Wellposedness for A Thermo-Piezo-Electric Coupling Model.

AANMPDE 2016, Strobl, Austria

Rainer Picard
(joint work with T. Mulholland, S. Trostorff & M. Waurick)

Technische Universität Dresden, Germany

The General Shape of Evolutionary Equations.

General Form: $\partial_0 \equiv \frac{\partial}{\partial t} \equiv \partial_t$

$$\partial_0 V + AU = f \text{ on }]0,\infty[,$$
$$V(0+) = \Phi,$$

in a suitable Hilbert space setting.

Without much loss of generality: $\Phi = 0$. Thus

$$\partial_0 \mathscr{M} U + AU = f \text{ on } \mathbb{R}. \qquad (1)$$

Evolutionary Equation in a simple, standard case: $\mathscr{M} = M_0 + \partial_0^{-1} M_1$ and A skew-selfadjoint,

$$(\partial_0 M_0 + M_1 + A)\, U = f.$$

The General Shape of Evolutionary Equations.

General Form: $\partial_0 \equiv \frac{\partial}{\partial t} \equiv \partial_t$

$$\partial_0 V + AU = f \text{ on }]0,\infty[,$$
$$V(0+) = \Phi,$$

in a suitable Hilbert space setting.

Without much loss of generality: $\Phi = 0$. Thus

$$\partial_0 \mathscr{M} U + AU = f \text{ on } \mathbb{R}. \quad (1)$$

Evolutionary Equation in a simple, standard case: $\mathscr{M} = M_0 + \partial_0^{-1} M_1$ and A skew-selfadjoint,

$$(\partial_0 M_0 + M_1 + A)\, U = f.$$

The General Shape of Evolutionary Equations.

General Form: $\partial_0 \equiv \frac{\partial}{\partial t} \equiv \partial_t$

$$\partial_0 V + AU = f \text{ on }]0,\infty[,$$

$$V(0+) = \Phi,$$

in a suitable Hilbert space setting.

Without much loss of generality: $\Phi = 0$. Thus

$$\partial_0 \mathscr{M} U + AU = f \text{ on } \mathbb{R}. \tag{1}$$

Evolutionary Equation in a simple, standard case: $\mathscr{M} = M_0 + \partial_0^{-1} M_1$ and A skew-selfadjoint,

$$(\partial_0 M_0 + M_1 + A)\, U = f.$$

The Time Derivative

Solution Theory: Does the operator

$$\left(\partial_0 \mathscr{M} + A\right)^{-1}$$

exist as a continuous linear mapping on a suitable Hilbert space?

Which *"suitable"* Hilbert space?

A weighted L^2-space $H_\rho(\mathbb{R}, H)$ constructed by completion of the space $\mathring{C}_1(\mathbb{R}, H)$ of differentiable H-valued functions with compact support w.r.t. $\langle \cdot | \cdot \rangle_{\rho,H}$ (norm: $|\cdot|_{\rho,H}$)

$$(\varphi, \psi) \mapsto \int_{\mathbb{R}} \langle \varphi(t) | \psi(t) \rangle_H \exp(-2\rho t)\, dt.$$

Time-differentiation ∂_0 as a closed operator in $H_\rho(\mathbb{R}, H)$ induced by

$$\mathring{C}_1(\mathbb{R}, H) \subseteq H_\rho(\mathbb{R}, H) \to H_\rho(\mathbb{R}, H),$$
$$\varphi \mapsto \varphi'.$$

The Time Derivative

Solution Theory: Does the operator

$$\left(\partial_0 \mathscr{M} + A\right)^{-1}$$

exist as a continuous linear mapping on a suitable Hilbert space?

Which *"suitable"* Hilbert space?

A weighted L^2-space $H_\rho(\mathbb{R}, H)$ constructed by completion of the space $\mathring{C}_1(\mathbb{R}, H)$ of differentiable H-valued functions with compact support w.r.t. $\langle \cdot | \cdot \rangle_{\rho,H}$ (norm: $|\cdot|_{\rho,H}$)

$$(\varphi, \psi) \mapsto \int_{\mathbb{R}} \langle \varphi(t) | \psi(t) \rangle_H \exp(-2\rho t)\, dt.$$

Time-differentiation ∂_0 as a closed operator in $H_\rho(\mathbb{R}, H)$ induced by

$$\begin{aligned} \mathring{C}_1(\mathbb{R}, H) \subseteq H_\rho(\mathbb{R}, H) &\to H_\rho(\mathbb{R}, H), \\ \varphi &\mapsto \varphi'. \end{aligned}$$

The Time Derivative

Solution Theory: Does the operator

$$(\partial_0 \mathscr{M} + A)^{-1}$$

exist as a continuous linear mapping on a suitable Hilbert space?

