# Multigrid methods for Isogeometric analysis

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# Outline

# Model problem

Elliptic model problem: Find $u \in H^1(\Omega)$:

$$-\Delta u + u = f \text{ in } \Omega, \qquad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega$$

Variational formulation: Find $u \in V$:

$$a(u, v) = \langle f, v\rangle \qquad \forall v \in V$$

where

$$a(u, v) = \int_\Omega (\nabla u \cdot \nabla v + uv)\, dx, \qquad \langle f, v\rangle = \int_\Omega fv\, dx.$$

Or as a linear system:

$$Au = f.$$

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## What is Isogeometric Analysis?

- **Idea:** One method that can be used for design (CAD) and numerical simulation
- **Technical:** B-spline (NURBS) based FEM

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# B-spline basis functions

Let $m \in \mathbb{N}$, $h = 1/m$ and let

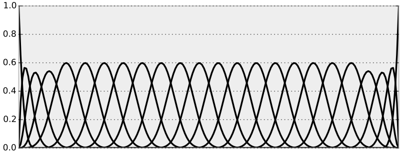
$$S_{p,h} := \{u \in C^{p-1}(0,1) : u|_{((j-1)h,jh)} \in \mathcal{P}^p \quad \forall j = 1, \ldots, m\},$$

denote the **spline space** over $[0,1]$ with degree $p$, maximum continuity $C^{p-1}$, and mesh size $h$.

We denote the standard **B-spline basis functions** by

$$S_{p,h} = \text{span}(\mathcal{B}), \qquad \mathcal{B} = \{\phi_1, \ldots, \phi_n\},$$

where $n = \dim S_{p,h} = m + p$.



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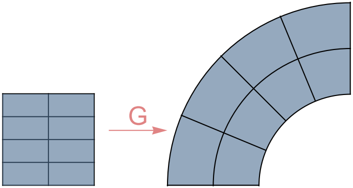
In higher dimensions, we form **tensor product** spline spaces:

$$S^2_{p,h} = S_{p,h} \otimes S_{p,h}, \qquad \phi_{j_1,j_2}(x,y) := \phi_{j_1}(x)\phi_{j_2}(y).$$

# Isogeometric Analysis

## Global geometry transformation




## More complicated domains:

- Multi-patch discretization with tensor-product patches

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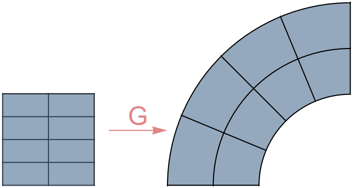


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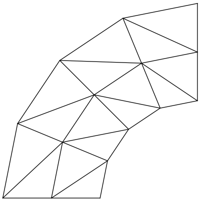
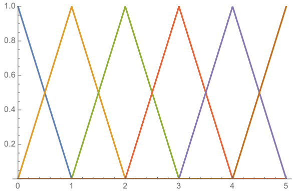
# Finite element method

## Courant element





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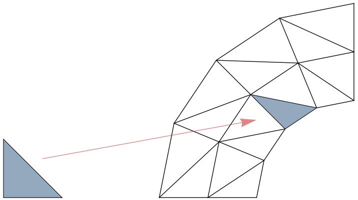

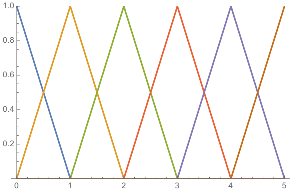

# Properties of the B-spline basis

- Non-negativity: $\phi_i(x) \geq 0$
- Partition of unity: $\sum_i \phi_i(x) = 1$
- Approximation power:

$$\|u - u_h\|_{L_2} \leq C_p h^p |u|_{H^p}$$

- $\dim S_{p,h} = n + p$, unlike $\dim S_{p,0,h} = n\,p + 1$
- Condition number (of the basis):

$$\kappa(M_{p,h}) = \mathcal{O}(2^{pd}) \qquad \kappa(K_{p,h}) = \mathcal{O}(h^{-2}2^{pd}),$$

where $M_{p,h}$ is the mass matrix and $K_{p,h}$ is the stiffness matrix.

