On Evolutionary Inclusions

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5.7.2016

9th Workshop on Analysis and Advanced Numerical Methods for Partial Differential Equations (not only) for Junior Scientists
(AANMPDE(JS)-9-16)
July 4-8, 2016, Strobl, Austria

On evolutionary inclusions

Ref.: An alternative approach to diff. inclusions in H-spaces, NA, 75(15), 5851-5865, 2012.

1 Motivation

Consider Maxwell's equations in gases; $\Omega \subseteq \mathbb{R}^3$:

$$\begin{aligned} \partial_0 \varepsilon E + J - \operatorname{curl} H &= f \\ \partial_0 \mu H + \operatorname{curl} E &= 0 \quad \text{in } \Omega \\ E \times n &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

J is related with E by:

$$\left.\begin{aligned} |E| < E_0 &\implies J = \sigma E \\ |E| = E_0 &\implies \exists \lambda \geq 0 : J = (\sigma + \lambda) E \end{aligned}\right\} (*)$$

where $E_0 > 0$ is the threshold of ionization,

$\varepsilon, \mu, \sigma \in L_\infty(\Omega)$.

Define:

$$h := \{ (u,v) \in L_2(\Omega)^3 \times L_2(\Omega)^3 \mid |u| \leq E_0, \ \operatorname{Re}\langle u | v \rangle = E_0 |v| \}.$$

Then (E, J) satisfy $(*)$ $\iff$ $(E, J - \sigma E) \in h$.

("$\Rightarrow$": $J = \sigma E \to J - \sigma E = 0 \implies 0 = \operatorname{Re}\langle E | J - \sigma E \rangle = E_0 | J - \sigma E | = 0.$

$J = (\sigma + \lambda) E, \ |E| = E_0 \implies \operatorname{Re}\langle E | J - \sigma E \rangle = \lambda |E|^2 \overset{!}{=} E_0 |J - \sigma E| = |E|^2 \lambda.$

"$\Leftarrow$": $|E| < E_0 \to \operatorname{Re} E_0 |J - \sigma E| = \operatorname{Re}\langle E | J - \sigma E \rangle \leq \underbrace{|E|}_{< E_0} |J - \sigma E| \to J = \sigma E.$

$|E| = E_0 \to \operatorname{Re}\langle E | J - \sigma E \rangle = E_0 | J - \sigma E| = |E| |J - \sigma E|$

$\to \exists \lambda \in \mathbb{R}: \lambda E = J - \sigma E. \implies J = (\sigma + \lambda) E.$

Hence: $|\lambda| |E|^2 = |E| |J - \sigma E| = \operatorname{Re}\langle E | \lambda E \rangle = \lambda |E|^2$

$$\rightsquigarrow \partial_0 \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} + \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl}_0 & 0 \end{pmatrix} \ni \left(\begin{pmatrix} E \\ H \end{pmatrix}, \begin{pmatrix} f \\ 0 \end{pmatrix} \right) \quad \to \lambda = |\lambda| \to \lambda \geq 0 \ )$$

Lemma 1: $h \subseteq L_2(\Omega)^3 \times L_2(\Omega)^3$ is monotone, i.e.

$$\forall (x,y),(u,v) \in h: \quad \mathrm{Re}\langle u-x | v-y\rangle \geq 0.$$

proof:
$$\begin{aligned}\mathrm{Re}\langle u-x|v-y\rangle &= E_0(|v|+|y|) - (\mathrm{Re}\langle u|y\rangle + \mathrm{Re}\langle x|v\rangle) \\ &\geq E_0(|v|+|y|) - (\underbrace{|u|}_{\leq E_0}|y| + \underbrace{|x|}_{\leq E_0}|v|) \\ &\geq 0.\end{aligned}$$

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Lemma 2: $\forall f \in L_2(\Omega) \ \exists (u,v) \in h: \quad u+v=f. \quad ((1+h)(u) = f)$.

proof: Let $f \in L_2(\Omega)$.

a) $|f| \leq E_0 \Rightarrow u = f, \ y = 0$

b) $|f| > E_0 \Rightarrow u = E_0 \frac{f}{|f|}, \quad v := (1 - E_0 \frac{1}{|f|})f$

$$\begin{aligned}\Rightarrow |u| = E_0, \ \mathrm{Re}\langle u|v\rangle &= \frac{E_0}{|f|}\langle f | f - E_0 \frac{f}{|f|}\rangle \\ &= E_0|f| - E_0^2 \\ &= E_0(|f| - E_0) \\ &= E_0 |(1-E_0\frac{1}{|f|})||f| = E_0|v|\end{aligned}$$

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Such relations are called maximal monotone.

(2) Maximal monotone relations

Let H Hilbert space.

Def.: $A \subseteq H \times H$ is called <u>monotone</u>, if $\forall (u,v),(x,y) \in A: \ \mathrm{Re}\langle u-x|v-y\rangle \geq 0$.

A is called <u>maximal monotone</u>, if. A monotone and

$$\forall B \subseteq H\times H; \ A \subseteq B, \ B \text{ monotone} \Rightarrow A = B.$$

Thm. (Minty '61)

Let $A \subseteq H \times H$ monotone. Then

$$\begin{aligned}A \text{ max. monotone} &\Longleftrightarrow \exists \lambda > 0: \forall f \in H \ \exists (u,v) \in A: \quad u + \lambda v = f. \\ &\Longleftrightarrow \forall \lambda > 0 \ \ \text{— — — —}\end{aligned}$$

Remark: If A max. mon. $\Rightarrow (\lambda + A)^{-1}: H \to H$ Lipschitz with.

$$|(\lambda + A)^{-1}|_{Lip} \underset{CSI}{\leq} \frac{1}{\lambda} \qquad (\lambda > 0).$$

Example:

a) Let $\varrho > 0$, $H_\varrho(\mathbb{R}) := \{ f : \mathbb{R} \to \mathbb{C} \mid f \text{ meas.}, \int_{\mathbb{R}} |f(t)|^2 e^{-2\varrho t} dt < \infty \}$.

