On Potential Methods for Porous Media Flows

Wolfgang L. Wendland

Institute for Applied Analysis and Numerical Simulation & SIMTECH, Univ. Stuttgart

joint work with Mirela Kohr (Babeş–Bolyai Univ., Cluj–Napoca, Romania)
Massimo Lanza de Cristoforis (Univ. of Padova, Italy),
Sergey E. Mikhailov (Brunel Univ. London, United Kingdom)

Strobl 2016, 3.–10.7.

Contents

1. The transmission problem

$$\Delta \mathbf{u} - \alpha \mathbf{u} - k|\mathbf{u}|\mathbf{u} - \beta(\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla p = \mathbf{f} \text{ and } \nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega_+;$$

$$\Delta \mathbf{u} - \nabla p = \mathbf{0} \text{ and } \nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega_-;$$

$\gamma_+\mathbf{u}_+ - \gamma_-\mathbf{u}_- = \mathbf{h}_0$ on $\Gamma = \partial\Omega$ connected boundary and

$$\partial^+_{n;\alpha}\big(\mathbf{u}_+, p_+; (\mathbf{f}_+ + \mathbb{E}_+(k|\mathbf{u}_+|\mathbf{u}_+ + \beta(\mathbf{u}_+\cdot\nabla)\mathbf{u}_+)\big) - \mu\partial^-_n(\mathbf{u}_-, p_-; \mathbf{f}_-) + \tfrac{1}{2}\mathcal{P}(\gamma_-\mathbf{u}_- + \gamma_+\mathbf{u}_+) = \mathbf{g}_0 \quad \text{on } \Gamma;$$

$$\mathbf{u}_-(\infty) = \mathbf{u}_\infty \;:\; \lim_{|\mathbf{x}|\to\infty} \int_{\mathbb{S}^2(\mathbf{x})} |\mathbf{u}_-\big(\mathbf{y}(\mathbf{x})\big) - \mathbf{u}_\infty| ds_y = 0, \qquad \text{(Leray)}.$$

$\Omega_+ \subset \mathbb{R}^3$ bounded Lipschitz; $\Omega_- := \mathbb{R}^3 \setminus \overline{\Omega}_+; \alpha, k, \beta \geq 0$;

$$\mathcal{P}(\mathbf{x}) \in L^\infty(\Gamma)^{3\times3}, \mathcal{P}(\mathbf{x}) = \mathcal{P}^*(\mathbf{x}), \forall\, \boldsymbol{\xi} \in \mathbb{R}^3 \;:\; \langle \mathcal{P}\boldsymbol{\xi}, \boldsymbol{\xi}\rangle_{\mathbb{R}^3} \geq 0;$$

$$\mathbb{S}^2(\mathbf{x}) = \{\mathbf{y}(\mathbf{x}) \;:\; |\mathbf{y}(\mathbf{x}) - \mathbf{x}| = 1\}\,;\, \mathbb{E}(\mathbf{v}) := \tfrac{1}{2}\big(\nabla\mathbf{v} + (\nabla\mathbf{v})^\top\big);$$

$$\langle \partial^\pm_{n;\alpha}(\mathbf{u}^\pm, p^\pm; \mathbf{f}^\pm), \boldsymbol{\varphi}\rangle_{L^2(\Gamma)} = 2\langle \mathbb{E}(\mathbf{u}), \mathbb{E}\gamma_\pm^{-1}\boldsymbol{\varphi}\rangle_{L^2(\Omega_\pm)}$$

$$+\alpha\langle \mathbf{u}_\pm, \gamma_\pm^{-1}\boldsymbol{\varphi}\rangle_{L^2(\Omega_\pm)} - \langle p_\pm, \operatorname{div}\,(\gamma_\pm^{-1}\boldsymbol{\varphi})\rangle_{L^2(\Omega_\pm)} + \langle \mathbf{f}^\pm, \gamma_\pm^{-1}\boldsymbol{\varphi}\rangle_{L^2(\Omega_\pm)}$$

2. Function spaces

$$
\begin{aligned}
H^s(\mathbb{R}^3) &:= \{f \mid \mathcal{F}^{-1}(1+|\boldsymbol{\xi}|^2)^{-s/2}\mathcal{F}f \,:\, f \in L^2(\mathbb{R}^3\},\\
H^s(\Omega) &:= \{f \in \mathcal{D}'(\Omega) \,:\, \exists F \in H^s(\mathbb{R}^3) \wedge F|_\Omega = f\},\\
\widetilde{H}^s(\Omega) &:= \{f \in H^s(\mathbb{R}^3) \wedge \mathrm{supp} f \subseteq \overline{\Omega}\},\\
\big(H^s(\Omega)\big)' &= \widetilde{H}^{-s}(\Omega)\,,\; H^{-s}(\Omega) = \big(\widetilde{H}^s(\Omega)\big)'\,,\; L^2(\Omega)\text{−duality;}
\end{aligned}
$$

$$
\varrho(\mathbf{x}) := (1+|\mathbf{x}|^2)^{1/2}\,,\; L^p(\varrho^{-1};\Omega_-) := \{f \mid \varrho^{-1}f \in L^p(\Omega_-)\},
$$

$$
\begin{aligned}
\mathcal{H}^1(\Omega_-) &:= \{f \in \mathcal{D}'(\Omega_-) \mid \varrho^{-1}f \in L^2(\Omega_-) \wedge \nabla f \in L^2(\Omega_-)\},\\
\| f \|^2_{\mathcal{H}^1(\Omega_-)} &:= (\| \varrho^{-1} f \|^2_{L^2(\Omega_-)} + \| \nabla f \|^2_{L^2(\Omega_-)}),\\
\mathring{\mathcal{H}}^1(\Omega_-) &= \overline{\mathcal{D}(\Omega_-)}^{\|\cdot\|_{\mathcal{H}^1(\Omega_-)}} \subset \mathcal{H}^1(\Omega_-),\\
\widetilde{\mathcal{H}}^1(\Omega_-) &:= \{v \in \mathcal{H}^1(\mathbb{R}^3) \wedge \mathrm{supp}\, v \subseteq \overline{\Omega}_-\},\\
\mathcal{H}^{-1}(\Omega_-) &= \big(\widetilde{\mathcal{H}}^1(\Omega_-)\big)'\,,\; \widetilde{\mathcal{H}}^{-1}(\Omega_-) = \big(\mathcal{H}^1(\Omega_-)\big)';
\end{aligned}
$$

Embedding: $\mathcal{H}^1(\Omega_-) \hookrightarrow L^6(\Omega_-)$ for $\Omega_- \subset \mathbb{R}^3$.