Which *"suitable"* Hilbert space?

A weighted L^2-space $H_\rho(\mathbb{R}, H)$ constructed by completion of the space $\mathring{C}_1(\mathbb{R}, H)$ of differentiable H-valued functions with compact support w.r.t. $\langle \cdot | \cdot \rangle_{\rho,H}$ (norm: $|\cdot|_{\rho,H}$)

$$(\varphi, \psi) \mapsto \int_{\mathbb{R}} \langle \varphi(t) | \psi(t) \rangle_H \exp(-2\rho t)\, dt.$$

Time-differentiation ∂_0 as a closed operator in $H_\rho(\mathbb{R}, H)$ induced by

$$\begin{aligned} \mathring{C}_1(\mathbb{R}, H) \subseteq H_\rho(\mathbb{R}, H) &\to H_\rho(\mathbb{R}, H), \\ \varphi &\mapsto \varphi'. \end{aligned}$$

The Time Derivative (as a strictly positive-definite operator)

Time-differentiation ∂_0 is a normal operator in $H_\rho(\mathbb{R}, H)$. For $\rho_0 \in]0,\infty[$, $\rho \in]\rho_0,\infty[$, we have

$$\mathfrak{Re}\,\partial_0 = \rho \geq \rho_0 > 0,$$

i.e.

∂_0 is a strictly (and uniformly w.r.t. ρ) positive definite operator

with respect to the real inner product

$$(\phi,\psi) \mapsto \mathfrak{Re}\,\langle\phi|\psi\rangle_{\rho,H}\,.$$

Basic Solution Theory in $H_\rho(\mathbb{R}, H)$

Evolutionary Problem:

$$(\partial_0 M_0 + M_1 + A)U = F \qquad \text{(Evo-Sys)}$$

Key-Question: When is $(\partial_0 M_0 + M_1 + A)$ (and its adjoint) (real) strictly positive definite in $H_\rho(\mathbb{R}, H)$?

Theorem

Let A be skew-selfadjoint and M_0, M_1 be continuous linear operators in H such that M_0 is selfadjoint and

$$\mathfrak{Re}(\partial_0 M_0 + M_1) = \rho M_0 + \mathfrak{Re}\, M_1 \geq c_0 > 0$$

for some $c_0 \in \mathbb{R}$ and all $\rho \in\,]\rho_0, \infty[$, where $\rho_0 \in\,]0, \infty[$ is sufficiently large.
Then well-posedness of (Evo-Sys) *follows for all $\rho \in\,]\rho_0, \infty[$.*

Basic Solution Theory in $H_\rho(\mathbb{R}, H)$

Evolutionary Problem:

$$(\partial_0 M_0 + M_1 + A)U = F \qquad \text{(Evo-Sys)}$$

Key-Question: When is $(\partial_0 M_0 + M_1 + A)$ (and its adjoint) (real) strictly positive definite in $H_\rho(\mathbb{R}, H)$?

Theorem

Let A be skew-selfadjoint and M_0, M_1 be continuous linear operators in H such that M_0 is selfadjoint and

$$\mathfrak{Re}(\partial_0 M_0 + M_1) = \rho M_0 + \mathfrak{Re} M_1 \geq c_0 > 0$$

for some $c_0 \in \mathbb{R}$ and all $\rho \in]\rho_0, \infty[$, where $\rho_0 \in]0, \infty[$ is sufficiently large.

Then well-posedness of (Evo-Sys) follows for all $\rho \in]\rho_0, \infty[$.

Basic Solution Theory in $H_\rho(\mathbb{R}, H)$

Evolutionary Problem:

$$(\partial_0 M_0 + M_1 + A)U = F \qquad \text{(Evo-Sys)}$$

Key-Question: When is $(\partial_0 M_0 + M_1 + A)$ (and its adjoint) (real) strictly positive definite in $H_\rho(\mathbb{R}, H)$?