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# Fast solver for $Au = f$

## Requirements:

- Fast solver must be robust in $h$
- Should behave well in $p$

## We know from finite element world:

- Multigrid converges robustly in $h$.
- Use $S_{p,H} \subset S_{p,h}$ for $H = 2h$, setup a $h$-multigrid with fixed $p$

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# Multigrid with Gauss-Seidel smoother

| $\ell \setminus p$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | $\geq 9$ |
|---|---|---|---|---|---|---|---|---|---|
| 8 | 10 | 12 | 37 | 127 | 462 | 1762 | 6531 | 21657 | $>$50k |
| 7 | 10 | 12 | 37 | 127 | 488 | 1856 | 7247 | 23077 | $>$50k |
| 6 | 10 | 12 | 39 | 131 | 485 | 1883 | 6723 | 23897 | $>$50k |

V-cycle multigrid, $\nu_{pre} + \nu_{post} = 1 + 1$, stopping criterion: $\ell^2$ norm of the initial residual is reduced by a factor of $\epsilon = 10^{-8}$

# Observations and problems

- Obtain $h$-robustness of the method
  $\kappa(A) = \mathcal{O}(h^{-2})$, $\qquad \kappa(M) = \mathcal{O}(1)$

- In $p$: bad condition number of the mass matrix:
  $\kappa(A)$ and $\kappa(M)$ grow exponentially in $p$

- **Idea:**
  Basis-independent method (mass-smoother)

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# Outline

# Multigrid framework

One step of the multigrid method applied to iterate $u^{(0,0)} = u^{(0)}$ and right-hand-side $f$ to obtain $u^{(1)}$ is given by:

- Apply $\nu$ **smoothing steps**

  $$u^{(0,m)} = u^{(0,m-1)} + \tau L^{-1}(f - Au^{(0,m-1)})$$

  for $m = 1, \ldots, \nu$.
- Apply **coarse-grid correction**
  - Compute defect and restrict to coarser grid
  - Solve problem on coarser grid
  - Prolongate and add result

  If realized exactly (two-grid method):

  $$u^{(1)} = u^{(0,\nu)} + I_H^h A_H^{-1} I_h^H (f - Au^{(0,\nu)})$$

- Two-grid convergence $\Rightarrow$ multigrid (W-cycle) convergence

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# Multigrid setup

- **Nested spaces:** $S_{p,H}(\Omega) \subset S_{p,h}(\Omega)$
- The prolongation $I_H^h$ is the canonical embedding
- The restriction is its transpose: $I_h^H = (I_H^h)^T$
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# Outline

# A $p$-robust estimate for high smoothness

$$\widetilde{S}_{p,h}(0,1) := \left\{ u \in S_{p,h}(0,1) \;:\; \begin{array}{l} \frac{\partial^{2i+1}}{\partial x^{2i+1}} u(0) = 0 \\ \frac{\partial^{2i+1}}{\partial x^{2i+1}} u(1) = 0 \end{array} \;\forall_{i \in \mathbb{Z} \text{ with } 1 \le 2i+1 < p} \right\}$$

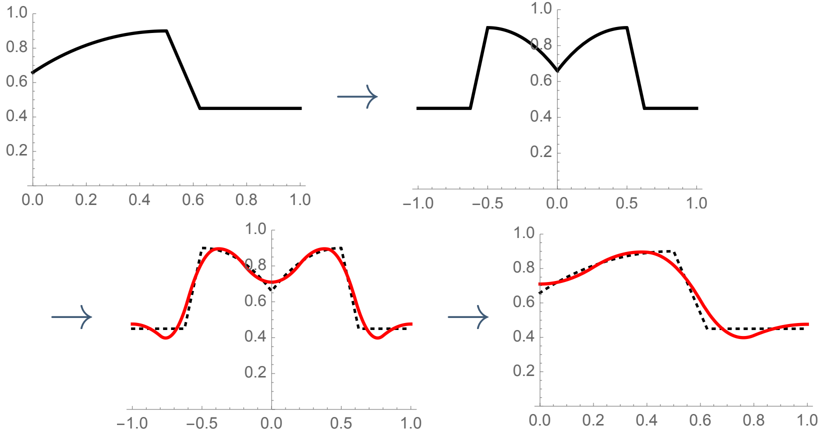
## Theorem (T., Takacs 2016)

*For each* $u \in H^1(\Omega)$, *each* $p \in \mathbb{N}$ *and each* $h$,

$$\|(I - \Pi)u\|_{L^2(\Omega)} \le \sqrt{2}\, h |u|_{H^1(\Omega)}$$

*is satisfied for* $\Pi$ *being the* $H^1$*-orthogonal projection into* $\widetilde{S}_{p,h}(\Omega)$.

