Inverse point source location for the Helmholtz equation

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Contents

The time-periodic wave equation

We consider the propagation of acoustic waves in a homogeneous medium

$$
\begin{aligned}
\partial_{tt} p(t) - \Delta p(t) &= f(t) \quad \text{in } \Omega,\\
\partial_t p(t) + \partial_\nu p(t) &= 0 \quad \text{on } \Gamma_{\mathrm{art}},\\
\partial_\nu p(t) &= 0 \quad \text{on } \Gamma_{\mathrm{D}}.
\end{aligned} \tag{1}
$$

$\partial\Omega = \Gamma_{\mathrm{art}} \cup \Gamma_{\mathrm{D}}$ and a source term f given by

$$f(t) = \hat{u}(t)\delta_{\hat{x}} \quad \hat{x} \in \Omega_c \subsetneq \Omega$$

and a time periodic signal

$$\hat{u} = \sum_{n=1}^{N} a_n \sin(-\omega_n t + \varphi_n)) = \sum_{n=1}^{N} \mathrm{Re}(\hat{u}_n \exp(-i\omega_n t)),$$

with frequencies ω_n, phaseshifts φ_n, amplitudes a_n and $\hat{u}_n = a_n \exp(i\varphi_n) \in \mathbb{C}$.

The Helmholtz equation

By a Fourier transform in t the solution p to (1) is given by

$$p(t) = \sum_{n=1}^{N} \mathrm{Re}(p_n \exp(-i\omega_n t)).$$

where the functions p_n are the solutions to the Helmholtz equation

$$\begin{aligned} -\Delta p_n - \omega_n^2 p_n &= \hat{u}_n \delta_{\hat{x}} \quad \text{in } \Omega, \\ \partial_\nu p_n - i\omega_n p_n &= 0 \quad \text{on } \Gamma_{\mathrm{art}}, \\ \partial_\nu p_n &= 0 \quad \text{on } \Gamma_{\mathrm{D}}, \end{aligned} \tag{2}$$

The inverse problem

Problem statement

Given some measurements of the acoustic pressure at the points $\{ x_k \}_{k=1,\dots,K}$, find the location $\hat{x} \in \Omega_c$ and coefficients $u_n \in \mathbb{C}^N$.

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- nonconvex optimization problem formulation, see Bermudez et. al 2004

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\begin{aligned}
\text{(3)} \qquad & \min_{x \in \Omega_c, \hat{u} \in \mathbb{C}^N} \quad \frac{1}{2} \sum_{k=1}^{K} |\vec{p}(x_k) - p_d^k|_{\mathbb{C}^N}^2 + \alpha |\hat{u}|_{\mathbb{C}^N}, \\
& \text{subject to} \quad -\Delta \vec{p} - \vec{\omega}^2 \vec{p} = \hat{u} \delta_x, \quad (+\text{BC})
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$$\text{(3)} \qquad \min_{x\in\Omega_c,\hat{u}\in\mathbb{C}^N} \quad \frac{1}{2}\sum_{k=1}^{K}|\vec{p}(x_k) - p_d^k|^2_{\mathbb{C}^N} + \alpha|\hat{u}|_{\mathbb{C}^N},$$
$$\text{subject to} \quad -\Delta\vec{p} - \vec{\omega}^2\vec{p} = \hat{u}\delta_x, \quad (+\text{BC})$$

- convex(ified) optimization problem formulation, see Bredies/Pikkarainen 2013

$$\text{(4)} \qquad \min_{u\in\mathcal{M}(\Omega_c,\mathbb{C}^N)} \quad \frac{1}{2}\sum_{k=1}^{K}|\vec{p}(x_k) - p_d^k|^2_{\mathbb{C}^N} + \alpha\|u\|_{\mathcal{M}(\Omega_c,\mathbb{C}^N)},$$
$$\text{subject to} \quad -\Delta\vec{p} - \vec{\omega}^2\vec{p} = u. \quad (+\text{BC})$$