Theorem

Let A be skew-selfadjoint and M_0, M_1 be continuous linear operators in H such that M_0 is selfadjoint and

$$\mathfrak{Re}\,(\partial_0 M_0 + M_1) = \rho M_0 + \mathfrak{Re}\, M_1 \geq c_0 > 0$$

for some $c_0 \in \mathbb{R}$ and all $\rho \in \,]\rho_0, \infty[$, where $\rho_0 \in \,]0, \infty[$ is sufficiently large.
Then well-posedness of (Evo-Sys) *follows for all $\rho \in \,]\rho_0, \infty[$.*

The Model Equations

Let $\Omega \subseteq \mathbb{R}^3$ be a non-empty open set. The equation of elasticity:

$$\partial_0^2 \rho_* u - \operatorname{Div} T = F_0, \tag{2}$$

here $u : \mathbb{R} \times \Omega \to \mathbb{R}^3$ displacement, $T : \mathbb{R} \times \Omega \to \operatorname{sym}\left[\mathbb{R}^{3\times 3}\right]$ stress tensor, $\rho_* : \Omega \to \mathbb{R}$ mass density, $F_0 : \mathbb{R} \times \Omega \to \mathbb{R}^3$ external force term.

Maxwell's equation:

$$\begin{aligned} \partial_0 B + \operatorname{curl} E &= F_3, \\ \partial_0 D - \operatorname{curl} H &= F_2 - \sigma E. \end{aligned} \tag{3}$$

Here, $B, D, E, H : \mathbb{R} \times \Omega \to \mathbb{R}^3$ are magnetic flux density, electric displacement, electric field and magnetic field, respectively. The functions $F_2, F_3 : \mathbb{R} \times \Omega \to \mathbb{R}^3$ are given source terms and $\sigma : \Omega \to \mathbb{R}$ denotes the resistance.

Boundary Conditions

Heat conduction: $$\partial_0 \Theta_0 \eta + \operatorname{div} q = F_4,$$

where $\eta : \mathbb{R} \times \Omega \to \mathbb{R}$ entropy density, $q : \mathbb{R} \times \Omega \to \mathbb{R}^3$ heat flux, $F_2 : \mathbb{R} \times \Omega \to \mathbb{R}$ external heat source, $\Theta_0 \in]0, \infty[$ reference temperature. Coupling in abstract form

$$\left(\partial_0 M_0 + M_1 + \begin{pmatrix} 0 & -\operatorname{Div} & 0 & 0 & 0 & 0 \\ -\operatorname{Grad} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\operatorname{curl} & 0 & 0 \\ 0 & 0 & \operatorname{curl} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \operatorname{div} \\ 0 & 0 & 0 & 0 & \operatorname{grad} & 0 \end{pmatrix} \right) \begin{pmatrix} v \\ T \\ E \\ H \\ \Theta_0^{-1} \theta \\ q \end{pmatrix} = F,$$

for suitable bounded operators M_0, M_1 on the Hilbert space $H := L^2(\Omega)^3 \oplus \operatorname{sym}\left[L^2(\Omega)^{3\times 3}\right] \oplus L^2(\Omega)^3 \oplus L^2(\Omega)^3 \oplus L^2(\Omega) \oplus L^2(\Omega)^3.$
Here, $v := \partial_0 u$ and $\theta : \mathbb{R} \times \Omega \to \mathbb{R}$ temperature.

Boundary Conditions

Boundary conditions?

We denote by $\mathring{C}_1(\Omega)$ the space of differentiable functions with compact support in Ω. Then we define the operator $\mathring{\operatorname{grad}}$ as the closure of

$$\begin{aligned} \mathring{C}_1(\Omega) \subseteq L^2(\Omega) &\to L^2(\Omega)^3 \\ \phi &\mapsto (\partial_1 \phi, \partial_2 \phi, \partial_3 \phi) \end{aligned}$$

as well as $\mathring{\operatorname{div}}$ as the closure of

$$\begin{aligned} \mathring{C}_1(\Omega)^3 \subseteq L^2(\Omega)^3 &\to L^2(\Omega) \\ (\phi_1, \phi_2, \phi_3) &\mapsto \sum_{i=1}^{3} \partial_i \phi_i. \end{aligned}$$

Boundary Conditions

It is $\mathring{\operatorname{div}} \subseteq -(\mathring{\operatorname{grad}})^*$. We set $\operatorname{div} := -\left(\mathring{\operatorname{grad}}\right)^*$ and $\operatorname{grad} := -\left(\mathring{\operatorname{div}}\right)^*$. Similarly, the operator $\mathring{\operatorname{curl}}$ is the closure of

$$\mathring{C}_1(\Omega)^3 \subseteq L^2(\Omega)^3 \to L^2(\Omega)^3$$
$$(\phi_1, \phi_2, \phi_3) \mapsto \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$$

and $\operatorname{curl} := \left(\mathring{\operatorname{curl}}\right)^* \supseteq \mathring{\operatorname{curl}}$.