# Idea behind $\widetilde{S}_{p,h}(0,1)$



# Inverse inequality

- A $p$-robust inverse inequality does not exist for $S_{p,h}(\Omega)$:

$$|u|_{H^1(\Omega)} \leq C\, h^{-1} \|u\|_{L^2(\Omega)} \rightarrow \textbf{not true} \text{ for all} u \in S_{p,h}(\Omega)$$

- Choose $u^*(x) := \max\{0, h - x\}^p$
- What about the space $\widetilde{S}_{p,h}(\Omega)$?

**Theorem (T., Takacs 2016)**

*For each $p \in \mathbb{N}$ and each $h$,*

$$|u|_{H^1} \leq 2\sqrt{3}h^{-1}\|u\|_{L^2}$$

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# Outline

# Robust multigrid for IgA

How to choose the smoother $L$ such that the two-grid/multigrid method **converges robustly** in $h$ and $p$?

- Standard smoothers (e.g., **Gauss-Seidel**) achieve $h$-robustness but scale poorly with $p$.
- Our previous concept: **mass smoother with low-rank boundary correction** is robust in $h$ and $p$, but only efficient up to 2D. *(Hofreither, T., Zulehner, CMAME 2016)*
- New idea: stable splitting of the spline space – **subspace correction**. Robust and efficient in arbitrary dimension.
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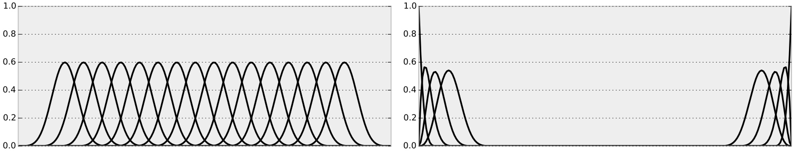
# Splittings of spline spaces

Any spline $u \in S_{p,h}(0,1)$ can be split into $u_I \in S^I_{p,h}(0,1)$ and $u_\Gamma \in S^\Gamma_{p,h}(0,1)$:

$$u = u_I + u_\Gamma$$




**Have:** Inverse inequality: $\|v\|_1 \leq c h^{-1} \|v\|_0 \qquad \forall v \in S^I_{p,h}(0,1)$.

**Problem:** Splitting is not stable.

$$c^{-1}\|u\|_1 \leq \|u_I\|_1 + \|u_\Gamma\|_1 \leq c\|u\|_1 \qquad \forall u \in S_{p,h}(0,1) \rightarrow \text{wrong!}$$

# The subspace $\widetilde{S}_{p,h}(0,1)$

We have seen that for $\widetilde{S}_{p,h}(0,1)$,

- an **approximation error estimate** and
- an **inverse inequality** holds.

**Define:**

$V := S_{p,h}(0,1)$
$V_0 := \widetilde{S}_{p,h}(0,1)$
$V_1$ is the $L^2$-orthogonal complement of $V_0$ in $V$

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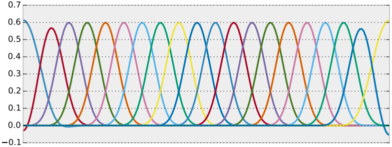
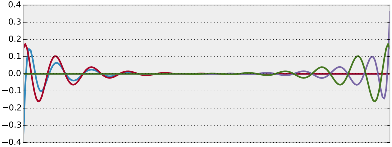
# Stability of the splitting based on $V_0$

Any spline $u \in V$ can be split into $u_0 \in V_0, u_1 \in V_1$: $u = u_0 + u_1$




Due to orthogonality, we have: $\|u\|_0^2 = \|u_0\|_0^2 + \|u_1\|_0^2 \qquad \forall u \in V.$

**Theorem (Hofreither, T. 2016)**

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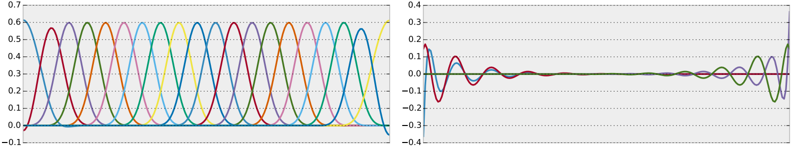


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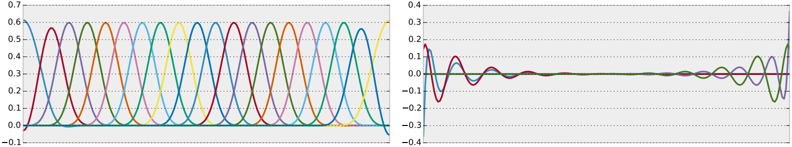