Inverse problem

Consider

$$\text{(5)} \qquad \min_{u \in \mathcal{M}(\Omega_c, \mathbb{C}^N)} J(p,u) := \frac{1}{2} \sum_{k=1}^{K} |p(x_k) - p_d^k|_{\mathbb{C}^N}^2 + \alpha \|u\|_{\mathcal{M}(\Omega_c,\mathbb{C}^N)},$$

$$\text{(6)} \qquad \text{subject to} \quad \begin{cases} -\Delta p_n - \omega_n^2 p_n = u_n|_\Omega, & x \in \Omega, \\ \partial_\nu p_n - i\omega_n p_n = 0, & x \in \Gamma_{art}, \\ \partial_\nu p_n = u_n|_{\Gamma_D}, & x \in \Gamma_D, \end{cases} \quad n = 1, 2, \ldots, N.$$

- $\Omega \subset \mathbb{R}^d$, $d \leq 3$, convex polygonal/polyhedral, $\Omega_c \subset \bar{\Omega}$ compact.
- $\Omega_c \cap \Gamma_{art} = \emptyset$.
- $\Gamma_D = \cup_i \overline{\Gamma_i}$, $\overline{\Gamma_i}$ plane face of Γ, $\Gamma_{art} = \partial\Omega \setminus \Gamma_D$.
- $\{x_k\}_{k=1}^K \subset \Omega \setminus \Omega_c$.

Wellposedness of State equation

Consider the following very weak formulation of (6)

(7) $$-(p, \Delta\varphi + \omega^2\varphi)_\Omega = \langle u, \chi_{\Omega_c}\varphi\rangle \quad \forall \varphi \in H^2(\Omega) \text{ with } \partial_\nu\varphi + \chi_{\Gamma_{art}} i\omega\varphi = 0 \text{ on}$$

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Existence of very weak solutions

For any $u \in \mathcal{M}(\Omega_c)$, there exists a unique very weak solution $p \in L^2(\Omega)$ to (6) and there holds

$$||p||_{L^2(\Omega)} \leq C||u||_{\mathcal{M}(\Omega_c)}$$

for some $C > 0$.

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Additional regularity

There holds $p \in W^{1,s}(\Omega) \cap \mathcal{C}(\Omega \setminus \mathcal{N}_\varepsilon(\Omega_c))$ for any $s < d/(d-1)$, $\varepsilon > 0$ where

$$\mathcal{N}_\varepsilon(\Omega_c) = \{\, x \in \Omega \mid \operatorname{dist}(x, \Omega_c) \leq \varepsilon \,\}\,.$$

Existence of optimal solutions

Control-to-observation operator

The operator S given by

$$S : \mathcal{M}(\Omega_c, \mathbb{C}^N) \to (\mathbb{C}^N)^K \quad S(u) = (p(x_1), p(x_2), \dots, p(x_K)).$$

is weak*-to-strong continuous. Moreover it is the dual of $S^* : (\mathbb{C}^N)^K \to \mathcal{C}(\Omega_c, \mathbb{C}^N)$ with $S^* q = \xi|_{\Omega_c}$, $\xi = (\xi_1, \dots, \xi_N)$ and

$$(8) \qquad \begin{cases} -\Delta \xi_n - \omega_n \xi_n = \sum_{k=1}^K q_{n,k} \delta_{x_k}, & x \in \Omega, \\ \partial_\nu \xi_n + i\omega_n \xi_n = 0, & x \in \Gamma_{art}, \qquad n \in \{1, 2, \dots, N\}, \\ \partial_\nu \xi_n = 0, & x \in \Gamma_D, \end{cases}$$

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Existence of optimal solutions

The optimal control problem (5) has an optimal solution $\widehat{u}$.

Existence of finitely supported solutions

Bound on the number of support points

There exists an optimal solution $\widehat{u}$ to (5) which consists of $N_d \leq 2NK$ point sources,

$$\widehat{u} = \sum_{j=1}^{N_d} \widehat{u}_j \delta_{\widehat{x}_j} \quad \text{where } \widehat{u}_j \in \mathbb{C}^N,\ \widehat{x}_j \in \Omega_c.$$

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$$U_{p_d,\alpha} = \overline{\operatorname{conv}\{\, u \in U_{p_d,\alpha} \mid u \text{ extremal } \}}^{\text{weak}-*}.$$

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- Show

$$\{\, u \in U_{p_d,\alpha} \mid u \text{ extremal } \} \subset \left\{ \sum_{j=1}^{2NK} u_j \delta_{x_j} \,\middle|\, u_j \in \mathbb{C}^N,\ x_j \in \Omega_c \right\}.$$

