Adaptive non-symmetric coupling of Finite Volume and Boundary Element Method

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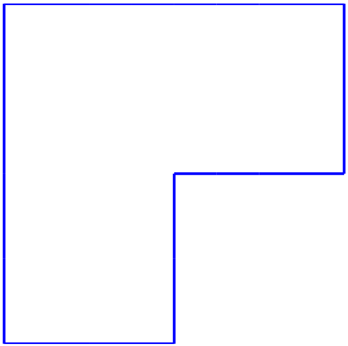
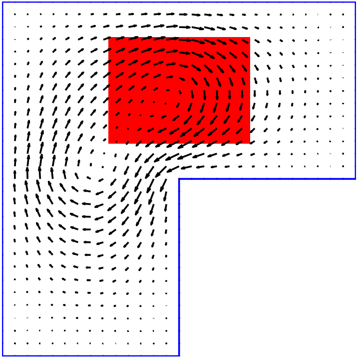
Motivation



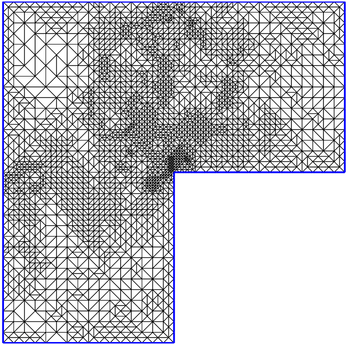
Interior domain Ω and complement Ω_e (boundary Γ)

- Example: flow on domain with reentrant corner

Motivation

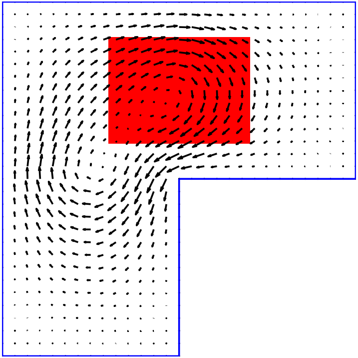


Vector field and source



Adaptively generated mesh

Motivation



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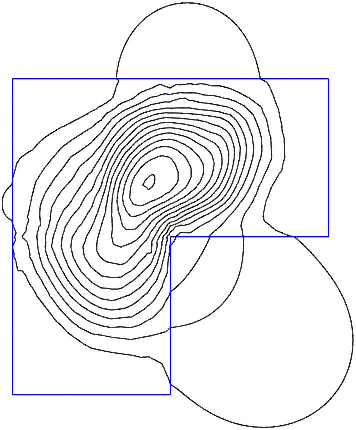
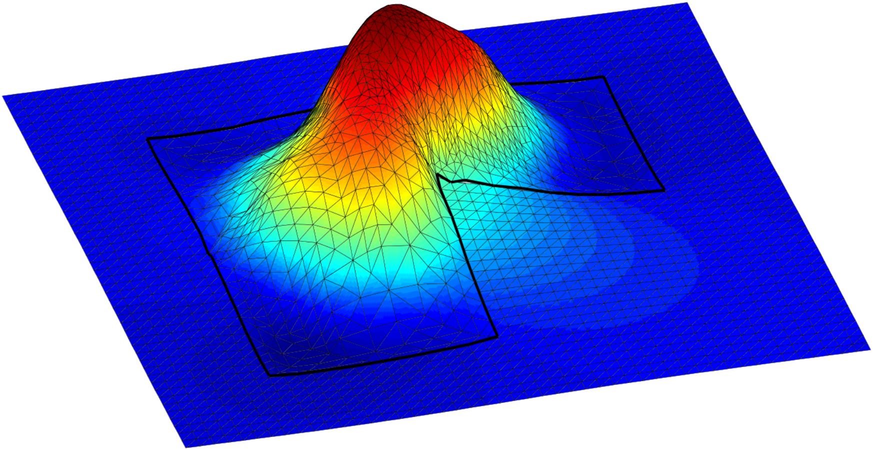


Contour lines

Computed solution



Requirements on our numerical method

Model problem for transport and flow in porous media

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Interior problem:

- Stable solution for convection dominated problems
- Flux conservation

$\Rightarrow$ Finite Volume Method

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Interior problem:

- Stable solution for convection dominated problems
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$\Rightarrow$ Finite Volume Method

Exterior problem:

- Treat unbounded domains without truncation

$\Rightarrow$ Boundary Element Method

Then: couple these methods at the boundary.

