Numerical Methods for p-Power Law Diffusion Problems

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Outline

References:

Reference:

- Toulopoulos, I., Wick, T., *2016. Numerical methods for power-law diffusion problems, RICAM preprint 2016-11.*

Context and motivations

▶ Overall Objective: space discretization + iterative process

p-type model
$$\begin{cases} -\mathrm{div}\mathbf{A}(\nabla u) &= f \quad \text{in } \Omega \subset \mathbb{R}^{d=2} \\ u &= u_D, \text{ on } \Gamma_D := \partial\Omega \end{cases}$$

the operator $\mathbf{A}(\nabla u) : \mathbb{R}^2 \to \mathbb{R}^2$ has p-type form

$$\mathbf{A}(\nabla u) = (\varepsilon^2 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u,$$

where $|.|^2 = (\cdot, \cdot)$, $p > 1$ and $\varepsilon > 0$ is a parameter.
! $\varepsilon = 0$ critical points $\nabla u = 0$, degenerate for $p > 2$ and singular $p < 2$.

- a basic model for describing many physical phenomena.
- efficient discretization techniques.

Weak formulation

find $u \in W_D^{1,p} := \{u \in W^{1,p}(\Omega) : v|_{\partial\Omega} = u_D\},\ s.t.\ B(u,\phi) = l_f(\phi),\ \forall \phi \in W_0^{1,p}(\Omega)$

$$B(u,\phi) = \int_\Omega \mathbf{A}(\nabla u) \cdot \nabla \phi \, dx,\ \ l_f(\phi) = \int_\Omega f \phi \, dx.$$

Assumption: $u \in V := W^{\ell,p}(\Omega)$ with $\ell \geq 2$ and $p > 1$,

Our main goal:

- **finite element discretization + error analysis**
  - $V_h^{(k)} := \{\phi_h \in C(\overline{\Omega}) : \phi_h|_E \in \mathbb{Q}_k(E),\ \forall E \in T_h\},\ k \geq 1$
- **iterative methods:**
  - Newton-like methods (residual-based and error-oriented),
  - Augmented Lagrangian (splitting and monolithic)

φ – structure problems (L. Diening and M. Růžička)

φ – structure: $\mathbf{A}(\nabla u) = \varphi'(\nabla u)\dfrac{\nabla u}{|\nabla u|}$

N-function $\varphi(t) := \int_0^t (\varepsilon^2 + s^2)^{\frac{p-2}{2}} s \, ds$

relevant functions

$$\widehat{\varphi}(t) := (\varepsilon + t)^{p-2}, \quad \mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2, \mathbf{F}(\mathbf{a}) = (\varepsilon^2 + |\mathbf{a}|^2)^{\frac{p-2}{4}} \mathbf{a},$$

Lemma, (L. Diening, Ettwein M. Růžička 2007, 2008 ...)

$$\begin{aligned}
(\mathbf{A}(\mathbf{P}) - \mathbf{A}(\mathbf{Q})) \cdot (\mathbf{P} - \mathbf{Q}) \quad &\sim \quad |\mathbf{F}(\mathbf{P}) - \mathbf{F}(\mathbf{Q})|^2, \\
&\sim \quad \widehat{\varphi}(|\mathbf{P}| + |\mathbf{Q}|)|\mathbf{P} - \mathbf{Q}|^2, \\
|\mathbf{A}(\mathbf{P}) - \mathbf{A}(\mathbf{Q})| \quad &\sim \quad \widehat{\varphi}(|\mathbf{P}| + |\mathbf{Q}|)|\mathbf{P} - \mathbf{Q}|,
\end{aligned}$$

holds for all $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^2$.

Known results

Lemma, (L. Diening and M. Růžička 2007)

$$\int_\Omega (\mathbf{A}(\nabla u) - \mathbf{A}(\nabla v)) \cdot (\nabla w - \nabla v)\, dx \leq \delta \int_\Omega |\mathbf{F}(\nabla u) - \mathbf{F}(\nabla v)|^2\, dx + c(\delta) \int_\Omega |\mathbf{F}(\nabla w) - \mathbf{F}(\nabla v)|^2\, dx.$$

Lemma, (finite element books)

Let $\mathcal{I}_h^k$ be the interpolation operator and let $u \in W^{l,p}(\Omega)$ with $k+1 \geq l$.

$$|u - \mathcal{I}_h^k u|_{W^{s,p}(\Omega)} \leq C h^{l-s} |u|_{W^{l,p}(\Omega)},\ 0 \leq s \leq l.$$