Optimality condition

Support condition

A measure $\widehat{u} \in \mathcal{M}(\Omega_c, \mathbb{C}^N)$ is a solution to (5) if and only if the adjoint state $\widehat{\xi} = S^*(S\widehat{u} - p_d)$ satisfies $\|\widehat{\xi}\|_{\mathcal{C}(\Omega_c, \mathbb{C}^N)} \leq \alpha$ and the polar decomposition $\mathrm{d}\widehat{u} = \widehat{u}' \,\mathrm{d}|\widehat{u}|$, with $\widehat{u}' \in L^1(\Omega_c, |\widehat{u}|, \mathbb{C}^N)$, satisfies

$$\alpha \widehat{u}' = -\widehat{\xi} \qquad |\widehat{u}|\text{-almost everywhere.}$$

Thereby, $\operatorname{supp} |\widehat{u}| \subset \{x \in \Omega_c : |\widehat{\xi}(x)|_{\mathbb{C}^N} = \alpha\}$ for each solution $\widehat{u}$.

Numerical solution, see Bredies/Pikkarainen 2013

1. Set $u^1 = 0$, i=1, $M = \frac{\sum_{k=1}^{K} |p_d^k|^2}{2\alpha}$, $\varphi(t) = \begin{cases} \beta t, & t \leq M \\ \frac{\beta}{2M}(t^2 + M^2), & else \end{cases}$.

while $s_i \neq 0$ **do**

2. Calculate $\xi^i = S^*(Su_k - p_d)$. Determine $x^i \in \Omega_c$ with $\langle -\xi^i, \delta_{x^i} \rangle = \|\xi^i\|_{\mathcal{C}(\Omega_c, \mathbb{C}^N)}$
3. Set $\theta^i = \begin{cases} 0, & \|\xi^i\|_{\mathcal{C}(\Omega_c,\mathbb{C}^N)} \leq \alpha \\ -\frac{M}{\beta}\xi^i(x_k), & else \end{cases}$
4. Set $s_i = \min\left\{1, \frac{\alpha\|u_k\|_{\mathcal{M}(\Omega_c,\mathbb{C}^N)} - \varphi(\theta^i) + \langle u^i - \theta^i \delta_{x^i}, \xi^i \rangle}{\|S(u^i - \theta^i \delta_{x^i})\|^2_{\mathbb{C}^{NK}}}\right\}$ and

$u_{i+\frac{1}{2}} = (1 - s_i)u^i + s_i\theta^i\delta_{x^i} = \sum_{k=1}^{N_d^{i+\frac{1}{2}}} \tilde{u}_k^{i+\frac{1}{2}} \delta_{x_k^{i+\frac{1}{2}}}$.

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5. Postprocess $u_{k+\frac{1}{2}}$ to get $u_{k+\frac{3}{4}} = \sum_{k=1}^{N_d^{i+\frac{3}{4}}} \tilde{u}_k^{i+\frac{3}{4}} \delta_{x_k^{i+\frac{3}{4}}}$, $N_d^{i+\frac{3}{4}} \leq 2NK$

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6. Get u_{k+1} by solving

$$\min_{\tilde{u}} J(S(u_{k+\frac{3}{4}}(\tilde{u})), u_{k+\frac{3}{4}}(\tilde{u})) \; s.t. \; u_{k+\frac{3}{4}}(\tilde{u}) = \sum_{k=1}^{N_d^{i+\frac{3}{4}}} \tilde{u}_k \delta_{x_k^{i+\frac{3}{4}}}$$

Postprocessing

1. Let $u = \sum_{k=1}^{N_d} \mathbf{u}_k \delta_{x_k}$, $N_d = 2NK + 1$

2. Set
$$u_k = u|_{\{x_k\}} = \mathbf{u}_k \delta_{x_k}, \text{ and } w_k = S(\mathbf{v}_k \delta_{x_k}), \text{ where } \mathbf{v}_k = \frac{\mathbf{u}_k}{\|\mathbf{u}_k\|}.$$

3. Find nontrivial solution $\lambda \in \mathbb{R}^{N_d}$ of $\sum_{k=1,\dots,N_d} \lambda_k w_k = 0$ with $\sum_{k=1,\dots,N_d} \lambda_k \geq 0$.