Consider

$$\partial_{0,c} : C_c^\infty(\mathbb{R}) \subseteq H_\varrho(\mathbb{R}) \longrightarrow H_\varrho(\mathbb{R})$$
$$\varphi \longmapsto \varphi'$$

$$\Rightarrow \quad \langle \partial_{0,c}\varphi | \psi \rangle_\varrho = \int_{\mathbb{R}} \varphi'(t)^* \psi(t)\, e^{-2\varrho t} dt$$
$$= -\int_{\mathbb{R}} \varphi(t) (\psi'(t) - 2\varrho \psi(t))\, e^{-2\varrho t} dt$$
$$= \langle \varphi | -\partial_0 \psi + 2\varrho \psi \rangle_\varrho$$

$\Rightarrow + \partial_{0,c} \subseteq -\partial_{0,c}^* + 2\varrho \Rightarrow \partial_{0,c}$ closable ; $\overline{\partial_{0,c}} =: \partial_0$.

and:

$$\operatorname{Re} \langle \partial_0 \varphi | \varphi \rangle_\varrho = \frac{1}{2} \left(\langle \partial_0 \varphi | \varphi \rangle_\varrho + \langle \varphi | \partial_0 \varphi \rangle_\varrho \right)$$
$$= \varrho |\varphi|_\varrho^2 \geq 0.$$

~~Moreover~~ $\to \partial_0 - \varrho$ monotone

Moreover: ∂_0 cont. invertible $\Rightarrow$

$\varrho + (\partial_0 - \varrho)$ onto

$\to \partial_0 - \varrho$ max. monotone.

b) $A : D(A) \subseteq H \to H$ skew-selfadjoint

$\Longrightarrow$ A max. mon. (since $1 + A$ onto and $\operatorname{Re} \langle Ax | x \rangle = 0 \quad (x \in D(A))$.)

3 Solution Theory:

Setting: Let $M_0, M_1 \in L(H)$, $A \subseteq H \times H$ max. mon.

Consider $(u, f) \in \overline{\partial_0 M_0 + M_1 + A}$ for given $f \in H_\varrho(\mathbb{R}; H)$.

AIM: Show that there is $c>0$, such that

$\overline{\partial_0 M_0 + M_1 + A} - c$ is max. monotone in $H_\varrho(\mathbb{R};H)$.

$\rightsquigarrow$ $(\partial_0 M_0 + M_1 + A)^{-1} = \left(c + (\partial_0 M_0 + M_1 + A - c)\right)^{-1}$ Lipschitz-cont. mapping

$\Longrightarrow$ well-posedness.

Idea: Show $\underline{\partial_0 M_0 + M_1 - c \text{ max. mon}}$ + pert. results.

Lemma: Let $M_0, M_1 \in L(H)$, M_0 s.a., s.th.

$$\exists c>0, \varrho_0>0\ \forall \varrho \geq \varrho_0 : \operatorname{Re}\langle(\varrho M_0 + M_1)x|x\rangle \geq c|x|^2.$$

$\Longrightarrow$ $\partial_0 M_0 + M_1 - c$ max. mon. in $H_\varrho(\mathbb{R};H)$ for each $\varrho \geq \varrho_0$.

proof: Observe

$$\begin{aligned}\operatorname{Re}\langle(\partial_0 M_0 + M_1)x|x\rangle &= \operatorname{Re}\langle \partial_0 M_0 x| \partial_0^* x\rangle + \operatorname{Re}\langle M_1 x|x\rangle \\ &= \operatorname{Re}\langle M_0 x|(-\partial_0 + 2\varrho)x\rangle + \operatorname{Re}\langle M_1 x|x\rangle \\ &= -\operatorname{Re}\langle x|\partial_0 M_0 x\rangle + \operatorname{Re}\langle(\varrho M_0 + M_1)x|x\rangle \\ &\quad - \operatorname{Re}\langle x|M_1 x\rangle\end{aligned}$$

$\Longrightarrow$ $\operatorname{Re}\langle(\partial_0 M_0 + M_1)x|x\rangle = \operatorname{Re}\langle(\varrho M_0 + M_1)x|x\rangle \geq c|x|^2$.

$\Longrightarrow$ $\partial_0 M_0 + M_1 - c$ monotone.

Same for $(\partial_0 M_0 + M_1)^*$ $\rightsquigarrow$ $\partial_0 M_0 + M_1$ onto. //

Thm: Let $M_0, M_1 \in L(H)$, M_0 s.a., s.th.

$$\exists c>0, \varrho_0 \geq 0\ \forall \varrho \geq \varrho_0 : \operatorname{Re}\langle(\varrho M_0 + M_1)x|x\rangle \geq c|x|^2.$$

Moreover let $A \subseteq H \times H$ max. mon with $(0,0) \in A$.

$\Longrightarrow$ $\overline{(\partial_0 M_0 + M_1 + A)} - c$ is max. mon. in $H_\varrho(\mathbb{R};H)$ for each $\varrho \geq \varrho_0$

Moreover $(\partial_0 M_0 + M_1 + A)^{-1}$ is causal.