FE approximation

The finite element approximation reads as follows:

find $u_h \in V_h^{(k)}$ such that for all $\phi_h \in V_h^{(k)}$ holds

$$B(u_h, \phi_h) = l_f(\phi_h),$$

! Similar discrete form for IgA methods

Quasi-interpolation estimates for $\mathcal{I}_h^k u\,,\, k \geq 1$

Theorem, (I. Toulopoulos and T. Wick 2015)

Let $u \in W^{l,p}(\Omega)$ with $l \geq 2,\ p > 1$, and let $\mathcal{I}_h^k u$ be the interpolant. Then,

$$\int_\Omega |\mathbf{F}(\nabla u) - \mathbf{F}(\nabla \mathcal{I}_h^k u)|^2\, dx \lesssim Ch^{2(l-1)}, \text{ for } k+1 \geq l.$$

Proof.

we set $\widehat{\Phi}(\tau) = \widehat{\varphi}(|\nabla u| + |\nabla \mathcal{I}_h^k u|)^{\frac{1}{p-2}} = \varepsilon + |\nabla u| + |\nabla \mathcal{I}_h^k u|$. By previous known inequalities:

$$\int_\Omega |\mathbf{F}(\nabla u) - \mathbf{F}(\nabla \mathcal{I}_h^k u)|^2\, dx \leq \int_\Omega |\widehat{\varphi}(\tau)|\, |\nabla u - \nabla \mathcal{I}_h^k u)|^2\, dx \leq \text{..Hölder...}$$

Error estimates

Theorem: final error estimate (I. Toulopoulos and T. Wick 2015)

Let $u \in V$ be the weak solution and let $u_h \in V_h^{(k)}$ be the FE solution. Then

$$\int_\Omega |\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_h)|^2 \, dx \lesssim Ch^{2(l-1)}, \quad k+1 \geq l$$

Proof.

$$\int_\Omega (\mathbf{A}(\nabla u_h) - \mathbf{A}(\nabla \mathcal{I}_h^k u)) \cdot \nabla \phi_h \, dx = \int_\Omega (\mathbf{A}(\nabla u) - \mathbf{A}(\nabla \mathcal{I}_h^k u)) \cdot \nabla \phi_h \, dx.$$

Choosing $\phi_h = u_h - \mathcal{I}_h^k u$ and $\delta > 0$ small enough in Youngs inequality,

$$\int_\Omega |\mathbf{F}(\nabla u_h) - \mathbf{F}(\nabla \mathcal{I}_h^k u)|^2 \, dx \leq c(\delta) \int_\Omega |\mathbf{F}(\nabla u) - \mathbf{F}(\nabla \mathcal{I}_h^k u)|^2 \, dx, ...$$

The discrete nonlinear system

The FE solution is expressed as $u_h = \sum_{i}^{N_h} U_i \phi_{h,i}(x)$, where the degrees of freedom, $\mathbf{U} = [U_1, ..., U_i, ..., U_{N_h}]$ satisfy the system of nonlinear equations

$$\mathbf{B}(\mathbf{U}) = \mathbf{f}, \tag{3.1}$$

where the entries $B_i(\mathbf{U})$ of $\mathbf{B}$ and f_i of $\mathbf{f}$ correspondingly are

$$B_i(\mathbf{U}) = B(u_h, \phi_{h,i}), \text{ and } f_i = l_f(\phi_{h,i}). \tag{3.2}$$

This nonlinear system (3.1) is in generally not well-conditioned .

our goal: to develop efficient nonlinear iterative methods for solving it.

Investigate the influence of the parameters p and ε^2 to the convergence

Eigenvalues of $\mathrm{B}'(\eta)$

We introduce the function $\mathbf{A}(\eta) = (\varepsilon^2 + \eta_1^2 + \eta_2^2)^{\frac{p-2}{2}}(\eta_1, \eta_2)$.

