AANMPDE(JS)

Strobl, July 7, 2016

An extended midpoint scheme for the Landau-Lifschitz-Gilbert equation in computational micromagnetics

Bernhard Stiftner

joint work with

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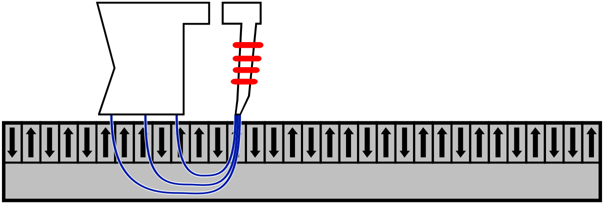
TU Wien
Institute for Analysis and Scientific Computing

Model problem

What is it all about?



- Writing information on hard drives

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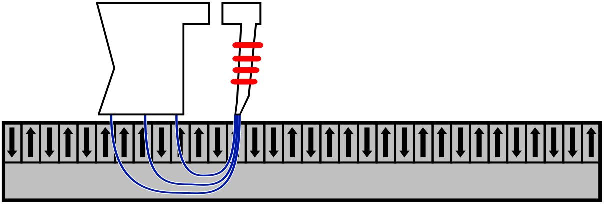
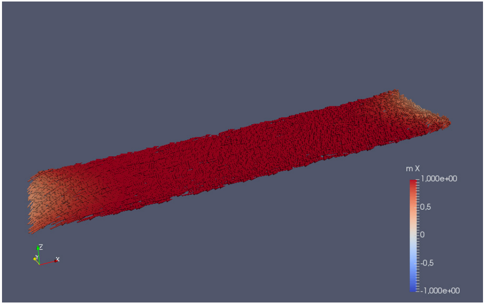


- Writing information on hard drives
- Describes evolution of magnetization $\boldsymbol{m} : \mathbb{R}^3 \to \mathbb{R}^3$ over time
- Ferromagnetic body $\Omega \subset \mathbb{R}^3$
- Constant modulus of magnetization

Numerical experiments



Schöberl: NGSolve Finite Element Library

Śmigaj, Betcke et al.: *ACM Trans. Math. Software*, 41, 2015

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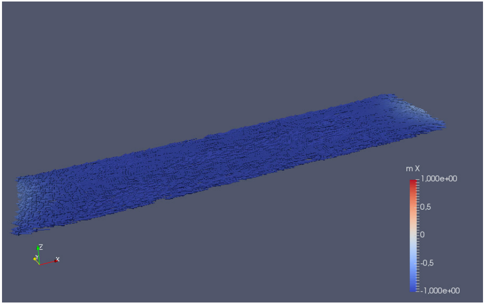

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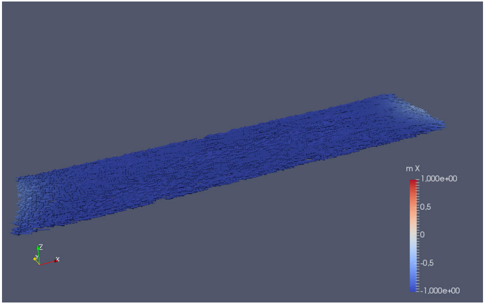

- FEM-part $\rightsquigarrow$ Netgen/NGSolve
- BEM-part $\rightsquigarrow$ BEM++

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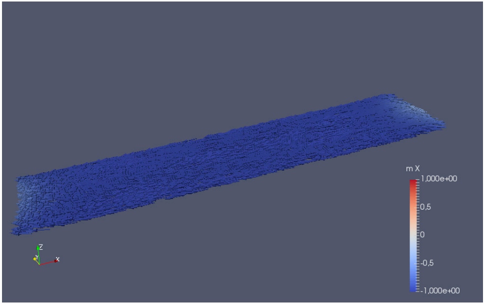

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Model problem

LLG equation

- $\partial_t \boldsymbol{m} = -\boldsymbol{m} \times \boldsymbol{h}_{\text{eff}} + \alpha \boldsymbol{m} \times \partial_t \boldsymbol{m}$ in $\Omega_T := (0,T) \times \Omega$
- $\partial_{\mathbf{n}} \boldsymbol{m} = 0$ on $(0,T) \times \partial\Omega$
- $\boldsymbol{m}(0) = \boldsymbol{m}^0$ on Ω, $|\boldsymbol{m}^0| = 1$ on Ω