Boundary Conditions

Analogously for $\mathring{\text{Grad}}$ and $\mathring{\text{Div}}$ as the closure of

$$\mathring{C}_1(\Omega)^3 \subseteq L^2(\Omega)^3 \to \mathrm{sym}\left[L^2(\Omega)^{3\times 3}\right]$$
$$(\phi_1, \phi_2, \phi_3) \mapsto \frac{1}{2}\left(\partial_j \phi_i + \partial_i \phi_j\right)_{i,j\in\{1,2,3\}}$$

and of

$$\mathrm{sym}\left[\mathring{C}_1(\Omega)^{3\times 3}\right] \subseteq \mathrm{sym}\left[L^2(\Omega)^{3\times 3}\right] \to L^2(\Omega)^3$$
$$(\phi_{ij})_{i,j\in\{1,2,3\}} \mapsto \left(\sum_{j=1}^{3} \partial_j \phi_{ij}\right)_{i\in\{1,2,3\}},$$

respectively. $\text{Grad} := -\left(\mathring{\text{Div}}\right)^*$ $\text{Div} := -\left(\mathring{\text{Grad}}\right)^*$.

Boundary Conditions

For smooth boundary $\partial\Omega$

$$u = 0 \text{ on } \partial\Omega$$

for $u \in D(\mathring{\text{grad}})$ or $u \in D(\mathring{\text{Grad}})$,

$$u \cdot n = 0 \text{ on } \partial\Omega$$

for $u \in D(\mathring{\text{div}})$ or $u \in D(\mathring{\text{Div}})$, where n denotes the exterior unit normal vector field on $\partial\Omega$ and

$$u \times n = 0 \text{ on } \partial\Omega,$$

for $u \in D(\mathring{\text{curl}})$.

Boundary Conditions

We will assume that $v = 0, E \times n = 0$ and $q \cdot n = 0$ on the boundary in the generalized sense. The spatial block operator is replaced by

$$\begin{pmatrix} 0 & -\operatorname{Div} & 0 & 0 & 0 & 0 \\ -\mathring{\operatorname{Grad}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\operatorname{curl} & 0 & 0 \\ 0 & 0 & \mathring{\operatorname{curl}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathring{\operatorname{div}} \\ 0 & 0 & 0 & 0 & \operatorname{grad} & 0 \end{pmatrix},$$

which is now a skew-selfadjoint operator on H.

The Model Equations

The material relations ($\mathscr{E} := \operatorname{Grad} u$ strain tensor)

$$\begin{aligned} T &= C\,\mathscr{E} - eE - \lambda\,\theta, \\ D &= e^{*}\mathscr{E} + \varepsilon E + p\,\theta, \\ B &= \mu\,H, \\ \eta &= \lambda^{*}\mathscr{E} + p^{*}E + \alpha\,\Theta_0^{-1}\,\theta. \end{aligned}$$

Here $C \in L\left(\mathrm{sym}\left[L^2(\Omega)^{3\times 3}\right]\right)$ is the elasticity tensor, $\varepsilon, \mu \in L\left(L^2(\Omega)^3\right)$ are the permittivity and permeability, respectively, $\alpha := \rho_* c \in L(L^2(\Omega))$ is the product of the mass density $\rho_* \in L^\infty(\Omega)$ and the specific heat capacity $c \in L(L^2(\Omega))$ and $\Theta_0 \in]0, \infty[$ reference temperature. The operators $e \in L\left(L^2(\Omega)^3; \mathrm{sym}\left[L^2(\Omega)^{3\times 3}\right]\right)$, $\lambda \in L\left(L^2(\Omega); \mathrm{sym}\left[L^2(\Omega)^{3\times 3}\right]\right)$, $p \in L(L^2(\Omega); L^2(\Omega)^3)$ are "coupling parameters".

The Model Equations

Relative temperature $\Theta_0^{-1}\,\theta$ as new unknown

$$
\begin{aligned}
T &= C\,\mathscr{E} - eE - (\lambda\Theta_0)\,\Theta_0^{-1}\theta,\\
D &= e^{*}\mathscr{E} + \varepsilon E + (p\Theta_0)\;\Theta_0^{-1}\theta,\\
B &= \mu\,H,\\
\Theta_0\eta &= (\Theta_0\lambda^{*})\,\mathscr{E} + (\Theta_0\,p^{*})\,E + \gamma_0\,\Theta_0^{-1}\theta,
\end{aligned}
$$

where we introduced the abbreviation

$$\gamma_0 := \Theta_0\,\alpha.$$

The Model Equations

Maxwell-Cattaneo-Vernotte modification

$$\partial_0 \kappa_1 q + \kappa_0^{-1} q + \operatorname{grad} \theta = 0,$$

for operators $\kappa_0, \kappa_1 \in L(L^2(\Omega)^3)$. To adapt the material relations to our framework we solve for $\mathscr{E}$ and obtain