4. Set $u_{new} = \sum_{k=1}^{N_d} \left(1 - \frac{\lambda_k}{\tau\|\mathbf{u}_k\|}\right) \mathbf{u}_k \delta_{x_k}$ with $\tau = \max_{k=1,\dots,N_d} \frac{\lambda_k}{\|\mathbf{u}_k\|} \geq 0$.

5. $\# supp\ u_{new} \leq N_d - 1$, $J(S(u_{new}), u_{new}) \leq J(S(u), u)$.

Convergence

Sublinear Convergence (Bredies13)

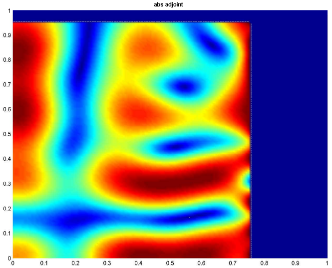
Let the sequence $\{u_i\}_{i\in\mathbb{N}}$ be generated by the generalized conditional gradient method. Then $\{u_i\}_{i\in\mathbb{N}}$ is a minimizing sequence for J, every u_i has at most $2NK$ support points and every subsequence of $\{u_i\}_{i\in\mathbb{N}}$ has a weak* convergent subsequence that converges to a minimizer $\widehat{u}$, $\#\operatorname{supp}\ \widehat{u} \leq 2NK$ of J. Furthermore

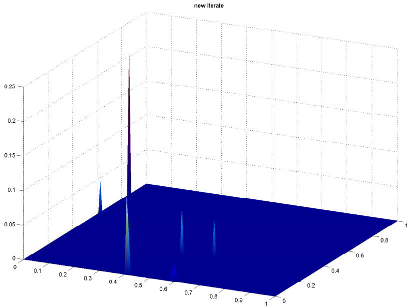
$$J(S(u_i), u_i) - J(S(\widehat{u}), \widehat{u}) \leq \frac{c}{i}$$

for a positive, i-independent constant c

Numerical Example

- Consider the unit square Ω, $\Omega_c = \{\, x \in \Omega \mid x_1 \leq 0.75,\ \ x_2 \leq 0.95 \,\}$, $x_k = (0.875, \frac{k}{16}),\ \ k = 1, \ldots, 15,\ \omega_1 = 13.$
- Try to recover a single Dirac located in $\hat{x} = (0.125, 0.75)$, $u_1 = 0.538 + 0.867i$
- Solve (5) for $\alpha_k = (0.6)^k$, $k = 3, \ldots, 33$





But...

Set $w_\Omega^n = \frac{1}{K}\sum_{k=1}^{K} |G_n^{x_k}|$, and consider as admissible set

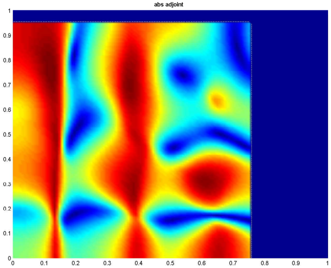
$$\mathcal{M}_w(\Omega_c, \mathbb{C}^N) = \left\{ u \in \mathcal{M}(\Omega_c, \mathbb{C}^N) \;\middle|\; \int_{\Omega_c} |w_\Omega u'| \,\mathrm{d}|u| < \infty \right\}.$$

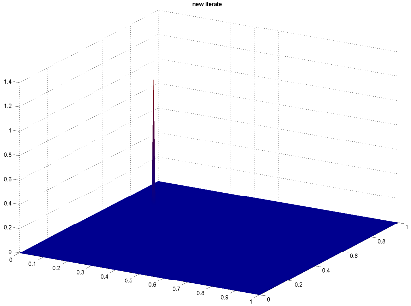
and the corresponding total variation norm

$$\|u\|_{\mathcal{M}_w(\Omega_c,\mathbb{C}^N)} = \int_{\Omega_c} |wu'| \,\mathrm{d}|u| = \int_{\Omega_c} \sqrt{\sum_{n=1}^{N} (w^n(x)|u_n'(x)|)^2} \,\mathrm{d}|u|(x)$$

And it works!

- Still a lot to do here, when should you use which weight, does it work in general?
- Maybe get rid of assumption that $\Omega_c \subsetneq \Omega$.





Last but not least...

Thank you for your attention!