Traces:

$\gamma_\pm : H^1(\Omega_\pm) \to H^{\frac{1}{2}}(\Gamma)\,,\ \exists\ \gamma_\pm^{-1}\ :\ H^{1/2}(\Gamma) \to H^1(\Omega_\pm),$
$\gamma_- : \mathcal{H}^1(\Omega_-) \to H^{\frac{1}{2}}(\Gamma)$ surjectively (Lang+Méndez 2006, Mikhailov 2011) .

3. Solution properties for Brinkman and Stokes systems

Lemma: (Amrouche+Razafison 2006) Let $\mathbf{u} \in \mathcal{D}'(\Omega_-)^3 \wedge \nabla \mathbf{u} \in L^2(\Omega)^{3\times 3}$. Then there exists $\mathbf{u}_\infty \in \mathbb{R}^3$ s.th. $\mathbf{u} - \mathbf{u}_\infty \in \mathcal{H}^1(\Omega_-)^3$,

$$\mathbf{u}_\infty = \tfrac{1}{4} \lim_{|\mathbf{x}|\to\infty} \int_{\mathbb{S}^2} \mathbf{u}\big(\mathbf{y}(\mathbf{x})\big) ds_y,$$

$\mathbf{u} \in L^6(\Omega_-)^3\,,\ \|\mathbf{u} - \mathbf{u}_\infty\|_{L^6(\Omega_-)^3} \leq C \|\nabla \mathbf{u}\|_{L^2(\Omega_-)^3}\,,$

$$\lim_{|\mathbf{x}|\to\infty} \int_{\mathbb{S}^2} |\mathbf{u}\big(\mathbf{y}(\mathbf{x})\big) - \mathbf{u}_\infty| ds_y = 0.$$

$$\mathcal{L}_\alpha(\mathbf{u}_\pm, p_\pm) := \Delta \mathbf{u}_\pm - \alpha \mathbf{u}_\pm - \nabla p_\pm,$$
$$H^1(\Omega_\pm, \mathcal{L}_\alpha) := \{(\mathbf{u}_\pm, p_\pm, \widetilde{f}_\pm) \in H^1(\Omega_\pm)^3 \times L^2(\Omega_\pm) \times \widetilde{H}^{-1}(\Omega_\pm)^3\}.$$

Lemma: Let $\alpha \geq 0$, $\mathcal{L}_\alpha(\mathbf{u}_\pm, p_\pm) = \widetilde{\mathbf{f}}_\pm, \mathrm{div}\ \mathbf{u}_\pm = 0$ in $\Omega_\pm$ and

$$(\mathbf{u}_\pm, p_\pm, \widetilde{\mathbf{f}}_\pm) \in H^1(\Omega_\pm, \mathcal{L}_\alpha).$$

Then Green's formula holds for all $\mathbf{w} \in H^1(\Omega_\pm)^3$:

$$\pm\langle \partial^\pm_{n;\alpha}(\mathbf{u}_\pm, p_\pm, \widetilde{\mathbf{f}}_\pm)\,,\, \gamma_\pm \mathbf{w}\rangle_\Gamma = 2\langle \mathbb{E}(\mathbf{u}_\pm), \mathbb{E}(\mathbf{w})\rangle_{\Omega_\pm} + \alpha\langle \mathbf{u}_\pm, \mathbf{w}_\pm\rangle_{\Omega_\pm} - \langle p_\pm, \mathrm{div}\ \mathbf{w}\rangle_{\Omega_\pm} + \langle \widetilde{\mathbf{f}}_\pm, \mathbf{w}\rangle_{\Omega_\pm}$$

and $\partial^\pm_{n;\alpha}(\mathbf{u}_\pm, p_\pm, \widetilde{\mathbf{f}}_\pm) \in H^{-1/2}(\Gamma)$.

Lemma: Let $\alpha = 0$ and $(\mathbf{u}, p) \in \mathcal{H}^1(\Omega_-, \mathcal{L}_0)$ be a solution of the Stokes system. Then Green's formula holds in Ω_- and $\partial_{n;0}(\mathbf{u}, p, \widetilde{\mathbf{f}_-}) \in H^{-1/2}(\Gamma)$.
If $\widetilde{\mathbf{f}}_- = \mathbf{0}$ then $\mathbf{u}(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-1})\,,\, \nabla\mathbf{u}(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-2})$
and $p(\mathbf{x}) \in \mathcal{O}(|\mathbf{x}|^{-2})$ for $|\mathbf{x}| \to \infty$.

4. Potential operators

Brinkman fundamental solution in $\mathbb{R}^3$:
(for $n = 2$ see e.g. Kohr & Pop 2004)

$$\mathcal{G}^{\alpha}_{jk}(\mathbf{z}) = \frac{1}{4\pi|\mathbf{z}|}\left\{\delta_{jk}A_1(\sqrt{\alpha}|\mathbf{z}|) + \frac{z_j z_k}{|\mathbf{z}|^2}A_2(\sqrt{\alpha}|\mathbf{z}|)\right\},\ P^s_j(\mathbf{z}) = \frac{1}{2\pi}\ \frac{z_j}{|\mathbf{z}|^3},$$

$$A_1(\varrho) = 2e^{-\varrho}(1+\varrho^{-1}+\varrho^{-2}) - 2\varrho^{-2} = 1 - \frac{4}{3}\varrho + \frac{3}{4}\varrho^2 - \dots,$$
$$A_2(\varrho) = -2e^{-\varrho}(1+3\varrho^{-1}+3\varrho^{-2}) + 6\varrho^{-2} = 1 - \frac{1}{4}\varrho^2 - \dots,$$

$$S^{\alpha}_{ijk}(\mathbf{z}) = P^s_j(\mathbf{z})\delta_{ik} + \partial_k\mathcal{G}^{\alpha}_{ij}(\mathbf{z}) + \partial_i\mathcal{G}^{\alpha}_{kj}(\mathbf{z})$$
$$P^{d,\alpha}_{ik}(\mathbf{z}) = \frac{1}{2\pi}\left\{\delta_{ik}(\alpha|\mathbf{z}|^2 - 2)\frac{1}{|\mathbf{z}|^3} + 6\frac{z_i z_k}{|\mathbf{z}|^5}\right\}.$$

Stokes fundamental solution: Set $\alpha = 0$.