$$\begin{aligned}
\mathscr{E} &= C^{-1} T + C^{-1} eE + C^{-1} (\lambda \Theta_0) \Theta_0^{-1} \theta, \\
D &= e^* C^{-1} T + \left(\varepsilon + e^* C^{-1} e\right) E + \left(p\Theta_0 + e^* C^{-1} \lambda \Theta_0\right) \Theta_0^{-1} \theta, \\
B &= \mu H, \\
\Theta_0 \eta &= \Theta_0 \lambda^* C^{-1} T + \left(\Theta_0 p^* + \Theta_0 \lambda^* C^{-1} e\right) E + \\
&\qquad + \left(\gamma_0 + \Theta_0 \lambda^* C^{-1} \lambda \Theta_0\right) \Theta_0^{-1} \theta.
\end{aligned}$$

The Model Equations

Material law operators:

$$M_0 := \begin{pmatrix} \rho_* & 0 & 0 & 0 & 0 & 0 \\ 0 & C^{-1} & C^{-1}e & 0 & C^{-1}\lambda\Theta_0 & 0 \\ 0 & e^* C^{-1} & \left(\varepsilon + e^* C^{-1} e\right) & 0 & \left(p\Theta_0 + e^* C^{-1}\lambda\Theta_0\right) & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & \Theta_0\lambda^* C^{-1} & \left(\Theta_0 p^* + \Theta_0\lambda^* C^{-1} e\right) & 0 & \left(\gamma_0 + \Theta_0\lambda^* C^{-1}\lambda\Theta_0\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & \kappa_1 \end{pmatrix}$$

and

$$M_1 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \kappa_0^{-1} \end{pmatrix}.$$

Well-Posedness

Theorem

Assume that $\rho_*, \varepsilon, \mu, C, \gamma_0$ *are selfadjoint and non-negative. Furthermore, we assume* $\rho_*, \mu, C, \gamma_0 \gg 0$ *as well as* $\rho\left(\varepsilon - \Theta_0 p \gamma_0^{-1} p^* \Theta_0\right) + \sigma, \rho\kappa_1 + \kappa_0^{-1} \gg 0$ *("*$\gg 0$*" short for uniformly strictly positive definite) for sufficiently large* $\rho > 0$*. Then,* M_0 *and* M_1 *satisfy the positive definiteness condition and hence, the corresponding problem of thermo-piezo-electricity is well-posed.*

Well-Posedness

Proof.

Obviously, M_0 selfadjoint. Moreover, $\rho_*, \mu, \rho\kappa_1 + \kappa_0^{-1} \gg 0$ for sufficiently large ρ. Left to show

$$\rho \begin{pmatrix} C^{-1} & C^{-1}e & C^{-1}\lambda\Theta_0 \\ e^*C^{-1} & \varepsilon + e^*C^{-1}e & p\Theta_0 + e^*C^{-1}\lambda\Theta_0 \\ \Theta_0\lambda^*C^{-1} & \Theta_0 p^* + \Theta_0\lambda^*C^{-1}e & \gamma_0 + \Theta_0\lambda^*C^{-1}\lambda\Theta_0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{pmatrix} \gg 0$$

for sufficiently large ρ. Congruent to

$$\rho \begin{pmatrix} C^{-1} & 0 & 0 \\ 0 & \varepsilon - \Theta_0 p\gamma_0^{-1}p^*\Theta_0 & 0 \\ 0 & 0 & \gamma_0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The latter operator is then strictly positive definite by assumption and so the assertion follows. □

A "Simplification"

Use electrostatics!

$E = -\mathring{\operatorname{grad}}\varphi$ for a potential $\varphi \in D(\mathring{\operatorname{grad}})$ and $D \in D(\operatorname{div})$ and we set $\psi := \operatorname{div} D$. Moreover, no conductivity term, i.e. $\sigma = 0$ and no magnetic field, then

$$\partial_0 \begin{pmatrix} \rho_* & 0 & 0 & 0 \\ 0 & C^{-1} & C^{-1}\lambda\Theta_0 & 0 \\ 0 & \Theta_0\lambda^* C^{-1} & (\gamma_0 + \Theta_0\lambda^* C^{-1}\lambda\Theta_0) & 0 \\ 0 & 0 & 0 & \kappa_1 \end{pmatrix} \begin{pmatrix} v \\ T \\ \Theta_0^{-1}\theta \\ q \end{pmatrix} +$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \kappa_0^{-1} \end{pmatrix} \begin{pmatrix} v \\ T \\ \Theta_0^{-1}\theta \\ q \end{pmatrix} + \begin{pmatrix} 0 & -\operatorname{Div} & 0 & 0 \\ -\mathring{\operatorname{Grad}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathring{\operatorname{div}} \\ 0 & 0 & \operatorname{grad} & 0 \end{pmatrix} \begin{pmatrix} v \\ T \\ \Theta_0^{-1}\theta \\ q \end{pmatrix} +$$