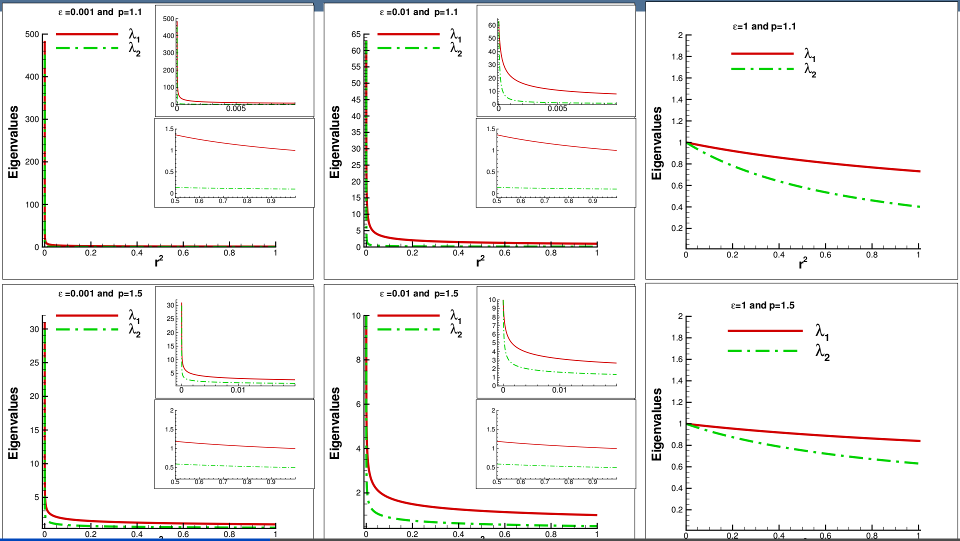
$$J_{\mathbf{A}} := \frac{\partial \mathbf{A}(\eta)}{\partial \eta} = \rho(\varepsilon, |\eta|) \begin{bmatrix} ((p-1)\eta_1^2 + \eta_2^2 + \varepsilon^2) & (p-2)\eta_1\eta_2 \\ (p-2)\eta_1\eta_2 & ((p-1)\eta_2^2 + \eta_1^2 + \varepsilon^2)) \end{bmatrix}$$

where $\rho(\varepsilon, |\eta|) = (\varepsilon^2 + \eta_1^2 + \eta_2^2)^{(p-4)/2}$.

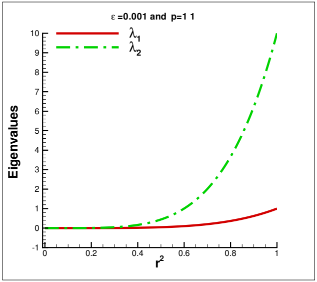
$$\begin{aligned} \lambda_1 =& (\varepsilon^2 + \eta_1^2 + \eta_2^2)^{(p-2)/2}, \\ \lambda_2 =& (\varepsilon^2 + \eta_1^2 + \eta_2^2)^{(p-4)/2}\big((p-1)(\eta_1^2 + \eta_2^2) + \varepsilon^2\big). \end{aligned}$$

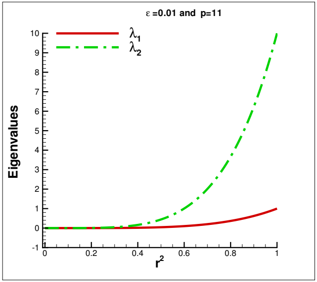
The properties of the eigenvalues of Jacobian $J_{\mathbf{A}} := \dfrac{\partial \mathbf{A}(\eta)}{\partial \eta}$, outline the behavior of the eigenvalues of the Newton Jacobian matrix.

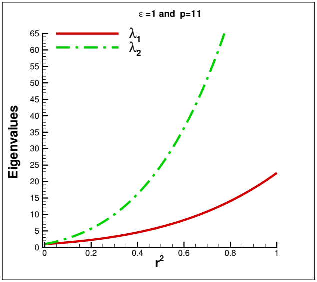
Graphs of λ_1 and λ_2 for $p < 2$



Graphs of λ_1 and λ_2 for $p = 11$







Newton method

$$B'(u)(v,w) = \int_\Omega (\varepsilon^2 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla v \cdot \nabla w \, dx + (p-2) \int_\Omega (\varepsilon^2 + |\nabla u|^2)^{\frac{p-4}{2}} (\nabla u \cdot \nabla v)(\nabla u \cdot \nabla w) \, dx,$$

$$\mathbf{B}'(\mathbf{V}) := B'(v_h)(\phi_{h,i}, \phi_{h,j}), \;\; \phi_{h,i}, \phi_{h,j} \in V_h^{(k)},$$

1. given $\mathbf{U}^0$, for $n = 0, 1, 2, \ldots$ compute $\mathbf{Z}^n$ from

   $\mathbf{B}'(\mathbf{U}^n)\mathbf{Z}^n = -\left(\mathbf{B}(\mathbf{U}^n) - \mathbf{f}\right)$, update: $\mathbf{U}^{n+1} = \mathbf{U}^n + \tau \mathbf{Z}^n$,

2. if $\|\mathbf{B}(\mathbf{U}^n) - \mathbf{f}\|_{L^2} \leq \; tolerance$ set $U^* = \mathbf{U}^{n+1}$ and exit other wise repeat step 2.