Model problem (2D)

Find $u \in H^1(\Omega)$ and $u_e \in H^1_{\ell oc}(\Omega_e)$ such that

$$\operatorname{div}(-\mathbf{A}\nabla u + \mathbf{b}u) + cu = f \qquad \text{in } \Omega,$$

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\operatorname{div}(-\mathbf{A}\nabla u + \mathbf{b}u) + cu &= f && \text{in } \Omega, \\
-\Delta u_e &= 0 && \text{in } \Omega_e, \\
u_e(x) &= C_\infty \log|x| + \mathcal{O}(1/|x|) && \text{for } |x| \to \infty,
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u &= u_e && \text{on } \Gamma,\\
(\mathbf{A}\nabla u - \mathbf{b}u)\cdot\mathbf{n} &= \frac{\partial u_e}{\partial \mathbf{n}} && \text{on } \Gamma^{in} \quad (\mathbf{b}\cdot\mathbf{n} < 0),\\
(\mathbf{A}\nabla u)\cdot\mathbf{n} &= \frac{\partial u_e}{\partial \mathbf{n}} && \text{on } \Gamma^{out} \quad (\mathbf{b}\cdot\mathbf{n} \geq 0).
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$\mathbf{A}$... s.p.d. diffusion matrix, $\mathbf{b}$... convectional velocity field,
c... reaction function, f... source term,
$C_\infty \in \mathbb{R}$... radiation constant, $\frac{1}{2}\operatorname{div} b + c \geq 0$

Well-posedness: [Erath, SINUM 2012]

Representation formula for exterior problem

The solution u_e of the exterior can be represented as (for $x \in \Omega_e$)

$$u_e(x) = -\int_\Gamma G(x-y)\frac{\partial u_e}{\partial \mathbf{n}}(y)\,ds_y + \int_\Gamma \frac{\partial_y}{\partial \mathbf{n}} G(x-y) u_e(y)\,ds_y,$$

with the fundamental solution of the Laplacian $-\Delta$

$$G(x-y) = \begin{cases} -\frac{1}{2\pi}\log|x-y| & \text{for } d=2, \\ \frac{1}{4\pi}\frac{1}{|x-y|} & \text{for } d=3. \end{cases}$$

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Taking traces, the representation formula becomes ($\phi := \partial u_e / \partial \mathbf{n}$)

$$u_e|_\Gamma = -\mathcal{V}(\phi) + (1/2 + \mathcal{K})\,(u_e|_\Gamma)$$

$\mathcal{V}$... single layer operator, $\mathcal{K}$... double layer operator.

Weak coupling formulation

Combine weak formulation of interior problem with Galerkin approach to integral equation:

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Find $u \in H^1(\Omega)$, $\phi \in H^{-1/2}(\Gamma)$ such that

$$\begin{aligned} \mathcal{A}(u,v) - \langle \phi, v\rangle_\Gamma &= (f,v)_\Omega, \\ \langle \psi, (1/2 - \mathcal{K})u\rangle_\Gamma + \langle \psi, \mathcal{V}\phi\rangle_\Gamma &= 0 \end{aligned}$$

for all $v \in H^1(\Omega)$, $\psi \in H^{-1/2}(\Gamma)$, where

$$\mathcal{A}(u,v) := (\mathbf{A}\nabla u - \mathbf{b}u, \nabla v)_\Omega + (cu, v)_\Omega + \langle \mathbf{b}\cdot\mathbf{n}\, u, v\rangle_{\Gamma^{out}}$$

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[Johnson, Nédélec, Math. Comp 1980] (for smooth boundary)
[Sayas, SINUM 2009] (for Lipschitz boundary, purely diffusive)
[Erath, Of, Sayas, Numer. Math. 2016] (this case)

Finite Volume Method

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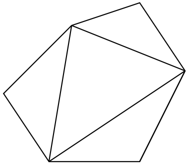
Here: vertex-centered FVM

- $u_h \in \mathcal{S}^1(\mathcal{T})$ (piecewise affine linear functions)

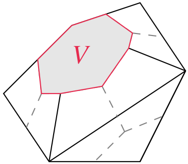


Primal mesh $\mathcal{T}$

Finite Volume Method

Here: vertex-centered FVM

- $u_h \in \mathcal{S}^1(\mathcal{T})$ (piecewise affine linear functions)
- Use Gaussian theorem over control volume V