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# Setting up the splitting in 1D

- Construction of $V_0$ and $V_1$ is local process on the boundary
- Basis functions away from the boundary are directly taken as basis functions in $V_0$
- For the first and last $p$ basis functions, we can use a SVD (for two $p \times p$ matrices) to set up the $\ell^2$-orthogonal splitting representing the basis functions for for $V_0$ and $V_1$ as linear combination of the $\phi_i$
- The vectors representing the basis functions on $V_1$ are pre-multiplied with $M^{-1}$ to obtain $L^2$-orthogonality

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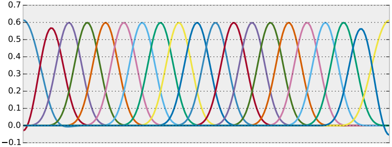
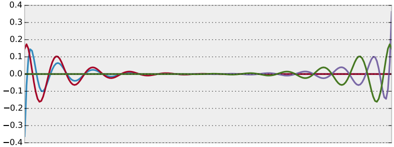
# Stability of the splitting based on $V_0$ (once more)

Any spline $u \in V$ can be split into $u_0 \in V_0, u_1 \in V_1$: $u = u_0 + u_1$

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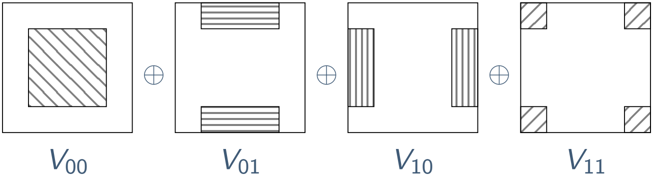
*holds, where $c$ does not depend on $h$ or $p$.*

# A stable splitting in 2D

The 2D tensor product spline space is given by

$$\begin{aligned} V^2 &= V \otimes V \\ &= (V_0 \oplus V_1) \otimes (V_0 \oplus V_1) \\ &= (V_0 \otimes V_0) \oplus (V_0 \otimes V_1) \oplus (V_1 \otimes V_0) \oplus (V_1 \otimes V_1) \\ &= V_{00} \oplus V_{01} \oplus V_{10} \oplus V_{11}. \end{aligned}$$



# A stable splitting in 2D

Let

$$Q_0 : V \to V_0, \qquad Q_1 : V \to V_1$$

denote the $L_2$-orthogonal projectors into $V_0$ and $V_1$. Then

$$Q_{\alpha_1,\alpha_2} := Q_{\alpha_1} \otimes Q_{\alpha_2} \quad : \quad V^2 \to V_{\alpha_1,\alpha_2}$$

is the $L_2$-orthogonal projector into $V_{\alpha_1,\alpha_2}$.

## Theorem (Hofreither, T. 2016)

*For any tensor product spline $u \in V^2$, we have*

$$c^{-1}\|u\|_1^2 \leq \sum_{(\alpha_1,\alpha_2)=(0,0)}^{(1,1)} \|Q_{\alpha_1,\alpha_2} u\|_1^2. \leq c\|u\|_1^2$$

*with a constant $c$ which does not depend on $h$ or $p$.*

# Stable splitting in arbitrary dimensions

For a multiindex $\alpha \in \{0,1\}^d$, we define projectors

$$Q_\alpha := Q_{\alpha_1} \otimes \ldots \otimes Q_{\alpha_d} \quad : \quad V^d \to V_{\alpha_1} \otimes \ldots \otimes V_{\alpha_d} =: V_\alpha$$

into the $2^d$ subspaces $V_\alpha$.

## Theorem (Hofreither, T. 2016)

*For any $d$-dimensional tensor product spline $u \in V^d$, we have*

$$c^{-1} \|u\|_1^2 \leq \sum_{\alpha=(0,\ldots,0)}^{(1,\ldots,1)} \|Q_\alpha u\|_1^2 \leq c \|u\|_1^2$$

*with a constant $c$ which does not depend on $h$ or $p$.*

# A smoother based on subspace correction

In each subspace $V_\alpha$, we apply a local smoothing operator $L_\alpha : V_\alpha \rightarrow V_\alpha'$. The overall operator is

$$L = \sum_\alpha Q_\alpha' L_\alpha Q_\alpha .$$

## Theorem (a variant of Hackbusch's analysis)

*Assume that we have an appropriate approxiamtion error estimate and*

$$\langle Av, v\rangle \le c\langle Lv, v\rangle \qquad \forall v \in V$$

*and*

$$\langle Lv, v\rangle \le c\langle (A + h^{-2} M^d) v, v\rangle \qquad \forall v \in V .$$

