Computational Performance of Controllability Techniques for Time-Periodic Waves Using Discrete Exterior Calculus

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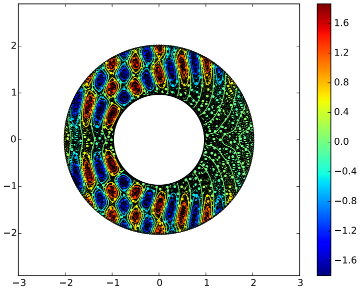
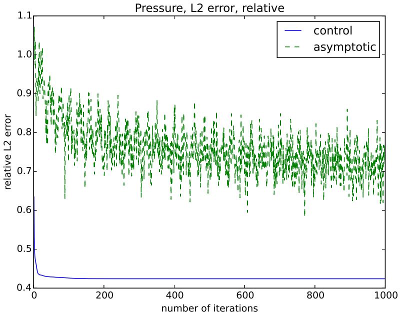
Wave propagation

- Phenomenon: propagation of time-harmonic waves.
- Analysis of time-harmonic waves in time-domain (not in frequency domain).
- As the time-domain analysis often leads to large problems, we accelerate the solution process via so-called controllability methods (Bristeau, Glowinski, and Periaux 1993, 1998).





Controllability techniques

- Controllability methods by Glowinki et al. were first formulated basing on variational methods.
- For a scalar wave equation in a mixed formulation, controllability algorithm was proposed by Glowinski and Rossi in 2006 (variational formulation).
- Theoretical framework on the controllability techniques has further been discussed by Pauly and Rossi in 2011, generalizing the theory for a generalized Maxwell's equation in the context of *differential forms.*
- Discretization of differential form formulation is naturally done with discrete exterior calculus (DEC) (Hirani 2003, doctoral dissertation).
- Finite element method (FEM) with Raviart-Thomas type elements provides natural discretization for the variational formulation (of the mixed wave problem).

Wave problem in a mixed formulation

- Exterior scalar wave problem with first-order approximation of absorbing boundary:

$$c^{-2}\frac{\partial v}{\partial t} - \text{div } \mathbf{p} = 0\,, \qquad \text{in} \quad \Omega \times (0,T)\,, \tag{1}$$

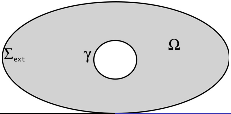
$$\frac{\partial \mathbf{p}}{\partial t} - \text{grad } v = 0\,, \qquad \text{in} \quad \Omega \times (0,T)\,, \tag{2}$$

$$v = g\,, \qquad \text{on} \quad \gamma \times (0,T)\,, \tag{3}$$

$$c^{-1}v + \mathbf{p}\cdot\mathbf{n} = 0\,, \qquad \text{on} \quad \Sigma_{\text{ext}} \times (0,T)\,, \tag{4}$$

$$v(0) = v(T)\,, \quad \mathbf{p}(0) = \mathbf{p}(T) \tag{5}$$

where $T > 0$ is the time period, Ω is a bounded domain in $\mathbb{R}^k$, γ is the boundary on the obstacle, and Σ_{ext} is the external boundary.
- Further on, v is a scalar function, $\mathbf{p}$ is a vector function, c is a known scalar parameter (propagation velocity of wave), g is a known source on γ and $\mathbf{n}$ is an outward unit normal vector.



Notice

We have also directly written the scalar wave problem in a mixed formulation, (1-5). Notice, that writing

$$v = \frac{\partial w}{\partial t} \, , \quad \mathbf{p} = \text{grad } w \, , \tag{6}$$

we may reduce (1-5) to a (classical formulation of) wave problem with respect to variable function w.

This presentation

- This presentation discusses computational performance of a controllability algorithm (Glowinski) for time-periodic solutions of a scalar wave equation (linear acoustics).
- Since, method used in spatial discretization contributes the performance of the controllability algorithm, this presentation addresses comparison of two discretizations,
  - the first one being based on FEM and
  - the second one on DEC (aka discrete differential forms).
  - The numerical performance of the mentioned controllability techniques for a time-periodic two-dimensional scalar wave equation has been studied with finite element spatial discretization (Kähkönen et al. 2011).
- The discussion concentrates on scattering of a plane wave in a two-dimensional setup paying also attention on sensitivity of the numerical solution on quality of the computation grid.