Newton potentials:

$$(\mathcal{N}_{\alpha;\Omega_\pm}\boldsymbol{\Psi})_j(\mathbf{x}) := -\int\limits_{\mathbf{y}\in\Omega_\pm} \mathcal{G}^\alpha_{jk}(\mathbf{x}-\mathbf{y})\Psi_k(\mathbf{y})d\mathbf{y}\colon \widetilde{H}^{-1}(\Omega_\pm)^3 \to H^1_{\mathrm{div}}(\Omega_\pm)^3,$$

$$(P^s_{\alpha;\Omega_\pm}\boldsymbol{\Psi})(\mathbf{x}) := -\int\limits_{\mathbf{y}\in\Omega_\pm} P^s_k(\mathbf{x}-\mathbf{y})\Psi_k(\mathbf{y})d\mathbf{y} : \widetilde{H}^{-1}(\Omega_\pm)^3 \to L^2(\Omega_\pm)^3.$$

Surface Potentials: $\boldsymbol{\Psi}(\mathbf{x}) = (\gamma_\pm^{-1}\mathbf{g})(\mathbf{x})\,,\ \mathbf{x}\in\Omega_\pm;$

$$(\mathbf{V}_{\alpha;\Gamma})_j(\mathbf{x}) := -\int\limits_{\mathbf{y}\in\Gamma} \mathcal{G}^\alpha_{jk}(\mathbf{x}-\mathbf{y})g_k(\mathbf{y})ds_{\mathbf{y}} \quad : H^{-1/2}(\Gamma)^3 \to H^1_{\mathrm{div}}(\Omega_\pm)^3,$$

$$(P^s_{\alpha;\Gamma}\mathbf{g})(\mathbf{x}) := -\int\limits_{\mathbf{y}\in\Gamma} P^s_k(\mathbf{x}-\mathbf{y})g_k(\mathbf{y})ds_{\mathbf{y}} \quad : H^{-1/2}(\Gamma)^3 \to L^2(\Omega_\pm)^3.$$

$$(\mathbf{W}_{\alpha;\Gamma})_k(\mathbf{x}) := -\int\limits_{\mathbf{y}\in\Gamma} S^\alpha_{ik\ell}(\mathbf{x}-\mathbf{y})n_\ell(\mathbf{y})h_i(\mathbf{y})ds_{\mathbf{y}} \quad : H^{1/2}(\Gamma)^3 \to H^1_{\mathrm{div}}(\Omega_\pm)^3,$$

$$(P^d_{\alpha;\Gamma}\mathbf{h})(\mathbf{x}) := \int\limits_{\mathbf{y}\in\Gamma} P^{d,\alpha}_{ik}(\mathbf{x}-\mathbf{y})n_k(\mathbf{y})h_i(\mathbf{y})ds_{\mathbf{y}} \quad : H^{1/2}(\Gamma)^3 \to L^2(\Omega_\pm)^3.$$

Boundary integral operators and jump relations at $\mathbf{x} \in \Gamma$:

$$(\gamma_\pm \mathbf{V}_{\alpha;\Gamma}\mathbf{g})_j(\mathbf{x}) =: (\mathcal{V}_{\alpha;\Gamma}\mathbf{g})_j(\mathbf{x}) = \int\limits_{\mathbf{y}\in\Gamma} \mathcal{G}^\alpha_{jk}(\mathbf{x}-\mathbf{y}) g_k(\mathbf{y}) ds_{\mathbf{y}} : \quad H^{-1/2}(\Gamma)^3 \to H^{1/2}(\Gamma)^3,$$

$$(\gamma_\pm \mathbf{W}_{\alpha;\Gamma}\mathbf{h})_j(\mathbf{x}) =: \pm\tfrac{1}{2}\delta_{jk} h_k(\mathbf{x}) + (\mathbf{K}_{\alpha;\Gamma}\mathbf{h})_j(\mathbf{x})$$
$$= \pm\tfrac{1}{2}\delta_{jk} h_k(\mathbf{x}) + \text{p.v.} \int\limits_{\mathbf{x}\neq\mathbf{y}\in\Gamma} S^\alpha_{ijk}(\mathbf{x}-\mathbf{y}) n_k(\mathbf{y}) h_i(\mathbf{y}) ds_{\mathbf{y}} : \quad H^{1/2}(\Gamma)^3 \to H^{1/2}(\Gamma)^3,$$

$$\partial^\pm_{n\alpha;\Gamma}(\mathbf{V}_{\alpha;\Gamma}\mathbf{g}, P^s_{\alpha;\Gamma}\mathbf{g})_j(\mathbf{x}) = \mp\tfrac{1}{2}\delta_{jk} g_k(\mathbf{x}) + (\mathbf{K}^*{}_{\alpha;\Gamma}\mathbf{g})_j(\mathbf{x}) : \quad \big(H^{-1/2}(\Gamma)\big)^3 \to \big(H^{1/2}(\Gamma)\big)^3,$$

$$\partial^\pm_{n\alpha;\Gamma}(\mathbf{W}_{\alpha;\Gamma}\mathbf{h}, P^d_{\alpha;\Gamma}\mathbf{h})_j(\mathbf{x}) = (\mathbf{D}_{\alpha;\Gamma}\mathbf{h})_j(\mathbf{x}) : \big(H^{1/2}(\Gamma)\big)^3 \to \big(H^{-1/2}(\Gamma)\big)^3;$$

Ker $\mathcal{V}_{\alpha;\Gamma} = \mathbb{R}\mathbf{n}$.