$$+\partial_0 \begin{pmatrix} \begin{pmatrix} 0 \\ C^{-1} eE \end{pmatrix} \\ \begin{pmatrix} (\Theta_0 p^* + \Theta_0\lambda^* C^{-1} e)\, E \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} F_0 \\ F_1 \\ F_4 \\ F_5 \end{pmatrix}.$$

A "Simplification"

To express E in terms of the other unknowns:

$$D = e^* C^{-1} T + \left(\varepsilon + e^* C^{-1} e\right) E + \left(p\Theta_0 + e^* C^{-1} \lambda \Theta_0\right) \Theta_0^{-1} \theta. \quad (4)$$

Setting

$$\begin{aligned}\Phi &:= e^* C^{-1} T + \left(p\Theta_0 + e^* C^{-1} \lambda \Theta_0\right) \left(\Theta_0^{-1} \theta\right) \\ &= e^* C^{-1} (T + \lambda \theta) + p\theta,\end{aligned}$$

$$D = (\varepsilon + e^* C^{-1} e) E + \Phi.$$

Using now that $\psi = \operatorname{div} D$ and $E = -\mathring{\operatorname{grad}}\varphi$ we get that

$$-\operatorname{div}\left(\varepsilon + e^* C^{-1} e\right) \mathring{\operatorname{grad}}\varphi + \operatorname{div}\Phi = \psi.$$

We assume that C, ε are selfadjoint and $\varepsilon + e^* C^{-1} e \gg 0$ and set $M := \sqrt{\varepsilon + e^* C^{-1} e}$.

A "Simplification"

Then, the latter equality can be written as

$$-\operatorname{div} M^2 \mathring{\operatorname{grad}} \varphi + \operatorname{div} M M^{-1} \Phi = \left| M \mathring{\operatorname{grad}} \right|^2 \varphi + \operatorname{div} M M^{-1} \Phi$$
$$= \psi,$$

which gives

$$\mathring{\operatorname{grad}} \varphi + M^{-1} \left(\left(M \mathring{\operatorname{grad}} \right) \left| M \mathring{\operatorname{grad}} \right|^{-2} \operatorname{div} M \right) M^{-1} \Phi =$$
$$= M^{-1} \left(M \mathring{\operatorname{grad}} \right) \left| M \mathring{\operatorname{grad}} \right|^{-2} \psi,$$

if we assume that $\psi \in D\left(\left| M \mathring{\operatorname{grad}} \right|^{-2} \right)$.

A "Simplification"

This suggests to replace

$$\begin{aligned} E &= -\mathring{\operatorname{grad}}\varphi \\ &= M^{-1}\left(\overline{\left(M\,\mathring{\operatorname{grad}}\right)\left|M\,\mathring{\operatorname{grad}}\right|^{-2}\operatorname{div} M}\right) M^{-1}\Phi + \\ &\quad - M^{-1}\left(M\,\mathring{\operatorname{grad}}\right)\left|M\,\mathring{\operatorname{grad}}\right|^{-2}\psi, \end{aligned}$$

With $P := P_{\overline{M\mathring{\operatorname{grad}}[L^2(\Omega)]}}$

$$E = -M^{-1}PM^{-1}\Phi - M^{-1}\left(M\,\mathring{\operatorname{grad}}\right)\left|M\,\mathring{\operatorname{grad}}\right|^{-2}\psi.$$

A "Simplification"

Hence,

$$\begin{pmatrix} \begin{pmatrix} 0 \\ C^{-1} eE \end{pmatrix} \\ \begin{pmatrix} \left(\Theta_0 p^* + \Theta_0 \lambda^* C^{-1} e\right) E \\ 0 \end{pmatrix} \end{pmatrix}$$

$$= - \begin{pmatrix} \begin{pmatrix} 0 \\ C^{-1} eM^{-1} PM^{-1} \Phi \end{pmatrix} \\ \begin{pmatrix} \left(\Theta_0 p^* + \Theta_0 \lambda^* C^{-1} e\right) M^{-1} PM^{-1} \Phi \\ 0 \end{pmatrix} \end{pmatrix} +$$

$$- \begin{pmatrix} \begin{pmatrix} 0 \\ C^{-1} eM^{-1} \left(M \operatorname{\mathring{grad}}\right) \left|M \operatorname{\mathring{grad}}\right|^{-2} \psi \end{pmatrix} \\ \begin{pmatrix} \left(\Theta_0 p^* + \Theta_0 \lambda^* C^{-1} e\right) M^{-1} \left(M \operatorname{\mathring{grad}}\right) \left|M \operatorname{\mathring{grad}}\right|^{-2} \psi \\ 0 \end{pmatrix} \end{pmatrix}.$$