Dual mesh $\mathcal{T}^*$

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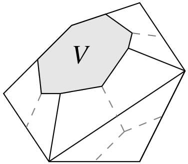

Dual mesh $\mathcal{T}^*$

$$\int_V f\,dx = \int_V \operatorname{div}(-\mathbf{A}\nabla u_h + \mathbf{b}u_h) + cu_h\,dx$$

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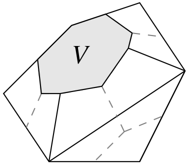

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$$\int_V f\,dx = \int_V \operatorname{div}(-\mathbf{A}\nabla u_h + \mathbf{b}u_h) + cu_h\,dx$$
$$= \int_{\partial V} (-\mathbf{A}\nabla u_h + \mathbf{b}u_h)\cdot\mathbf{n}\,ds + \int_V cu_h\,dx$$

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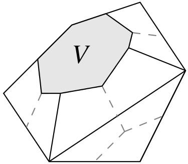

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&= \int_{\partial V\setminus\Gamma} (-\mathbf{A}\nabla u_h + \mathbf{b}u_h)\cdot\mathbf{n}\,ds + \int_{\partial V\cap\Gamma} -(\mathbf{A}\nabla u_h - \mathbf{b}u_h)\cdot\mathbf{n}\,ds \\
&+ \int_V cu_h\,dx
\end{aligned}
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FVM-BEM coupling

Find

- $u_h \in \mathcal{S}^1(\mathcal{T})$
- $\phi_h \in \mathcal{P}^0(\mathcal{E}_\Gamma)$ (piecewise constant functions on boundary mesh)

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for all $V \in \mathcal{T}^*$ and $\psi_h \in \mathcal{P}^0(\mathcal{E}_\Gamma)$.

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$$\langle \psi_h, (1/2 - \mathcal{K}) u_{eh} \rangle_\Gamma + \langle \psi_h, \mathcal{V}\phi_h \rangle_\Gamma = 0,$$

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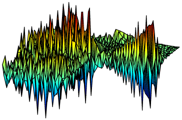
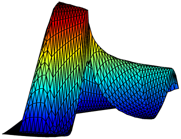
Existence, uniqueness, convergence under some model restrictions: [Erath, Of, Sayas, Numer. Math. 2016] (For smooth enough input: $\mathcal{O}(h)$)

Upwind stabilization

How do we get stable solutions for convection dominated problems?
→ Upwind stabilization





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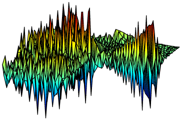
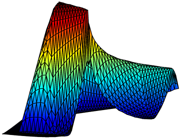
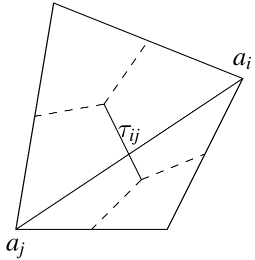




In the term $\int_{\partial V \setminus \Gamma} \mathbf{b} \cdot \mathbf{n} u_h$ replace u_h along τ_{ij} by

$$u_{h,ij} := \lambda_{ij} u_h(a_i) + (1 - \lambda_{ij}) u_h(a_j),$$

λ_{ij} depends on $\mathbf{A}$, $\mathbf{b}$, τ_{ij} and $|\tau_{ij}|$.



Residual a posteriori error estimator

Use robust computable error estimator $\eta(u_h)$ to steer an adaptive algorithm

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- Efficiency and Reliability:

$$C_{\text{eff}}\eta(u_h) \leq |||u - u_h|||_\Omega + \|\phi - \phi_h\|_{H^{-1/2}(\Gamma)} \leq C_{\text{rel}}\eta(u_h)$$

- Measure error in energy (semi-)norm for robustness:

$$|||v|||_\Omega^2 := \|\mathbf{A}^{1/2}\nabla v\|_{L^2(\Omega)}^2 + \|\left(\frac{1}{2}\operatorname{div}\mathbf{b} + c\right)^{1/2} v\|_{L^2(\Omega)}^2$$