5. The linear transmission problem where $\mathbf{k}{=}\beta = 0$

Theorem: The linear problem with $k = \beta = 0$ has for given data

$$\widetilde{f}_\pm \in \widetilde{H}^{-1}(\Omega_\pm)^3\,,\; \mathbf{h}_0 \in H^{1/2}(\Gamma)^3\,,\; \mathbf{g}_0 \in H^{-1/2}(\Gamma)^3\,,\; \mathbf{u}_\infty \in \mathbb{R}^3$$

a unique solution $(\mathbf{u}_+, p_+) \in H^1_{\mathrm{div}}(\Omega_+)^3 \times L^2(\Omega_+)$,
$(\mathbf{u}_-, p_-) \in \mathcal{H}^1_{\mathrm{div}}(\Omega_-)^3 \times L^2(\Omega_-)$ and

$$\{\|\mathbf{u}_+\|_{H^1(\Omega_+)^3} + \|p_+\|_{L^2(\Omega_+)} + \|\mathbf{u}_- - \mathbf{u}_\infty\|_{H^1(\Omega_-)^3} + \|p_-\|_{L^2(\Omega_-)}\}$$
$$\leq C\{\|\widetilde{\mathbf{f}}_+\|_{\widetilde{H}^1(\Omega_+)^3} + \|\widetilde{\mathbf{f}}_-\|_{\widetilde{H}^{-1}(\Omega_-)^3} + \|\mathbf{h}_0\|_{H^{1/2}(\Gamma)^3} + \|\mathbf{g}_0\|_{H^{-1/2}(\Gamma)^3}\}$$

with $C = C(\Omega_+, \mathcal{P}, \alpha, \mu)$, satisfying

$$\lim_{|\mathbf{x}|\to\infty} \int_{\mathbb{S}^2} |\mathbf{u}_-\big(\mathbf{y}(\mathbf{x})\big) - \mathbf{u}_\infty| ds_{\mathbf{y}} = 0$$

and

$(\mathbf{u}_-(\mathbf{x}) - \mathbf{u}_\infty) = \mathcal{O}(|\mathbf{x}|^{-1})\,,\; \nabla\mathbf{u}_-(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-2})\,,\; p_-(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-2})$ for $|\mathbf{x}| \to \infty$.

Sketch of the proof:

1. Ansatz: $\mathbf{u}_\pm = \mathbf{v}_\pm + \mathbf{w}_\pm$, $p_{\mathbf{v},\pm} + \pi_\pm$; $\alpha = 0$ in Ω_-;
where $\mathbf{v}\pm = \mathcal{N}_{\alpha,\Omega_\pm}\widetilde{f}_\pm$, $p_{\mathbf{v},\pm} = P^s_{\alpha;\Omega_\pm}\widetilde{f}_\pm$ satisfying the homogeneous Brinkman system in Ω_+ and Stokes system in Ω_-.

2. $\mathbf{w}_\pm = \mathbf{V}_{\alpha;\Gamma}\boldsymbol{\varphi} + \mathbf{W}_{\alpha;\Gamma}\boldsymbol{\Phi}, \pi_\pm = P^s_{\alpha;\Gamma}\boldsymbol{\varphi} + P^d_{\alpha;\Gamma}\boldsymbol{\Phi}$,

$$\gamma_+\mathbf{w}_+ - \gamma_-\mathbf{w}_- = \mathbf{h}_{00} = \mathbf{h}_0 - \gamma_+\mathbf{v}_+ + \gamma_-\mathbf{v}_-,$$

$$\partial^+_{n;\alpha}(\mathbf{w}_+,\pi_+) - \mu\partial^-_{n;\alpha}(\mathbf{w}_-,\pi_-) + \tfrac{1}{2}\mathcal{P}(\gamma_+\mathbf{w}_+ + \gamma_-\mathbf{w}_-) = \mathbf{g}_{00}$$
$$= \mathbf{g}_0 - \partial^+_{n;\alpha}(\mathbf{v}_+,\pi_+) + \mu\partial_{n;0}(\mathbf{v}_-,\pi_-) - \tfrac{1}{2}\mathcal{P}(\gamma_+\mathbf{v}_+ + \gamma_-\mathbf{v}_-$$

3. Transmission conditions on Γ yield for $\boldsymbol{\Phi} \in H^{1/2}(\Gamma)$, $\boldsymbol{\varphi} \in H^{-1/2}(\Gamma)$ the Fredholm equations $\quad \boldsymbol{\Phi} = K_{\alpha,0;\Gamma}\boldsymbol{\Phi} + \mathcal{V}_{\alpha,0;\Gamma}\boldsymbol{\varphi} - \mathbf{h}_{00}$,

$$\{\tfrac{1}{2}(1+\mu)\mathbf{I} + (1-\mu)K^*_\Gamma\}\boldsymbol{\varphi} = -(1-\mu)D_{0,\Gamma}\boldsymbol{\Phi} - D_{\alpha,0;\Gamma}\boldsymbol{\Phi} - K^*_{\alpha,0,\Gamma}\boldsymbol{\varphi}$$
$$-\tfrac{1}{2}\mathcal{P}\big((\mathcal{V}_{\alpha,\Gamma} + \mathcal{V}_{0,\Gamma})\boldsymbol{\varphi} + (K_{\alpha,\Gamma} + K_{0,\Gamma})\big)\boldsymbol{\Phi} + \mathbf{g}_{00}.$$

The operator $\{\dots\}$ is invertible on $H^{-1/2}(\Gamma)$ and $D_{\alpha,0;\Gamma} := D_{\alpha,\Gamma} - D_{0,\Gamma}$, $K^*_{\alpha,0,\Gamma} := K^*_{\alpha,\Gamma} - K^*_{0,\Gamma} -$ and $\mathcal{P}(\dots)$ are compact.

4, Green's theorem and vanishing energy for $\mathbf{h}_{00} = \mathbf{g}_{00} = \mathbf{0}$ implies uniqueness.