Spatial discretization with FEM

In FEM-discretized form, the equations read

$$\int_\Omega (c^{-2} \frac{\partial v_h}{\partial t} - \text{div } \mathbf{p}_h) \, w_h \, \mathrm{d}x = 0$$

$$\int_\Omega (\frac{\partial \mathbf{p}_h}{\partial t} \cdot \mathbf{q}_h + v_h \cdot \text{div } \mathbf{q}_h) \, \mathrm{d}x + c \int_{\Sigma_{ext}} \mathbf{p}_h \cdot \mathbf{n} \, \mathbf{q}_h \cdot \mathbf{n} \, \mathrm{d}s = \int_\gamma g \, \mathbf{q}_h \cdot \mathbf{n} \, \mathrm{d}s$$

where $v_h, w_h \in V_h$, $\mathbf{p}_h, \mathbf{q}_h \in P_h$,

$$V_h = \left\{ v \in L^2(\Omega) : T \in \mathscr{T}, v|_T \in P_0(T) \right\},$$

$$P_h = \{ p \in H(\text{div}; \Omega) : \mathbf{q}|_T \in RT_1(T) \},$$

and RT_1 is the lowest order Raviart-Thomas element. $\mathscr{T}$ is the finite element triangulation.

Spatial discretization with DEC

In DEC-discretized form, the equations read

$$\frac{\partial v}{\partial t} - c^2 \mathbb{M}_k \mathbb{D}_{k-1} f = 0\,, \qquad \text{in} \quad \Omega_h \times (0,T)\,, \tag{7}$$

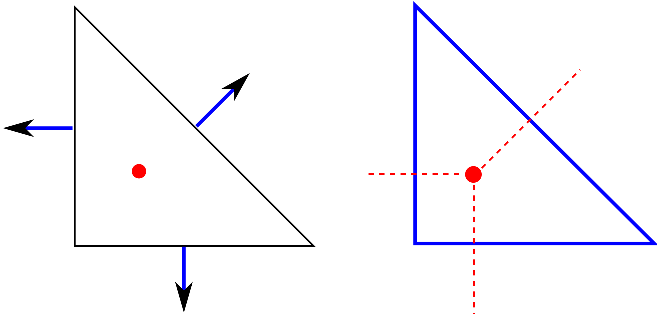
$$\mathbb{M}_{k-1} \frac{\partial f}{\partial t} - (-1)^k \mathbb{D}_{k-1}^{\mathsf{T}} v = 0\,, \qquad \text{in} \quad \Omega_h \times (0,T)\,, \tag{8}$$

$$v = g \qquad \text{on} \quad \gamma_h \times (0,T)\,, \tag{9}$$

$$v + \mathbb{C}\, f = 0\,, \qquad \text{on} \quad \Sigma_{\mathrm{ext},h} \times (0,T)\,,. \tag{10}$$

where v is associated with discrete 0-forms (on dual mesh) and f (flux) is associated with discrete $(k-1)$-forms (primal mesh), $f = \left\{ \int_{l_j} \mathbf{p} \cdot \mathbf{n}\, dl \right\}_j$,
$\mathbb{M}_\ell$ is the discrete Hodge star that maps discrete primal ℓ-forms to corresponding dual $(k-\ell)$-forms. $\mathbb{D}_\ell$ is discrete exterior derivative operating on discrete ℓ-forms.
(Relation $\mathbb{D}_0^{dual} = -(1)^k \mathbb{D}_{k-1}^{\mathsf{T}}$ has been used.) Here, k is the spatial dimension (in the examples, $k=2$);
$\mathbb{C} = c\, \mathbb{M}_{k-1,\mathrm{bound.}}$ and $\mathbb{M}_{k-1,\mathrm{bound.}}$ is now a discrete Hodge star that maps *boundary* primal $(k-1)$-forms to dual 0-forms.

Degrees of freedom in FEM and in DEC



Time discretization

Time discretization is done in a leapfrog manner. Magnitude v is discretized on the set $\{t_n\}_{n=0}^N$ and magnitude f is discretized on the set $\{t_{n+1/2}\}_{n=0}^N$.
As we assume time-harmonicity from the solutions, time-stepping schemes can be written as exact schemes:

$$f^{n+1} = \frac{f^{n+1/2} + f^{n+3/2}}{2\cos(\omega\Delta t/2)} \quad (11)$$

$$\left(\frac{\partial f}{\partial t}\right)^{n+1} = \frac{f^{n+3/2} - f^{n+1/2}}{\frac{2}{\omega}\sin(\omega\Delta t/2)} \quad (12)$$

$$v^{n+1/2} = \frac{v^{n} + v^{n+1}}{2\cos(\omega\Delta t/2)} \quad (13)$$

$$\left(\frac{\partial v}{\partial t}\right)^{n+1/2} = \frac{v^{n+1} - v^{n}}{\frac{2}{\omega}\sin(\omega\Delta t/2)} \quad (14)$$