A "Simplification"

Using now the definition of Φ, we can write

$$
\begin{aligned}
&-\left(\begin{pmatrix} 0 \\ C^{-1}eM^{-1}PM^{-1}\Phi \end{pmatrix} \\ \begin{pmatrix} \left(\Theta_0 p^* + \Theta_0\lambda^* C^{-1}e\right)M^{-1}PM^{-1}\Phi \\ 0 \end{pmatrix}\right) \\
&= -\left(\begin{pmatrix} 0 \\ C^{-1}eM^{-1}PM^{-1}e^*C^{-1}T \end{pmatrix} \\ \begin{pmatrix} \left(\Theta_0 p^* + \Theta_0\lambda^* C^{-1}e\right)M^{-1}PM^{-1}e^*C^{-1}T \\ 0 \end{pmatrix}\right) + \\
&\quad -\left(\begin{pmatrix} 0 \\ C^{-1}eM^{-1}PM^{-1}\left(p\Theta_0 + e^*C^{-1}\lambda\Theta_0\right)\left(\Theta_0^{-1}\theta\right) \end{pmatrix} \\ \begin{pmatrix} \left(\Theta_0 p^* + \Theta_0\lambda^* C^{-1}e\right)M^{-1}PM^{-1}\left(p\Theta_0 + e^*C^{-1}\lambda\Theta_0\right)\left(\Theta_0^{-1}\theta\right) \\ 0 \end{pmatrix}\right) \\
&= -W_0 \begin{pmatrix} v \\ T \\ \Theta_0^{-1}\theta \\ q \end{pmatrix}
\end{aligned}
$$

with

$$
W_0 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & C^{-1}eM^{-1}PM^{-1}e^*C^{-1} & C^{-1}eM^{-1}PM^{-1}\left(p\Theta_0 + e^*C^{-1}\lambda\Theta_0\right) & 0 \\ 0 & \left(\Theta_0 p^* + \Theta_0\lambda^* C^{-1}e\right)M^{-1}PM^{-1}e^*C^{-1} & \left(\Theta_0 p^* + \Theta_0\lambda^* C^{-1}e\right)M^{-1}PM^{-1}\left(p\Theta_0 + e^*C^{-1}\lambda\Theta_0\right) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
$$

A "Simplification"

Summarizing,

$$\partial_0 \left(\left(\begin{pmatrix} \rho_* & 0 & 0 & 0 \\ 0 & M_{11} & M_{12} & 0 \\ 0 & M_{12}^* & M_{22} & 0 \\ 0 & 0 & 0 & \kappa_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \kappa_0^{-1} \end{pmatrix} + \begin{pmatrix} 0 & -\operatorname{Div} & 0 & 0 \\ -\operatorname{G\mathring{r}ad} & 0 & 0 & 0 \\ 0 & 0 & 0 & \operatorname{d\mathring{i}v} \\ 0 & 0 & \operatorname{grad} & 0 \end{pmatrix} \right) \right) \begin{pmatrix} v \\ T \\ \Theta_0^{-1}\theta \\ q \end{pmatrix} = G,$$

with

$$M_{11} := C^{-1} - C^{-1} e M^{-1} P M^{-1} e^* C^{-1}$$

$$M_{12} := C^{-1}\lambda\Theta_0 - C^{-1} e M^{-1} P M^{-1} \left(p\Theta_0 + e^* C^{-1}\lambda\Theta_0\right)$$

$$= M_{11}\lambda\Theta_0 - C^{-1} e M^{-1} P M^{-1} p\Theta_0$$

$$M_{22} := \left(\gamma_0 + \Theta_0\lambda^* C^{-1}\lambda\Theta_0\right) - \left(\Theta_0 p^* + \Theta_0\lambda^* C^{-1} e\right) M^{-1} P M^{-1} \left(p\Theta_0 + e^* C^{-1}\lambda\Theta_0\right)$$

and the right-hand side has to be adjusted to

$$G := \begin{pmatrix} F_0 \\ F_1 + C^{-1} e M^{-1} \left(M \operatorname{g\mathring{r}ad}\right) \left|M \operatorname{g\mathring{r}ad}\right|^{-2} \partial_0 \psi \\ F_4 + \left(\Theta_0 p^* + \Theta_0 \lambda^* C^{-1} e\right) M^{-1} \left(M \operatorname{g\mathring{r}ad}\right) \left|M \operatorname{g\mathring{r}ad}\right|^{-2} \partial_0 \psi \\ F_5 \end{pmatrix}.$$