6. The nonlinear Darcy–Forchheimer–Brinkman transmission problem

Lemma: Let $\overset{\circ}{E}_{\pm}(\mathbf{w})(\mathbf{x}) := \begin{cases} \mathbf{w}(\mathbf{x}) & \text{for } \mathbf{x} \in \Omega_{\pm}, \\ \mathbf{0} & \text{for } \mathbf{x} \in \overline{\Omega}_{\mp}. \end{cases}$ Then the nonlinear mappings $\mathcal{I}_{k,\beta,\Omega_+}(\mathbf{v}) := \overset{\circ}{E}_{+}\big(k|\mathbf{v}|\mathbf{v} + \beta(\mathbf{v}\cdot\nabla)\mathbf{v}\big)$ are positively homogeneous of order 2 and for $\mathbf{v} \in H^1_{\mathrm{div}}(\Omega_+)^3$ into $L^{3/2}(\Omega_+)^3$ and $\widetilde{H}^{-1}(\Omega_+)^3$ are bounded:

$$\|\mathcal{I}_{k,\beta,\Omega_+}(\mathbf{v})\|_{L^{3/2}(\Omega_+)^3} \le c_1' \|\mathbf{v}\|^2_{H^1(\Omega_+)^3},$$

$$\|\mathcal{I}_{k,\beta,\Omega_+}(\mathbf{v})\|_{\widetilde{H}^{-1}(\Omega_+)^3} \le c_1 \|\mathbf{v}\|^2_{H^1(\Omega_+)^3}$$

and for $\mathbf{v}, \mathbf{w} \in H^1(\Omega_+)^3$ it holds

$$\|\mathcal{I}_{k,\beta,\Omega_+}(\mathbf{v}) - \mathcal{I}_{k,\beta,\Omega_+}(\mathbf{w})\|_{\widetilde{H}^{-1}(\Omega_+)^3} \le c_1\big(\|\,\mathbf{v}\,\|_{H^1(\Omega_+)^3} + \|\,\mathbf{w}\,\|_{H^1(\Omega_+)^3}\big)\|\mathbf{v} - \mathbf{w}\|_{H^1(\Omega_+)^3}.$$

Theorem: For the nonlinear problem

$$\Delta\mathbf{u} - \alpha\mathbf{u} - k|\mathbf{u}|\mathbf{u} - \beta(\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla p = \mathbf{f} \text{ and } \nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega_+;$$

$$\Delta\mathbf{u} - \nabla p = \mathbf{0} \text{ and } \nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega_-;$$

$$\gamma_+\mathbf{u}_+ - \gamma_-\mathbf{u}_- = \mathbf{h}_0 \quad \text{on } \Gamma = \partial\Omega \text{ and}$$

$$\partial^+_{n;\alpha}\big(\mathbf{u}_+, p_+; (\mathbf{f}_+ + \mathbb{E}_+(k|\mathbf{u}_+|\mathbf{u}_+ + \beta(\mathbf{u}_+\cdot\nabla)\mathbf{u}_+)\big) - \mu\partial^-_n(\mathbf{u}_-, p_-; \mathbf{f}_-) \\ + \tfrac{1}{2}\mathcal{P}(\gamma_-\mathbf{u}_- + \gamma_+\mathbf{u}_+) = \mathbf{g}_0 \quad \text{on } \Gamma;$$

$$\lim_{|\mathbf{x}|\to\infty} \int_{\mathbb{S}^2(\mathbf{x})} |\mathbf{u}_-\big(\mathbf{y}(\mathbf{x})\big) - \mathbf{u}_\infty| ds_y = 0$$

there are two positive constants ζ and η depending on $\Omega_+, \mathcal{P}, \alpha, \mu, k, \beta$ such that for all given data with

$$\|(\widetilde{\mathbf{f}}_+, \widetilde{\mathbf{f}}_-, \mathbf{h}_0, \mathbf{u}_\infty)\|_{X_0} := \\ \big\{\|\widetilde{\mathbf{f}}_+\|_{\widetilde{H}^{-1}(\Omega_+)^3} + \|\widetilde{\mathbf{f}}_-\|_{\widetilde{H}^{-1}(\Omega_+)^3} + \|\mathbf{h}_0\|_{H^{1/2}(\Gamma)^3} + \|\mathbf{g}_0\|_{H^{-1/2}(\Gamma)^3} + |\mathbf{u}_\infty|\big\} \le \zeta,$$

the nonlinear problem has a unique solution satisfying

$$\|(\mathbf{u}_+, p_+, \mathbf{u}_- - \mathbf{u}_\infty, p_-)\|_Y := \\ \big\{\|\mathbf{u}_+\|_{H^1(\Omega_+)^3} + \|p_+\|_{L^2(\Omega_+)} + \|\mathbf{u}_- - \mathbf{u}_\infty\|_{H^1(\Omega_-)^3} + \|p_-\|_{L^2(\Omega_-)}\big\} \le \eta$$

Corollary: The solution satisfies the apriori estimate

$$\|(\mathbf{u}_+, p_+, \mathbf{u}_- - \mathbf{u}_\infty, p_-)\|_Y \leq C\|(\widetilde{\mathbf{f}}_+, \widetilde{\mathbf{f}}_-, \mathbf{h}_0, \mathbf{g}_0, \mathbf{u}_\infty)\|_{X_0}$$

with $C = C(\Omega_+, \mathcal{P}, \alpha, \mu)$. Moreover

$$\lim_{|\mathbf{x}|\to\infty} \int_{\mathbb{S}^2} |\mathbf{u}_-\big(\mathbf{y}(\mathbf{x})\big) - \mathbf{u}_\infty| ds_y = 0.$$

Sketch of the proof:

Let $\mathbf{v} \in H^1(\Omega_+)$ be given and solve for $\mathbf{u}$:

$$\Delta\mathbf{u} - \alpha\mathbf{u} - \nabla p = \widetilde{f}_+ + k|\mathbf{v}|\mathbf{u} + \beta(\mathbf{v}\cdot\nabla)\mathbf{u}\,,\ \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega_+,$$
$$\Delta\mathbf{u} - \nabla p = \mathbf{0}\,,\ \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega_-.$$
$$\gamma_+\mathbf{u}_+ - \gamma_-\mathbf{u}_- = \mathbf{h}_0,$$
$$\partial^+_{n;\alpha}\big(\mathbf{u}_+, p_+; \big(\widetilde{\mathbf{f}}_+ + \mathbb{E}_+(k|\mathbf{v}|\mathbf{u} + \beta(\mathbf{v}\cdot\nabla)\mathbf{u})\big) - \mu\nabla^-_{n;0}(\mathbf{u}_-, p_-, \widetilde{\mathbf{f}}_-) +$$
$$+\mathcal{P}(\gamma_- u_- + \gamma_+ u_+) = g_0 \text{ on } \Gamma,$$
$$\lim_{|\mathbf{x}|\to\infty} \int_{\mathbb{S}^2(\mathbf{x})} |\mathbf{u}_-\big(\mathbf{y}(\mathbf{x})\big) - \mathbf{u}_\infty| ds_y = 0,$$

This defines a mapping $\mathbf{v} \mapsto \mathbf{u} =: \mathcal{U}(\mathbf{v}) : B_\zeta \to B_\zeta$ where $B_\zeta \subset H^1_{\mathrm{div}}(\Omega_+)^3$ is a closed ball of radius ζ. The a priori estimates are uniform with respect to $\mathbf{v}$ and they allow us to find $\zeta > 0$ such that $\mathcal{U}$ becomes in B_ζ a contraction if the given data are small enough. Then the Banach–Cacciapoli theorem yields the unique existence of $\mathbf{u} \in B_\zeta$ with $\mathcal{U}(\mathbf{u}) = \mathbf{u}$.

7. The problem on a compact Riemannian connected boundaryless orientable manifold $\mathcal{M}_{n-1} \subset \mathbb{R}^n$.

In every chart of a finite atlas let for $\boldsymbol{\xi} \in \mathcal{U}_\tau \subset \mathbb{R}^{n-1}$, $\mathbf{x} = \mathbf{x}(\boldsymbol{\xi})$, the basis of the tangent bundle be: $\partial_j = \frac{\partial}{\partial \xi_j}\mathbf{x}$, $j = 1, \dots, n-1$;

fundamental tensor: $g_{jk} = \partial_j \mathbf{x} \cdot \partial_k \mathbf{x}$, $g^{\ell m} g_{mk} = \delta^\ell_k$; $g = \det g_{jk}$.

For $\mathbf{u}(\boldsymbol{\xi}) = u^j \partial_j$ tangential field in the tangent bundle,
div $\mathbf{u} = \frac{1}{\sqrt{g}} \partial_k(\sqrt{g} u^k)$, $(\mathbf{grad}\, p) = (g^{jk} \partial_j p) \partial_k$ and $u_\ell = g_{\ell k} u^k$.

Riemannian or Levi–Civita connection:
$\Gamma^\ell_{jk} := g^{\ell m} (\partial_j \partial_k \mathbf{x}) \cdot \partial_m \mathbf{x}$ Christoffel symbol of 2nd kind,
covariant derivative: $\nabla_{\mathbf{u}} \mathbf{v} := u^j (\partial_j v^k + \Gamma^k_{j\ell} v^\ell) \partial_k$

deformation tensor: $(\mathrm{Def}\, \mathbf{u})_{\ell k} = \frac{1}{2}(\partial_k u_\ell - \Gamma^m_{\ell k} u_m + \partial_\ell u_k - \Gamma^m_{k\ell} u_m)$
adjoint: $(\mathrm{Def}^* \mathbf{A})_k = -(\mathrm{div} \mathbf{A})_k = -g^{\ell t} A_{\ell k; t} = -g^{\ell t} (\partial_t A_{\ell k} - \Gamma^m_{t\ell} A_{mk})$

Assumption: $\mathcal{M}_{n-1}$ does not admit any nontrivial Killing field X, i.e. satisfying Def $X = 0$.

Darcy–Forchheimer–Brinkman system on $\mathcal{M}_{n-1}$:

$$(\mathbf{L}\mathbf{u})_k := 2(\mathrm{Def}^*\mathrm{Def}\,\mathbf{u})_k = -g^{\ell s}\{\partial_s(\partial_k u_\ell - \Gamma^m_{\ell k}u_m + \partial_\ell u_k - \Gamma^m_{k\ell}u_m) \\ -\Gamma^m_{s\ell}(\partial_k u_m - \Gamma^j_{mk}u_j + \partial_m u_k - \Gamma^j_{km}u_j)\}$$

$$\mathbf{L}\mathbf{u} + \mathcal{P}\mathbf{u} + k|\mathbf{u}|\mathbf{u} + \beta\nabla_{\mathbf{u}}\mathbf{u} + \mathbf{grad}\,p = \mathbf{f}, \mathrm{div}\,\mathbf{u} = 0 \quad \text{in } \Omega \subset \mathcal{M}_{n-1}.$$

Let $\Omega_+ \subset \mathcal{M}$ be a Lipschitz domain with exterior normal vector $\mathbf{n}$ on $\Gamma = \partial\Omega_+$ and $\Omega_- := \Omega \setminus \overline{\Omega}_+$ also Lipschitz.

Conormal derivative (hydrodynamic tension) $\mathbf{t}^\pm_{\mathcal{P}}$ and Stokes theorem:

$$\mathbf{t}^\pm_{\mathcal{P}} : H^1(\Omega_\pm) \times L_2(\Omega_\pm) \times \widetilde{H}^{-1}(\overline{\Omega}_\pm) \to H^{-1/2}(\Gamma),$$

$$\pm\int_\Gamma \mathbf{t}^\pm_{\mathcal{P}}(\mathbf{u}, p; \widetilde{\mathbf{f}}_\pm) \cdot \boldsymbol{\varphi} d\sigma_\Gamma = 2\int_{\Omega_\pm} \widetilde{E}_\pm \mathrm{Def}\;\mathbf{u} : \mathrm{Def}(\gamma_\pm^{-1}\boldsymbol{\varphi})ds +$$
$$+\int_{\Omega_\pm} \widetilde{E}_\pm \mathcal{P}\mathbf{u} \cdot \gamma_\pm^{-1}\boldsymbol{\varphi}ds + \int_{\Omega_\pm} \widetilde{E}_\pm p\,\mathrm{div}\,(\gamma_\pm^{-1}\boldsymbol{\varphi})ds - \int_{\Omega_+} \widetilde{\mathbf{f}}_\pm \cdot \gamma_\pm^{-1}\boldsymbol{\varphi}ds.$$

Let λ on $\mathcal{M}_{n-1}$ be a symmetric semidefinit matrix–valued function.