Discrete problem

- Discrete problem after space and time discretizations will be as

$$\mathbb{A}v^{n+1} = \mathbb{A}v^{n} + \mathbb{B}f^{n+1/2} \tag{15}$$

$$\mathbb{C}f^{n+3/2} = \mathbb{C}f^{n+1/2} + \mathbb{D}v^{n+1} \tag{16}$$

  where (15) is exactly same equation for both FEM and DEC discretizations used. Furthermore, $\mathbb{A}$ is a diagonal matrix and thus its inverse matrix is directly accessible.
- In DEC discretization, also $\mathbb{C}$ is diagonal (if diagonal Hodge star is used).
- For FEM, $\mathbb{C}$ does not become diagonal, and thus for each time step, a linear equation system needs to be solved.
- Clearly, matrix multiplication is cheaper operation than solving a linear equation, which makes DEC much more efficient. But will there be problems with accuracy?

Conjugate Gradient Method

We are to minimize a functional

$$J(u_1, u_2) = \frac{1}{2}\left[c^{-2}\|v(T) - u_1\|^2_{L^2(\Omega)} + \|f(T) - u_2\|^2_{L^2(\Omega)}\right] \qquad (17)$$

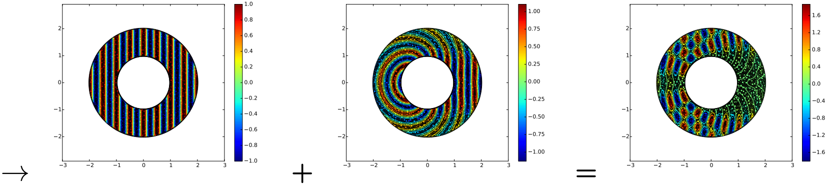
i.e. the difference between the initial conditions of the time-dependent problem and the corresponding variables after one time period.
The least squares minimization is performed by the conjugate gradient (CG) method. The gradient needed by the CG-method is gained by solving the adjoint problem.

Implementation and considered example

- In the implementation of the algorithm (spatial discretizations), we utilized PyDEC software (published by Bell and Hirani (2012)) and FEniCS software.
- As an example, we consider a circular scatterer with "sound-soft" boundary, and solve numerically the scattered wave as the incident wave is supposed to be given.
- As incident wave, we used plane wave with expression

$$v_{\text{in}} = \Re(e^{i(kx-\omega t)}) = \cos(kx - \omega t) \tag{18}$$

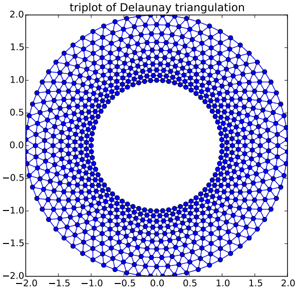
Circular scatterer

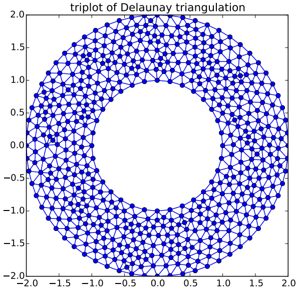


- The analytical solution of the scattered plane wave for a circular scatterer can be expressed with series solution of Bessel and Hankel functions in unbounded domain.
- Since the numerical solution is computed with first order artificial boundary condition, the approximative boundary condition produces a modelling error.

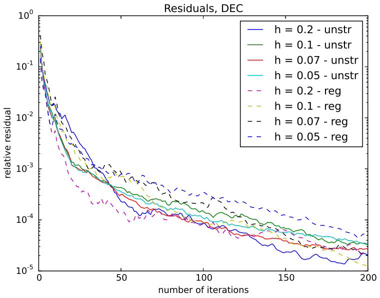
Different meshes

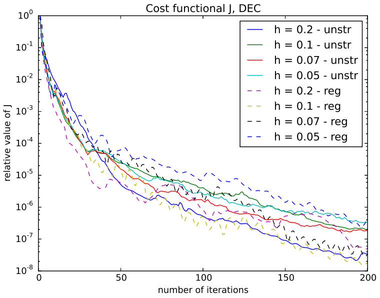
Regular and unstructured



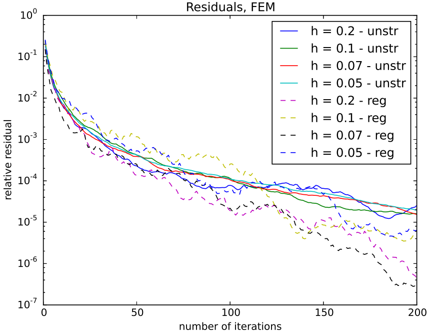


Convergence (DEC)