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- Gilbert damping constant $\alpha > 0$

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  - Exchange field: models microcrystalline effects, $C_{\mathrm{ex}} > 0$

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  - External field: $\boldsymbol{f} : \Omega \to \mathbb{R}^3$

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- Effective field $\boldsymbol{h}_{\text{eff}}$:
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  - External field: $\boldsymbol{f} : \Omega \to \mathbb{R}^3$
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- $\partial_t \boldsymbol{m} \cdot \boldsymbol{m} = 0 \iff \frac{1}{2} \frac{\mathrm{d}}{\mathrm{dt}} |\boldsymbol{m}|^2 = 0$ in $\Omega_T \implies |\boldsymbol{m}| = 1$ in Ω_T

Challenges

- Non-linear PDE
- Non-convex constraint $|\boldsymbol{m}| = 1$
- Stray field leads to coupled problem on $\mathbb{R}^3$ $\rightsquigarrow$ embedded problem

Carbou, Fabrie: *Differential Integral Equations* 5, 2001

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Dumas, Sueur: *Commun. Math. Phys.* 330, 2014

Challenges

- Non-linear PDE
- Non-convex constraint $|\boldsymbol{m}| = 1$
- Stray field leads to coupled problem on $\mathbb{R}^3$ $\rightsquigarrow$ embedded problem
- Weak-strong uniqueness property of LLG:
  - Ex. unique strong solution up to some $T^\star > 0$
  - Ex. global weak solution, uniqueness may fail
  - Weak solution is unique up to $T^\star$

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Numerical integrator

Time and space discretization

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- Space discretization:
  - Piecewise affine functions in every dimension $\rightsquigarrow$ $\boldsymbol{V}_h$
  - $|\boldsymbol{\varphi}_h(\boldsymbol{z})| = 1$ for all nodes $\rightsquigarrow$ $\boldsymbol{\mathcal{M}}_h$

Midpoint scheme

LLG equation

- $\partial_t \boldsymbol{m} = -\boldsymbol{m} \times \boldsymbol{h}_{\mathrm{eff}} + \alpha \boldsymbol{m} \times \partial_t \boldsymbol{m}$

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Algorithm (Bartels, Prohl, '06)

- Input: $\boldsymbol{m}_h^0 \in \boldsymbol{\mathcal{M}}_h$.
- For $0 < i < M-1$, find $\boldsymbol{m}_h^{i+1} \in \boldsymbol{\mathcal{M}}_h$ s.t. for all $\boldsymbol{\varphi}_h \in \boldsymbol{V}_h$

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- $\boldsymbol{h}_{\text{eff}}(t_i + \frac{k}{2}) \approx \boldsymbol{h}_h^{i+\frac{1}{2}} \in \boldsymbol{V}_h$

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- $\boldsymbol{h}_{\mathrm{eff}} := C_{\mathrm{ex}}\Delta\boldsymbol{m} + \boldsymbol{f} + \boldsymbol{\pi}(\boldsymbol{m})$

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Midpoint with full effective field

System at each time step

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- $\boldsymbol{h}_h^{i+\frac{1}{2}} = C_{\mathrm{ex}} \Delta_h \overline{\boldsymbol{m}}_h^{i+\frac{1}{2}} + \boldsymbol{\Pi}_h(\boldsymbol{f}_h^{i+\frac{1}{2}}) + \boldsymbol{\Pi}_h(\boldsymbol{\pi}_h(\overline{\boldsymbol{m}}_h^{i+\frac{1}{2}}))$

Bartels, Prohl: *SIAM J. Numer. Anal.*, 44, 2006

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- Solve non-linear system for every time-step $\rightsquigarrow$ fixed-point iteration

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- Solve non-linear system for every time-step $\rightsquigarrow$ fixed-point iteration
- Evaluate $\boldsymbol{\pi}_h(\cdot)$ in every iteration $\rightsquigarrow$ FEM-BEM problem

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Midpoint with full effective field