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- Element-wise contributions:

$$\eta = \left(\sum_{T\in\mathcal{T}} \eta_T^2\right)^{1/2}$$

Residual a posteriori error estimator II

$$\eta_T^2 := \eta_T^R + \eta_T^{J_i} + \eta_T^{J_e} + \eta_T^{BEM} + \eta_T^{Up},$$

where

- η_T^R... residual contribution $(\mu_T^2 \|f - \operatorname{div}(-\mathbf{A}\nabla u_h + \mathbf{b}u_h) - cu_h\|^2_{L^2(T)})$
- $\eta_T^{J_i}$... jumps in the interior $(\alpha_E^{-1/2} \min\{h_E \alpha_E^{-1/2}, \beta_E^{-1/2}\} \| [(-\mathbf{A}\nabla u_h)|_{E,T} - (-\mathbf{A}\nabla u_h)|_{E,T'}] \cdot \mathbf{n}\|^2_{L^2(E)})$
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Additionally, if an upwind stabilization is used:

- η_T^{Up}... measures upwind error on edges of the dual mesh

Reliability and Efficiency

Theorem (Reliability)

Under some restrictions on the eigenvalues of $\mathbf{A}$ *there holds:*
$|\!|\!| u - u_h |\!|\!|_\Omega + \|\phi - \phi_h\|_{H^{-1/2}(\Gamma)} \leq C_{\mathrm{rel}}\eta.$

C_{rel} is robust (indep. of the number of elements or variaton of the model data).

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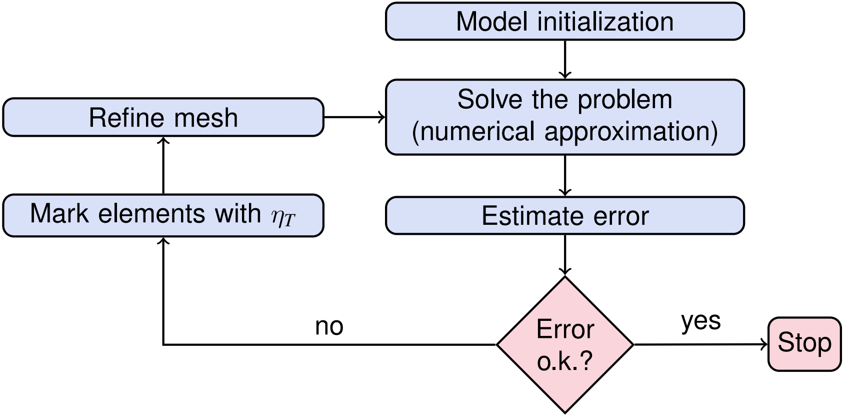
Theorem (Efficiency)

If the mesh on the boundary is quasi-uniform:
η *is also a lower bound (up to higher order terms and a constant).*

The constant is semirobust (additionally depends on the local Péclet number $\alpha_T^{-1}\|\mathbf{b}\|_{L^\infty(T)}h_T$).

Proofs: [Erath, S., Preprint 2016], robust estimates from [Erath, SINUM, 2013]

Adaptive Algorithm



Dörfler marking ($0 \leq \theta \leq 1$):

Find minimal subset $\widetilde{\mathcal{T}} \subset \mathcal{T}$ such that $\sum_{T \in \widetilde{\mathcal{T}}} \eta_T^2 \geq \theta \sum_{T \in \mathcal{T}} \eta_T^2$.

Numerical example with analytical solution

Setup for numerical example:

- L-shaped domain
- $\mathbf{A} = \begin{pmatrix} 10 + \cos x_1 & 160\, x_1 x_2 \\ 160\, x_1 x_2 & 10 + \sin x_2 \end{pmatrix}$
- No convection, reaction
- Prescribed solution in Ω (polar coordinates):

$$u(x_1, x_2) = r^{2/3} \sin(2\varphi/3)$$

- Solution in Ω_e:

$$u_e(x_1, x_2) = \log \sqrt{(x_1 + 0.125)^2 + (x_2 - 0.125)^2}$$

- Calculate right hand side f (and jumps) accordingly

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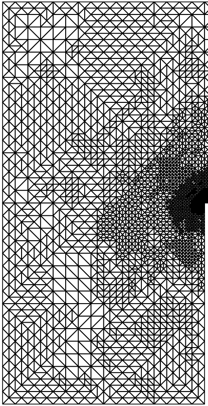

Figure: Adaptively generated mesh $\mathcal{T}^{(6)}$ with 6314 elements.