Theorem: The transmission problem on $\mathcal{M}_{n-1}$:

$$\begin{aligned}
&\mathbf{L}\mathbf{u}_+ + \mathcal{P}\mathbf{u} + \mathbf{grad}\, p_+ = \widetilde{\mathbf{f}}_+|_{\Omega_+} - (k|\mathbf{u}_+|\mathbf{u}_+ + \beta\nabla\mathbf{u}_+\mathbf{u}_+),\\
&\qquad\qquad \operatorname{div}\mathbf{u}_+ = 0 \text{ in } \Omega_+,\\
&\quad \mathbf{L}\mathbf{u}_- + \quad \mathbf{grad}\, p_- = \widetilde{\mathbf{f}}_-|_{\Omega_-} - \nabla_{\mathbf{u}_-}\mathbf{u}_-\,, \quad \operatorname{div}\mathbf{u}_- = 0 \quad \text{in } \Omega_-,\\
&\quad \mu\gamma_+\mathbf{u}_+ - \gamma_-\mathbf{u}_- = \mathbf{h} \quad \text{on } \Gamma,\\
&\mathbf{t}^+_{\mathcal{P}}\big(\mathbf{u}_+ . p_+, \widetilde{\mathbf{f}}_+ - \widetilde{E}_+(k|\mathbf{u}_+|\mathbf{u}_+ + \beta\nabla_{\mathbf{u}_+}\mathbf{u}_+)\big)\\
&\qquad\qquad -\mathbf{t}^-_0\big(\mathbf{u}_-, p_-; \widetilde{\mathbf{f}}_- - \widetilde{E}_-(\nabla_{\mathbf{u}_-}\mathbf{u}_)\big) + \lambda\gamma_+\mathbf{u}_+ = \mathbf{g}
\end{aligned}$$

with sufficiently small given data
$(\widetilde{\mathbf{f}}_+, \widetilde{\mathbf{f}}_-, \mathbf{h}, \mathbf{g}) \in \widetilde{H}^{-1}(\Omega_+) \times \widetilde{H}^{-1}(\Omega_-) \times H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$
has a unique solution $(\mathbf{u}_\pm, p_\pm) \in H^1(\Omega_\pm) \times L_2(\Omega_\pm)$.

The proof is based on Newton and boundary potentials in $\Omega_\pm$, and that the Killing fields are zero. Fundamental tensors can be constructed with A. Pomp's Levi functions as approximation of the fundamental tensors.

8. More general spaces, $\mathbf{0 < s < 1}$:

$$H^{s+\frac{1}{2}}_{\mathrm{div}}(\Omega_\pm) := \left\{\mathbf{u} \in H^{s+\frac{1}{2}}(\Omega_\pm) \wedge \operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega_\pm\right\},$$

$$H^{s-\frac{1}{2}}_{*}(\Omega_+) := \left\{p \in H^{s+\frac{1}{2}}(\Omega_+) \wedge \int_{\Omega_+} p ds = 0\right\},$$

$$H^{s}_{\mathbf{n}}(\Gamma) := \left\{\mathbf{v} \in H^{s}(\Gamma) \wedge \int_{\Gamma} \mathbf{v}\cdot\mathbf{n}\, d\sigma = 0\right\},$$

Solutions: $(\mathbf{u}_+, p_+) \in H^{s+\frac{1}{2}}_{\mathrm{div}}(\Omega_+) \times H^{s-\frac{1}{2}}_{*}(\Omega_+)$,

$(\mathbf{u}_-, p_-) \in H^{s+\frac{1}{2}}_{\mathrm{div}}(\Omega_-) \times H^{s-\frac{1}{2}}(\Omega_-)$;

Given data:

$(\widetilde{\mathbf{f}}_+, \widetilde{\mathbf{f}}_-, \mathbf{h}, \mathbf{g}) \in \widetilde{H}^{s-3/2}(\Omega_+) \times \widetilde{H}^{s-3/2}(\Omega_-) \times H^{s}_{\mathbf{n}}(\Gamma) \times H^{s-1}(\Gamma)$.

References

[1] Adams, R.A.: Sobolev Spaces. Academic Press, New York 1975.

[2] Agranovich, M.S.: Sobolev Spaces, their Generalizations and Elliptic Problems in Smooth and Lipschitz Domains (Russian, Moscow, Izdat. MZEMO 2013) Springer Cham, Heidelberg 2015.

[3] Alliot, F., Amrouche, C.: *Weak solutions for the exterior Stokes problem in weighted Sobolev spaces*, Math. Methods Appl. Sci. **23** (2000) 575–600.

[4] Amrouche, C., Meslameni, M., Neăsová, Š: *The stanionary Oseen equations in an exterior domain: An approach in weighted Sobolev spaces*, J. Differential Equations **256** (2014) 1955–1986.

[5] Aubin, T.: Some Nonlinear Problems in Riemannian Geometry. Springer–Verlag, Berlin, Heidelberg 1998.

[6] Behzadan, A., Holst, M.: *Multiplication in Sobolev spaces, revisited.* arXiv:1512.07379v1 [math.AP] (2015).

[7] Buffa, A., Costabel, M., Sheen, D.: *On traces for* H (curl Ω) *in Lipschitz domains.* J. Mathematical Anal. Appl. **276** (2002) 845–867.