A "Simplification"

Theorem

Let C, M, ρ_*, κ_1 be selfadjoint and non-negative such that $C, M, \rho_*, \nu\kappa_1 + \kappa_0^{-1} \gg 0$ for sufficiently large ν and P be an orthogonal projector. We set $Q := PM^{-1}e^*C^{-\frac{1}{2}}$ and assume that

$$1 - Q^*Q \gg 0,$$

$$\gamma_0 - \Theta_0 p^* M^{-1} P (1 - QQ^*)^{-1} PM^{-1} p\Theta_0 \gg 0.$$

Then, the thermo-piezo-electric system with quasi-static electric interaction is well-posed.

A "Simplification"

Proof

We need to verify our solvability condition. For doing so, it suffices to consider the block operator sub-matrix

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{22} \end{pmatrix}.$$

Noting that $M_{11} = C^{-1} - C^{-\frac{1}{2}} Q^* Q C^{-\frac{1}{2}} = C^{-\frac{1}{2}} (1 - Q^* Q) C^{-\frac{1}{2}}$, we obtain that M_{11} is boundedly invertible. Hence, by a symmetric Gauss step

$$\begin{pmatrix} M_{11} & 0 \\ 0 & M_{22} - M_{12}^* M_{11}^{-1} M_{12} \end{pmatrix},$$

which is strictly positive definite if and only if $M_{22} - M_{12}^* M_{11}^{-1} M_{12} \gg 0$.

A "Simplification"

Proof (continued)

We have

$$M_{22} = \gamma_0 + \Theta_0 \lambda^* M_{11} \lambda \Theta_0 + \\ - \left(\Theta_0 p^* M^{-1} P M^{-1} p \Theta_0 + 2 \mathfrak{Re} \left(\Theta_0 p^* M^{-1} Q C^{-\frac{1}{2}} \lambda \Theta_0 \right) \right)$$

and

$$M_{12} = M_{11} \lambda \Theta_0 - C^{-\frac{1}{2}} Q^* M^{-1} p \Theta_0 .$$

A "Simplification"

Proof (continued)

Thus,

$$M_{12}^* M_{11}^{-1} M_{12} = M_{12}^* \left(\lambda \Theta_0 - M_{11}^{-1} C^{-\frac{1}{2}} Q^* M^{-1} p \Theta_0\right)$$
$$= \Theta_0 \lambda^* M_{11} \lambda \Theta_0 - 2\mathfrak{Re}\left(\Theta_0 \lambda^* C^{-\frac{1}{2}} Q^* M^{-1} p \Theta_0\right) +$$
$$+ \Theta_0 p^* M^{-1} Q C^{-\frac{1}{2}} M_{11}^{-1} C^{-\frac{1}{2}} Q^* M^{-1} p \Theta_0.$$

Hence, we get

$$M_{22} - M_{12}^* M_{11}^{-1} M_{12} = \gamma_0 +$$
$$-\Theta_0 p^* \left(M^{-1} P M^{-1} + M^{-1} Q C^{-\frac{1}{2}} M_{11}^{-1} C^{-\frac{1}{2}} Q^* M^{-1}\right) p \Theta_0.$$

A "Simplification"

Proof (ending)

Using that $M_{11}^{-1} = C^{\frac{1}{2}} \left(1 - Q^* Q\right)^{-1} C^{\frac{1}{2}}$ we obtain

$$QC^{-\frac{1}{2}} M_{11}^{-1} C^{-\frac{1}{2}} Q^* = Q \left(1 - Q^* Q\right)^{-1} Q^* = -1 + \left(1 - QQ^*\right)^{-1}$$

and since $Q = PQ$ we have

$$QC^{-\frac{1}{2}} M_{11}^{-1} C^{-\frac{1}{2}} Q^* = -P + P \left(1 - QQ^*\right)^{-1} P$$

and thus,

$$M_{22} - M_{12}^* M_{11}^{-1} M_{12} = \gamma_0 +$$
$$-\Theta_0 p^* M^{-1} P \left(1 - QQ^*\right)^{-1} PM^{-1} p\Theta_0,$$

which is strictly positive definite by assumption. □