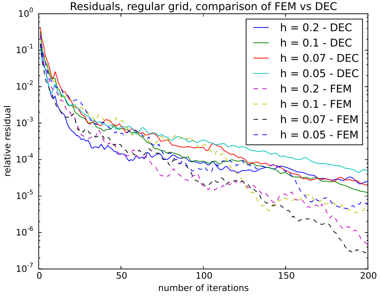


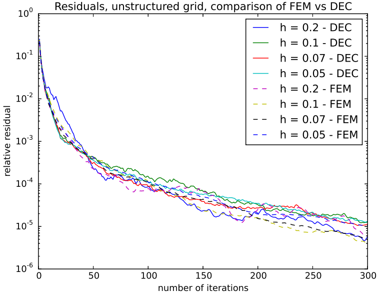


Convergence (FEM)



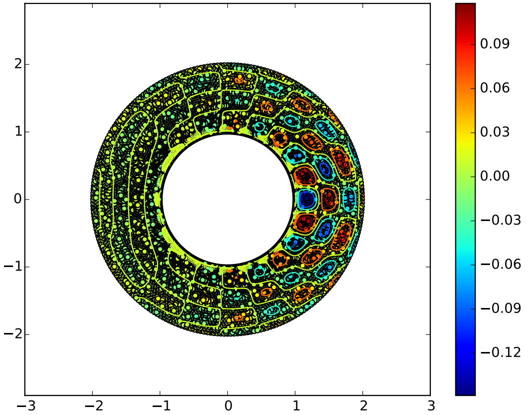
Convergence, comparison





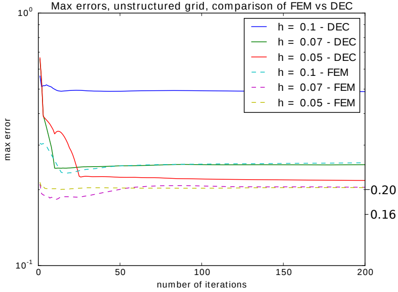
Error of function v

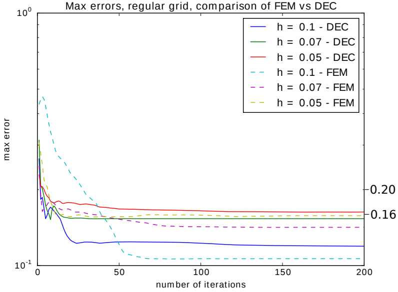
Numerical vs analytical



Max error for function v

Numerical vs analytical





Summary

- DEC and FEM spatial discretizations give approximately same rate of convergence in the controllability algorithm.
- In DEC, the grid quality significantly affects the error of the solution, as expected.
  - The same holds also for FEM.
- Efficiency: DEC leads to discretization, where one only needs to operate with matrix multiplications by each time step, but in FEM (with lowest order Raviart-Thomas elements combined with piecewise constant elements) we are to solve a linear equation system with each time step.

For Further Reading I

- M. O. Bristeau, R. Glowinski, J. Periaux 1993. Numerical Simulation of High Frequency Scattering Waves Using Exact Controllability Methods. Nonlinear Hyperbolic Problems: Theoretical, Applied, and Computational Aspects, Volume 43 of the series Notes on Numerical Fluid Mechanics (NNFM) pp 86-108, Vieweg+Teubner Verlag.
- M. O. Bristeau, R. Glowinski, J. Periaux 1998. Controllability Methods for the Computation of Time-Periodic Solutions; Application to Scattering. Journal of Computational Physics, Volume 147, Issue 2, 10 December 1998, Pages 265–292.

For Further Reading II

- R. Glowinski, and T. Rossi 2006. A mixed formulation and exact controllability approach for the computation of the periodic solutions of the scalar wave equation. (I) controllability problem formulation and related iterative solution. C. R. Math. Acad. Sci. Paris, 343(7):493–498.
- D. Pauly and T. Rossi 2011. Theoretical considerations on the computation of generalized time-periodic waves. Advances in Mathematical Sciences and Applications, 21(1):105–131.
- A. N. Hirani 2003. Discrete Exterior Calculus. PhD thesis, California Institute of Technology.

For Further Reading III

Sami Kähkönen, Roland Glowinski, Tuomo Rossi, and Raino Mäkinen. Solution of time-periodic wave equation using mixed finite-elements and controllability techniques. Journal of Computational Acoustics, 19(4):335–352, 2011.