[8] Chkadua, O., Mikhailov, S.E., Natroshvili, D.: *Localized boundary–domain singular integral equations based on harmonic parametrix for divergence–form elliptic PDEs with variable matrix coefficients.* Integr. Eqn. Oper. Theory **76** (2013) 509–547.

[9] Costabel, M.: *Boundary integral operators on Lipschitz domains: Elementary results.* SIAM J. Math. Anal **19** (1988) 613–626.

[10] Dindoš, M., Mitrea, M.: *The stationary Navier–Stokes system in nonsmooth manifolds: The Poisson problem in Lipschitz and C^1 domains.* Arch. Ration. Mech. Anal. **174** (2004) 1–47.

[11] Fabes, E., Mendes, O.,Mitrea, M.: *Boundary layers on Sobolev–Besov spaces and Poisson's equation for the Laplacian on Lipschitz domains.* J. Funct. Anal. **159** (1998) 323–369.

[12] Gagliardo, E.: *Caratterizzazione delle trace sulla frontiera relative ad alcune classi di funzione in n variabili.* Rendiconti Sem. Mat. della Università di Padova **27** (1975) 284–305.

[13] Galdi, G.P.: An Introduction to the Mathematical Theory of the Navier–Stokes Equations I, Linearised Steady Problems. Springer–Verlag, Berlin 1998.

[14] Gutt, G., Kohr, M., Pintea, C., Wendland, W.L.: *On the transmission problems for the Oseen and Brinkman system on Lipschitz domains in compact Riemannian manifolds.* DOI 10.1002/mana.201400365 (2014).

[15] Han, Houde: *Boundary integro–differential equations of elliptic boundary value problems and their numerical solution.* Sci. Sin. Ser. **A3** (1988) 1153–1165.

[16] Hsiao, G.C., Wendland, W.L.: *A finite element method for an integral equation of the first kind.* J. Math. Anal. Appl. **58** (1977) 449–491.

[17] Hsiao, G.C., Wendland, W.L.: Boundary Integral Equations. Springer–Verlag Berlin 2008.

[18] Kang, Feng: *Finite element method and natural boundary reduction.* Proc. Internat. Congress of Mathematicians. Warszawa 1983, 1439–1453.

[19] Kohr, M, Lanza di Cristoforis, C., Wendland, W.L.: Nonlinear Neumann–transmission problem for Stokes and Brinkman equations on Euclidian Lipschitz domains. Potential Anal. **38** (2013) 1123–1171.

[20] Kohr, M., Lanza de Cristoforis, M., Wendland, W.L.: *Boundary value problems of Robin type for the Brinkman and Darcy–Forchheimer–Brinkman systems in Lipschitz domains.* J. Math. Fluid Mech. **16** (2014) 595–630.

[21] Kohr, M., Lanza de Cristoforis, M., Wendland, W.L.: *On the Robin–transmission boundary value problem for the nonlinear Darcy–Forchheimer–Brinkman and Navier–Stokes system.* J. Math. Fluid Mech, **18** (2016), 293–329.

[22] Kohr, M., Mikhailov, S.E., Wendland, W.L.: Transmission problems for the Navier–Stokes and Darcy–Forchheimer–Brinkman systems in Lipschitz domains on compact Riemannian manifolds. J.Math. Fluid Mech. DOI 10.1007/s 00021-016-0273-6.

[23] Kohr, M., Pop, L.: Viscous Incompressible Flow for Low Reynold Numbers. WIT Press, Southampton 2004.

[24] Krutitskii, P.A, Medkova, D.: Neumann and Robin problems in a cracked domain with jump conditions on cracks. J. Math. Anal. Appl. **301** (2005) 99–114.

[25] Ladyzhenskaja, O.A.: The Mathematical Theory of Viscous Incompressible Flow. Gordon and Breach, New York 1963.

[26] Mc Lean, W.: Strongly Elliptic Systems and Boundary Integral Equations. Univ. Press, Cambridge 2000.

[27] Medkova, D.: L^q–solution of the Robin problem for the Oseen system. Preprint MU AU ČRI M-2014-34, Praha 2014.

[28] Mikhailov, S.: *About traces, extensions and co–normal derivative operators on Lipschitz domains.* In: Integral Methods in Science and Engineering. (C. Constanda et al. eds.) Birkhäuser, Boston (2007) 151–162.

[29] Mitrea, M., Monniaux, S., Wright, M.: *The Stokes operator with Neumann boundary conditions in Lipschitz domains* J. Math. Sci. **176** (2011) 409–457.

[30] Mitrea, M., Wright, M.: Boundary Value Problems for the Stokes System in Arbitrary Lipschitz Domains. Astérisque **344**, 2012.

[31] Nedelec, J.C., Planchard, J.: *Une Méthode variationelle d'elements pour la résolution numérique d'un problème extérieur dans* $\mathbb{R}^3$. RAIRO, Anal. Numer. **7** (1973) 105–129.

[32] Nield, D.A., Bejan, A.: Convection in Porous Media. 4th ed. Springer, New York 2013.

[33] Pomp, A.: The Boundary–Domain Integral Method for Elliptic Systems with an Application to Shells. Lecture Notes in Mathematics **1683**, Springer Verlag, Berlin 1998.

[34] Runst, T., Sickel, W.: Sobolev Spaces of Fractional Order, Nemytskij Operators and Nonlinear Partial Differential Equations. De Gruyter, Berlin, 1996.

[35] Stephan, E.P. Wendland, W.L.: A hypersingular boundary integral method for two–dimensional screen and crack problems. Arch. Rational Mech. Anal. **112** (1990) 363–390.

[36] Tartaglione, A.: *On the Stokes problem with slip boundary conditions,* Comm. Applied and Ind. Mathematics **1** (2010) 186–205.

[37] Varnhorn, W.: The Stokes Equation. Akademie–Verlag, Berlin 1994.

[38] Wendland, W.L., Zhu, J.: The boundary element method for three-dimensional Stokes flows exterior to an open surface. Math. Comput. Modelling **15** (1991) 19–41.

[39] Zeidler, E.: Nonlinear Functional Analysis and its Applications I, Fixed Point Theorems. Springer–Verlag, Berlin 1986.

Thank you for your attention!