Condition number B'

Lemma: bounds for $\kappa(\mathbf{B}')$

$$\kappa(\mathbf{B}') = \frac{\lambda_{Max}}{\lambda_{min}} \leq \begin{cases} C(\varepsilon + \|\nabla u_h\|_\infty)^{p-2}\, \varepsilon^{2-p}\, h^{-2} & \text{for } p > 2, \\ C(\varepsilon + \|\nabla u_h\|_\infty)^{2-p}\, \varepsilon^{p-2}\, h^{-2} & \text{for } 1 < p < 2, \end{cases}$$

$p > 1:$

$$\frac{B'(v_h)(w_h, w_h)}{\|\mathbf{w}\|^2} \leq \ldots C(\varepsilon + \|\nabla u_h\|_\infty)^{p-2} \frac{\|\nabla w_h\|^2_{L^2(\Omega)}}{h^{-2}\|w_h\|^2_{L^2(\Omega)}} \leq \ldots$$

$$\frac{B'(v_h)(w_h, w_h)}{\|\mathbf{w}\|^2} \geq \ldots C\varepsilon^{p-2} \frac{\|\nabla w_h\|^2_{L^2(\Omega)}}{h^{-2}\|w_h\|^2_{L^2(\Omega)}} \geq \ldots . \geq C\varepsilon^{p-2} h^2.$$

Newton's method - residual-based and error-oriented

- Difference in determining τ and stopping criterion.
- Residual-based (rather classical):

$$res_{n+1} := \|\mathbf{B}(\mathbf{U}^{n+1}) - \mathbf{f}\| \leq TOL, \tag{3.3}$$

- Error-oriented (P. Deuflhard 2011)

$$\|\mathbf{Z}^{n}\| \leq TOL, \quad \text{or} \quad \|\mathbf{Z}^{n+1}_{simp}\| \leq TOL, \tag{3.4}$$

where $\mathbf{Z}^{n+1}_{simp}$ is a Newton update from solving a simplified problem, in which only the residual is updated and the matrix kept.

smooth solutions, $|\nabla u|\big|_\Omega > 0$

$$\Omega = (0, \frac{\pi}{2})^2 \; u(x, y) = \sin(x)$$

-	$k = 1$ and $p = 1.01$							
-	$\varepsilon = 1.0E-3$				$\varepsilon = 1.0E-4$			
$\frac{h_0}{2^j}$	$\Vert F - F_h\Vert$	r	$U_h^0 = 0$	$P(U_{2h}^N)$	$\Vert F - F_h\Vert$	r	U_h^0	$P(U_{2h}^N)$
j=0	3.64e-02	-	13	13	3.64e-02	-	16	16
j=1	2.15e-02	0.76	16	8	1.96e-02	0.88	18	7
j=2	1.28e-02	0.75	23	9	1.05e-02	0.9	22	8
j=3	6.73e-03	0.93	28	9	5.69e-03	0.9	26	7
j=4	3.31e-03	1.03	32	8	3.17e-03	0.85	32	8
j=4	1.65e-03	1.0	34	7	1.78e-03	0.83	43	10
j=5	8.54e-04	0.96	36	6	9.36e-04	0.94	47	8
CPU			9s	1.8s			13s	2.1s

smooth solutions, $|\nabla u|\big|_\Omega > 0$

$\Omega = (0, \frac{\pi}{2})^2 \; u(x, y) = \sin(x)$

-	$k = 1$ and $p = 1.01$							
-	$\varepsilon = 1.0E-2$				$\varepsilon = 1$			
$\frac{h_0}{2^j}$	$\|F - F_h\|$	r	U_h^0	$P(U_{2h}^N)$	$\|F - F_h\|$	r	U_h^0	$P(U_{2h}^N)$
j=0	4.64e-02	-		12	2.770e-02	-		12
j=1	2.277e-02	1.02		8	1.385e-02	0.99		5
j=2	1.168e-02	0.96		9	6.926e-03	1.0		5
j=3	6.052e-03	0.95		10	3.463e-03	1.0		3
j=4	3.040e-03	0.99		8	1.731e-03	1.0		3
j=5	1.519e-03	1.0		5	8.658e-04	1.0		3
j=6	7.598e-04	1.0		4	4.329e-04	1.0		2
CPU				8.s				5.2s