*Then the two-grid method with smoother based on $L$ converges with a rate which depends only on $c$.*

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## Theorem (Hofreither, T. 2016)

*Assume that we have an appropriate approxiamtion error estimate and for every $\alpha \in \{0,1\}^d$ we have*

$$\langle A_\alpha v_\alpha, v_\alpha \rangle \le c \langle L_\alpha v_\alpha, v_\alpha \rangle \qquad \forall v_\alpha \in V_\alpha$$

*and*

$$\langle L_\alpha v_\alpha, v_\alpha \rangle \le c \langle (A_\alpha + h^{-2} M_\alpha^d) v_\alpha, v_\alpha \rangle \qquad \forall v_\alpha \in V_\alpha.$$

*Then the two-grid method with smoother based on $L$ converges with a rate which depends only on $c$.*

# Construction of the subspace smoothers (2D)

Let $M$ and $K$ denote the 1D mass and stiffness operators. Then

$$A = K \otimes M + M \otimes K + M \otimes M.$$

The restriction to the subspace $V_{\alpha_1,\alpha_2}$ is

$$A_\alpha = K_{\alpha_1} \otimes M_{\alpha_2} + M_{\alpha_1} \otimes K_{\alpha_2} + M_{\alpha_1} \otimes M_{\alpha_2}.$$

The robust inverse inequality in $V_0$ states that

$$K_0 \leq ch^{-2} M_0.$$

We want

$$c^{-1} A_\alpha \leq L_\alpha \leq c(A_\alpha + h^{-2} M_\alpha^d).$$

# Construction of the subspace smoothers (2D)

Using $K_0 \leq ch^{-2}M_0$, we estimate:

$$A_{00} = K_0 \otimes M_0 + M_0 \otimes K_0 + M_0 \otimes M_0 \lesssim h^{-2} M_0 \otimes M_0$$

$$A_{01} = K_0 \otimes M_1 + M_0 \otimes K_1 + M_0 \otimes M_1 \lesssim M_0 \otimes (h^{-2} M_1 + K_1)$$

$$A_{10} = K_1 \otimes M_0 + M_1 \otimes K_0 + M_1 \otimes M_0 \lesssim (h^{-2} M_1 + K_1) \otimes M_0$$

So we choose

$$L_{00} := h^{-2} M_0 \otimes M_0 \qquad L_{01} := M_0 \otimes (h^{-2} M_1 + K_1)$$

$$L_{10} := (h^{-2} M_1 + K_1) \otimes M_0 \qquad L_{11} := A_{11}$$

$L_{00}, L_{01}, L_{10}$ have tensor product structure. Invert using

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}.$$

$L_{11}$ lives in the small space $V_{11}$ ($\mathcal{O}(p^2)$ dofs) – invert directly.

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# Robust convergence

By construction, the subspace smoothers satisfy

$$\langle A_\alpha v_\alpha, v_\alpha\rangle \le c\langle L_\alpha v_\alpha, v_\alpha\rangle \qquad \forall v_\alpha \in V_\alpha$$

and

$$\langle L_\alpha v_\alpha, v_\alpha\rangle \le c\langle (A_\alpha + h^{-2} M_\alpha^d) v_\alpha, v_\alpha\rangle \qquad \forall v_\alpha \in V_\alpha.$$

In both cases, $c$ does not depend on $h$ or $p$.

**Theorem (Hofreither, T. 2016)**

*The two-grid method with the subspace correction smoother $L$ converges with a rate which does not depend on $h$ or $p$.*

The extension to W-cycle multigrid is standard.

# Computational costs

Construction is easily extended to $d$ dimensions.

Complexity analysis:

| | |
|---|---|
| **Setup costs** | $\mathcal{O}(np^2 + p^{3d})$ |
| **Application costs** | $\mathcal{O}\left(n^d p + \max_{k=0,\ldots,d} n^k p^{2(d-k)}\right) = \mathcal{O}\left(n^d p + p^{2d}\right)$ |
| **Stiffness matrix costs** | $\mathcal{O}\left(n^d p^d\right)$ |

For $d \geq 2$ and $p^2 \lesssim n$, both setup and application of the smoother are **not more expensive** than applying the stiffness matrix.

