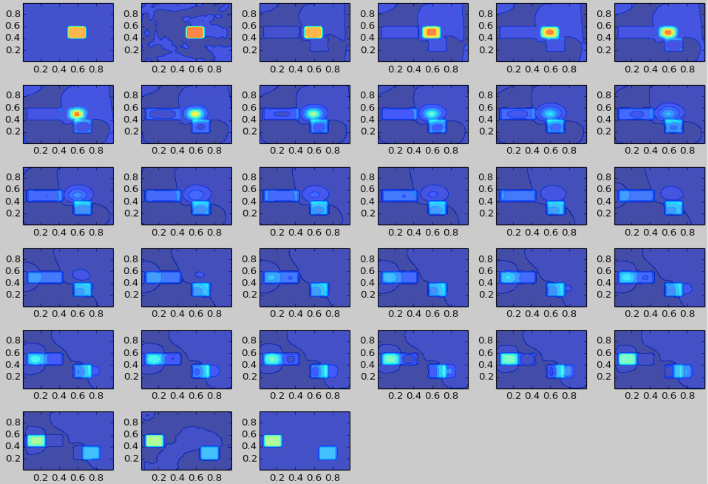
2 Examples (with higher and lower initial density):

- Initial distribution: $\rho_0(\boldsymbol{x}) = 0.4$ for $\boldsymbol{x} \in [0.5, 0.7] \times [0.4, 0.6]$ and $\rho_0(\boldsymbol{x}) = 0$ elsewhere; terminal distribution: $\rho_T(\boldsymbol{x}) = 0.3$ for $\boldsymbol{x} \in [0.05, 0.25] \times [0.4, 0.6]$, $\rho_T(\boldsymbol{x}) = 0.1$ for $\boldsymbol{x} \in [0.65, 0.85] \times [0.2, 0.4]$ and $\rho_T(\boldsymbol{x}) = 0$ elsewhere.
- Initial distribution: $\rho_0(\boldsymbol{x}) = 0.8$ for $\boldsymbol{x} \in [0.5, 0.7] \times [0.4, 0.6]$ and $\rho_0(\boldsymbol{x}) = 0$ elsewhere; terminal distribution: $\rho_T(\boldsymbol{x}) = 0.5$ for $\boldsymbol{x} \in [0.05, 0.25] \times [0.4, 0.6]$, $\rho_T(\boldsymbol{x}) = 0.3$ for $\boldsymbol{x} \in [0.65, 0.85] \times [0.2, 0.4]$ and $\rho_T(\boldsymbol{x}) = 0$ elsewhere.

Numerical results



Numerical results



Summary and outlook

Summary:

- Optimal control approach to the modeling of pedestrian dynamics
- Space-time solver based on Benamou-Brenier and Haber-Horesh

Outlook:

- Improve the model as well the solver
- More numerical results, e.g., include obstacles in the domain, finer space-time meshes
- Preconditioning
- Adaptive methods in space and time
- New minimal time optimization problem for evacuation

References

1 R. L. Hughes. A continuum theory for the flow of pedestrians. *Transportation Research Part B: Methodological*, 36(6):507–535, 2002.

2 M. Burger, M. Di Francesco, P. Markowich and M-T. Wolfram. Mean field games with nonlinear mobilities in pedestrian dynamics. A continuum theory for the flow of pedestrians. *DCDS B*, 19, 2014.

3 J.-D. Benamou and Y. Brenier. A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem. *Numerische Mathematik*, 84(3):375–393, 2000.

4 E. Haber and R. Horesh. A multilevel method for the solution of time dependent optimal transport. *Numerical Mathematics: Theory, Methods and Applications*, 8(1):97–111, 2015.

5 J. Nocedal and S. Wright. Numerical optimization. *Springer Science & Business Medial*, 2006.

6 M. Di Francesco, P. A. Markowich, J.-F. Pietschmann, and M.-T. Wolfram. On the Hughes' model for pedestrian flow: The one-dimensional case. *Journal of Differential Equations*, 250(3):1334–1362, 2011.

7 E. Carlini, A. Festa, F. Silva and M.-T. Wolfram. A Semi-Lagrangian scheme for a modified version of the Hughes model for pedestrian flow. *Preprint*, 2016.

Thank you for your attention!