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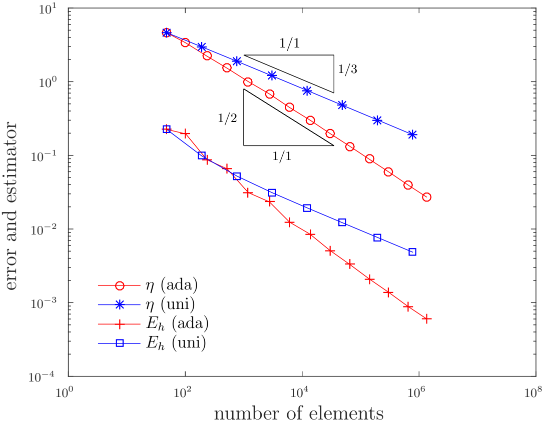

Figure: Convergence plot, the adaptive strategy yields a higher convergence rate.

A more practical example

Setup for numerical example:

- L-shaped domain
- $\mathbf{A} = \alpha \mathbf{I}$, $\alpha = \begin{cases} 0.5 & \text{for } x_1 > 0, \\ 10 & \text{for } x_2 \leq 0, \\ 50 & \text{else,} \end{cases}$
- $\mathbf{b} = (15000, 10000)^T$,
- $c = 0.01$,
- $f(x_1, x_2) = \begin{cases} 50 & \text{for } -0.2 \leq x_1 \leq -0.1, \quad -0.2 \leq x_2 \leq -0.05, \\ 0 & \text{else,} \end{cases}$
- t_0 and u_0 set to zero.

(and radiation condition $u_e(x) = a_\infty + \mathcal{O}(1/|x|)$ for $|x| \to \infty$.)

A more practical example

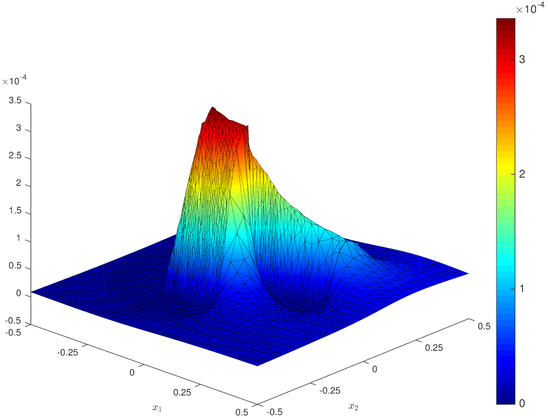

Figure: Computed solution on adaptive mesh with 3471 elements.

A more practical example

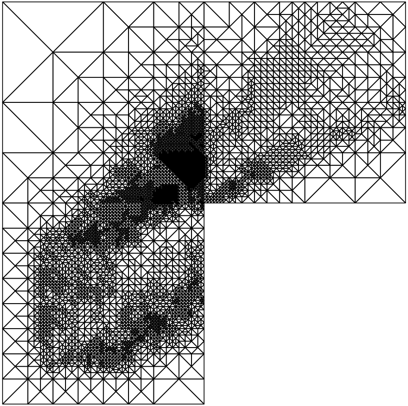

Figure: Adaptively generated mesh $\mathcal{T}^{(6)}$ with 8451 elements.

A more practical example

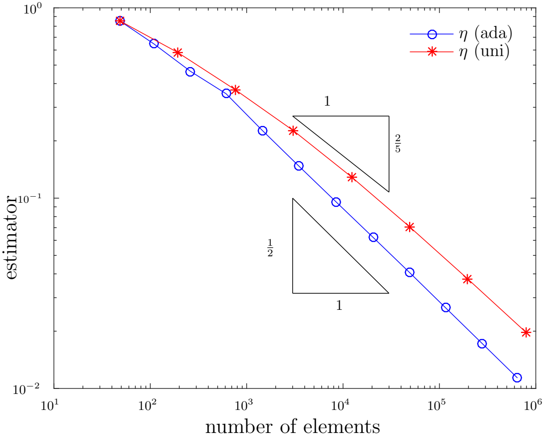

Figure: Convergence plot, the adaptive strategy yields a higher convergence rate.

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Thank you for your attention!

C. Erath and R. Schorr: An adaptive non-symmetric finite volume and boundary element coupling method for a fluid mechanics interface problem.
Preprint (2016), available online: `http://arxiv.org/abs/1605.07031`.

C. Erath, G. Of, and F.-J. Sayas: A non-symmetric coupling of the finite volume method and the boundary element method.
To appear in: Numer. Math., available online: `http://arxiv.org/abs/1509.00440`.

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