# Outline

# Iteration numbers: $d = 1$

| $\ell \diagdown p$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 9 | 27 | 33 | 34 | 34 | 33 | 33 | 33 | 32 | 31 | 31 |
| 8 | 27 | 33 | 34 | 34 | 32 | 33 | 33 | 31 | 30 | 30 |
| 7 | 27 | 33 | 34 | 34 | 32 | 33 | 33 | 31 | 28 | 30 |

V-cycle multigrid, $\nu_{pre} + \nu_{post} = 1 + 1$, stopping criterion: $\ell^2$ norm of the initial residual is reduced by a factor of $\epsilon = 10^{-8}$

# Iteration numbers: $d = 2$

| $\ell \setminus p$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 8 | 34 | 38 | 39 | 39 | 39 | 38 | 38 | 37 | 37 | 36 |
| 7 | 34 | 38 | 39 | 39 | 38 | 38 | 37 | 36 | 36 | 34 |
| 6 | 34 | 38 | 38 | 38 | 37 | 37 | 35 | 34 | 34 | 32 |
| 5 | 34 | 36 | 37 | 34 | 34 | 32 | 30 | 28 | 26 | 24 |
| 4 | 34 | 33 | 32 | 28 | 25 | 21 | 19 | 16 | 13 | 11 |
| 3 | 38 | 25 | 21 | 15 | 11 | 9 | 7 | - | - | - |

Iteration numbers using standard Gauss-Seidel smoother:

| $\ell \setminus p$ | 1 | 2 | 3 | 4 | 5 | 6 | $\geq 7$ |
|---|---|---|---|---|---|---|---|
| 8 | 10 | 12 | 37 | 127 | 462 | 1762 | $>$5k |

# Iteration numbers: $d = 3$

| $\ell \diagdown p$ | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| 5 | 44 | 43 | 42 | 39 |
| 4 | 39 | 36 | 32 | 29 |
| 3 | 30 | 42 | 18 | 23 |
| 2 | 16 | 23 | - | - |
| 1 | - | - | - | - |

Iteration numbers using standard Gauss-Seidel smoother:

| $\ell \diagdown p$ | 2 | 3 | 4 | $\geq 5$ |
|---|---|---|---|---|
| 5 | 38 | 240 | 1682 | $>$5k |

# Multigrid solver

- Single-patch solver (some DD approach might be used for multi-patch domains)
- **Robust convergence rates**, robust number of smoothing steps
- **Optimal computational complexity** in the sense: "same complexity as the multiplication with the stiffness matrix $A$"
- Mild dependence of the rates on $d$ (not fully analyzed)
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# Literature

S. Takacs and T. Takacs.
Approximation error estimates and inverse inequalities for B-splines of maximum smoothness.
*$M^3AS$, 26 (7), p. 1411 - 1445*, 2016.

C. Hofreither, S. Takacs and W. Zulehner.
A Robust Multigrid Method for Isogeometric Analysis using Boundary Correction.
*CMAME, available online*, 2016.

C. Hofreither and S. Takacs.
Robust Multigrid for Isogeometric Analysis using Subspace Correction.
*In preparation*, 2016.

## Thanks for your attention!

# Construction of the subspace smoothers (3D)

Using $K_0 \leq ch^{-2} M_0$, we estimate:

$$
\begin{aligned}
A_{000} &= K_0 \otimes M_0 \otimes M_0 + M_0 \otimes K_0 \otimes M_0 + M_0 \otimes M_0 \otimes K_0 + M_0 \otimes M_0 \otimes M_0 \\
&\lesssim h^{-2} M_0 \otimes M_0 \otimes M_0 =: L_{000} \\
A_{100} &= K_1 \otimes M_0 \otimes M_0 + M_1 \otimes K_0 \otimes M_0 + M_1 \otimes M_0 \otimes K_0 + M_1 \otimes M_0 \otimes M_0 \\
&\lesssim (K_1 + h^{-2} M_1) \otimes M_0 \otimes M_0 =: L_{100} \\
A_{110} &= K_1 \otimes M_1 \otimes M_0 + M_1 \otimes K_1 \otimes M_0 + M_1 \otimes M_1 \otimes K_0 + M_1 \otimes M_1 \otimes M_0 \\
&\lesssim (K_1 \otimes M_1 + M_1 \otimes K_1 + h^{-2} M_1 \otimes M_1) \otimes M_0 =: L_{110} \\
A_{111} &= K_1 \otimes M_1 \otimes M_1 + M_1 \otimes K_1 \otimes M_1 + M_1 \otimes M_1 \otimes K_1 + M_1 \otimes M_1 \otimes M_1 \\
&=: L_{111}
\end{aligned}
$$