System at each time step

- $\langle \mathrm{d_t}\, \boldsymbol{m}_h^{i+1}, \boldsymbol{\varphi}_h \rangle_h = -\langle \overline{\boldsymbol{m}}_h^{i+\frac{1}{2}} \times \boldsymbol{h}_h^{i+\frac{1}{2}}, \boldsymbol{\varphi}_h \rangle_h + \alpha \langle \overline{\boldsymbol{m}}_h^{i+\frac{1}{2}} \times \mathrm{d_t}\, \boldsymbol{m}_h^{i+1}, \boldsymbol{\varphi}_h \rangle_h$
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- Solve non-linear system for every time-step $\rightsquigarrow$ fixed-point iteration
- Evaluate $\boldsymbol{\pi}_h(\cdot)$ in every iteration $\rightsquigarrow$ FEM-BEM problem

Bartels, Prohl: *SIAM J. Numer. Anal.*, 44, 2006

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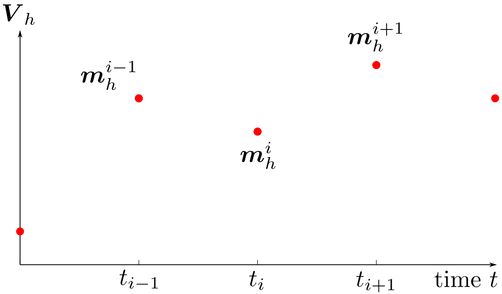
Main result

Interpolation in time

- Input: Sequence of approximations $(\boldsymbol{m}_h^i)_{i=0}^M$



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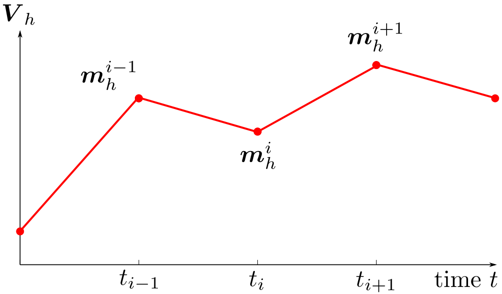

- Output: $\boldsymbol{m}_{hk} \in \boldsymbol{H}^1(\Omega_T)$

Main result

- Assumptions:
  - $\boldsymbol{m}_h^0 \to \boldsymbol{m}^0$ in $\boldsymbol{H}^1(\Omega)$ as $h \to 0$
  - $\boldsymbol{\pi}_h(\boldsymbol{m}_h^{i+1}, \boldsymbol{m}_h^i, \boldsymbol{m}_h^{i-1}) \to \boldsymbol{\pi}(\boldsymbol{m}(t_i))$ in $\boldsymbol{L}^2(\Omega)$ as $h \to 0$

Theorem (Praetorius, Ruggeri, S.' 16)

Ex. $\boldsymbol{m} \in \boldsymbol{H}^1(\Omega_T)$ s.t.

- ex. subsequence with $\boldsymbol{m}_{hk} \rightharpoonup \boldsymbol{m}$ in $\boldsymbol{H}^1(\Omega_T)$ unconditionally
- $\boldsymbol{m}$ is a weak solution of LLG

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- Other couplings: Maxwell-LLG (?), Spin-Diffusion-LLG (?)

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Conclusion

- Evaluate $\boldsymbol{\pi}_h \rightsquigarrow$ BEM-problem has to be solved
- New approach: $\boldsymbol{\pi}_h : \boldsymbol{V}_h^3 \to \boldsymbol{V}_h$
  - Adams-Bashforth: (formally) second order in time
- Unconditionally convergent numerical integrator
- Same technique on ELLG $\rightsquigarrow$ Decoupled system

Thanks for listening!

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Weak solution of LLG

LLG equation

- $\partial_t \boldsymbol{m} = -\boldsymbol{m} \times (C_{\mathrm{ex}} \Delta \boldsymbol{m} + \boldsymbol{f} + \boldsymbol{\pi}(\boldsymbol{m})) + \alpha \boldsymbol{m} \times \partial_t \boldsymbol{m}$

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Weak solution

- $\boldsymbol{m} \in \boldsymbol{H}^1(\Omega_T)$, $|\boldsymbol{m}| = 1$ a.e. in Ω_T
- $\boldsymbol{m}(0) = \boldsymbol{m}^0$
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- $\boldsymbol{m} \in \boldsymbol{L}^\infty(\boldsymbol{H}^1(\Omega))$ + energy estimate =: physical weak solution