Table : Example 1: The results for the different values of ε.

smooth solutions, $|\nabla u|\big|_{\Omega} > 0$

-	$k = 2$ and $p = 1.1$ and $p = 1.01$					
-	$\varepsilon = 1.0E-04\ p = 1.01$		$\varepsilon = 1.0E-01\ p = 1.01$		$\varepsilon = 1.0E-01\ p = 1.1$	
$\frac{h_0}{2^j}$	r	$U_h^0 = P(U_{2h}^N)$	r	$U_h^0 = P(U_{2h}^N)$	r	$U_h^0 = P(U_{2h}^N)$
j=0	-	20	-	11	-	13
j=1	2.01	4	2.0	2	2.0	2
j=2	1.99	4	1.99	2	2.0	2
j=3	2.0	5	2.0	1	2.0	1
j=3	2.0	5	2.0	1	2.0	1
j=4	2.0	5	2.0	1	2.0	1

Table : Example 1: The results for $k = 2$ for different choices of ε.

smooth solutions, $|\nabla u|\big|_{\Omega} > 0$

-	$k = 1$ and $k = 2$ for $p = 11$					
-	$\varepsilon = 1.0E-04$ $p = 11$		$\varepsilon = 1.0E-04$ $p = 11$		$\varepsilon = 0.01$ $p = 11$	
$\frac{h_0}{2^j}$	r	$P(U_{2h}^N)$	r	$P(U_{2h}^N)$	r	$P(U_{2h}^N)$
j=0	-	9	-	3	-	3
j=1	0.99	24	1.99	3	1.99	3
j=2	0.99	19	1.99	3	1.99	3
j=3	0.99	20	1.99	2	1.99	2
j=3	0.99	19	1.99	2	1.99	2
j=4	0.99	17	1.99	1	1.99	1
j=5	0.99	17	1.99	1	1.99	1

Table : Example 1: The results for $k = 2$ and $k = 1$ for $p = 11$.

smooth solutions, $\exists x_0 : |\nabla u(x_0)| = 0$

$\Omega = (0, \frac{\pi}{2})^2$ and $u(x, y) = \sin(2\pi(x + y))$

$p=$ / ε	1.01		1.1		1.3		1.5		1.8		2.25		3	
	r	N	r	N	r	N	r	N	r	N	r	N	r	N
10E-4							1.3	8	1.08	9	1	6	1	7
10E-3					1.8	11	1.2	8	1.1	9	1	6	1	7
10E-2	1	12	1	12	1.4	12	1.2	8	1.1	6	1	5	1	7
10E-1	1	10	1	9	1	9	1	6	1	5	1	5	1	7
1	1	4	1	4	1	5	1	5	1	5	1	5	1	4

$p=$ / ε	4.3		11	
	r	N	r	N
10E-4	1	10	1	20
10E-3	1	10	1	20
10E-2	1	9	1	19
10E-1	1	6	1	18
1	1	5	1	9

Augmented Lagrangian form

Transform the original problem into a saddle-point problem.

Let $\frac{1}{p} + \frac{1}{p'} = 1$, and the space $W \subset W_D^{1,p} \times (L^p(\Omega))^2$ by

$$W = \{(v, q)|(v, q) \in W_D^{1,p} \times (L^p(\Omega))^2 : \nabla v - \mathbf{q} = 0\}.$$

Following Glowinski-Tallec-1989, introduce $\mathcal{L}_r$ for $r > 0$,

$$\mathcal{L}_r(v, \mathbf{q}, \lambda) = \frac{1}{p} \int_\Omega (\varepsilon^2 + |\mathbf{q}|^2)^{p/2}\, dx - \int_\Omega fv\, dx + \frac{r}{2} \int_\Omega |\nabla v - \mathbf{q}|^2\, dx + \int_\Omega \lambda \cdot (\nabla v - \mathbf{q})\, dx,\ r > 0,$$

saddle-point problem: find$\{u, \mathbf{q}, \lambda\} \in W_D^{1,p} \times (L^p(\Omega))^2 \times (L^{p'}(\Omega))^2$

such that $\mathcal{L}_r(u, \mathbf{q}, \mu) \le \mathcal{L}_r(u, \mathbf{q}, \lambda) \le \mathcal{L}_r(v, \mathbf{w}, \lambda),$

$\forall \{v, \mathbf{w}, \mu\} \in W_D^{1,p} \times (L^p(\Omega))^2 \times (L^{p'}(\Omega))^2$

Augmented Lagrangian iterative method

Algorithm

procedure FOR $\lambda^0 \in (L^{p'}(\Omega))^2$ GIVEN

for each iteration step $n > 0 \in N$, find u^n, $\mathbf{q}^n$ and λ^{n+1} **do**

$$\text{ALG}\begin{cases} \underbrace{\begin{aligned} &-r\Delta u^n - \nabla\cdot\lambda^n + r\nabla\cdot\mathbf{q}^n = f,\\ &(\varepsilon^2 + |\mathbf{q}^n|^2)^{\frac{p-2}{2}}\mathbf{q}^n + r\mathbf{q}^n - r\nabla u^n - \lambda^n = 0,\\ &\lambda^n - \rho_n(\nabla u^n - \mathbf{q}^n) = \lambda^{n-1}, \end{aligned}}_{\text{monolithic}} \\ \text{update: } \underbrace{\lambda^{n+1} = \lambda^n + \rho_n(\nabla u^n - \mathbf{q}^n),}_{\text{splitting, ALG1}} \end{cases}$$

end for

end procedure

Numerical examples $p < 2$

-	$k = 1$, $\varepsilon = 1$							
-	s AL $p = 1.01$		s AL $p = 1.5$		m AL $p = 1.01$		m AL $p = 1.5$	
$\frac{h_0}{2^j}$	r	$P(U_{2h}^N)$	r	$P(U_{2h}^N)$	r	$P(U_{2h}^N)$	r	$P(U_{2h}^N)$
j=0	-	4	-	4	-	5	-	4
j=1	1.1	4	1.1	4	1.0	5	1.0	4
j=2	1.0	3	1.2	3	0.86	3	0.98	2
j=3	1.3	3	1.2	2	1.00	2	1.0	2
j=4	1.3	2	1.1	2	1.00	2	1.0	2
j=5	1.2	1	1.0	1	1.00	1	1.0	1

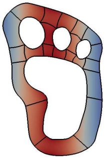
Table : Example 2: $\Omega = (0, \frac{\pi}{2})^2$ and $u(x, y) = \sin(2\pi(x + y))$

Numerical examples $p > 2$

-	$k = 1, \varepsilon = 1$							
-	s AL $p = 3$		s AL $p = 4.3$		m AL $p = 3$		m AL $p = 4.3$	
$\frac{h_0}{2^j}$	r	$P(U_{2h}^N)$	r	$P(U_{2h}^N)$	r	$P(U_{2h}^N)$	r	$P(U_{2h}^N)$
j=0	-	4	-	4	-	7	-	12
j=1	1.2	4	1.4	3	1.0	2	1.2	3
j=2	1.1	4	1.3	3	1.0	2	1.0	2
j=3	1.0	4	1.2	1	1.0	2	1.0	2
j=4	1.0	4	1.1	1	1.0	2	1.0	2

Table : Example 2: $\Omega = (0, \frac{\pi}{2})^2$ and $u(x, y) = \sin(2\pi(x + y))$

IETI-DP solvers in IgA (C.Hofer and I. T. 2016)



$$\begin{bmatrix} K_e & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_e \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \widetilde{K}_e & \widetilde{B}^T \\ \widetilde{B} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbf{f}}_e \\ 0 \end{bmatrix},$$

$$F_S \boldsymbol{\lambda} = \mathbf{d},$$

-	multi-patch IgA, using $k = 2$, $\mathbf{U}_0 = 0$			
-	$p = 1.1$	$p = 1.5$	$p = 1.5$ and $k = 4$	$p = 4$
Iterations	N	N	N	N
$\varepsilon = 10^{-3}$	19	7	7	8
$\varepsilon = 1$	4	4	3	4

Table : Example: $u(x, y) = \sin(2\pi x)$

Conclusions

1. p-type model
2. FE discretization, error analysis
3. Nonlinear iterative methods (Newton and ALG)
4. Numerical examples
   - validation of error estimates for several ε and p and $k \geq 1$
   - comparison between the nonlinear iterative methods
   - the influence of ε, p on the performance of the methods

Thank you for your attention

This work was supported by Austrian Science Fund (FWF) under the grant FWF-NFN S11703-N23.

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