Riccati based feedback stabilization to trajectories for parabolic equations

Duy Phan-Duc and Sérgio S. Rodrigues

Johann Radon Institute for Computational and Applied Mathematics (RICAM)
Austrian Academy of Sciences (ÖAW)
Linz, Austria

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Outline

Introduction

We consider controlled parabolic equations, for time $t \geq 0$, in a smooth domain $\Omega \in \mathbb{R}^d$ with boundary $\Gamma = \partial\Omega$, with d a positive integer, either of the form

$$\partial_t y - \nu \Delta y + f(y, \nabla y) + \sum_{i=1}^{M} u_i \Phi_i = 0; \qquad y|_\Gamma = g;$$

or in the form

$$\partial_t y - \nu \Delta y + f(y, \nabla y) = 0; \qquad y|_\Gamma = g + \sum_{i=1}^{M} u_i \Psi_i.$$

In the variables $(t, x, \bar{x}) \in (0, +\infty) \times \Omega \times \Gamma$, the unknown in the equation is the function $y = y(t, x) \in \mathbb{R}$; the diffusion coefficient $\nu > 0$; the functions $g = g(t, \bar{x}) \in \mathbb{R}$ and $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ are fixed.

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In system (1) the functions $\Phi_i = \Phi_i(x)$ are given and will play the role of controllers, while in system (2) that role will be played by the given functions $\Psi_i = \Psi_i(\bar{x})$.

Here M is a positive integer and in both systems, $u = u(t) \in \mathbb{R}^M$ is a control vector function at our disposal to be found.

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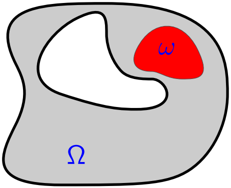
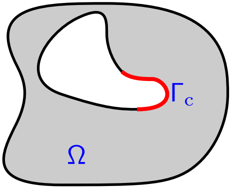
We are interested in controllers which are supported in a small domain, either $\operatorname{supp} \Phi_i \subset \omega \subseteq \Omega$ or $\operatorname{supp} \Psi_i \subset \Gamma_c \subseteq \Gamma$, where ω and Γ_c are given open subsets of Ω and Γ, respectively.



Internal controls Φ_i supported in $\omega \subseteq \Omega$.



Boundary controls Ψ_i supported in $\Gamma_c \subseteq \Gamma$.

Internal controllers

In our simulation, we will focus on the 2D case, considering our domain to be the unit ball. We define a rectangular sub-domain where we denote the internal controllers $\omega = \left(0, \frac{1}{2}\right) \times \left(0, \frac{1}{3}\right)$. In the figure below, we plot 4 piecewise-constant controllers in sub-domain ω.

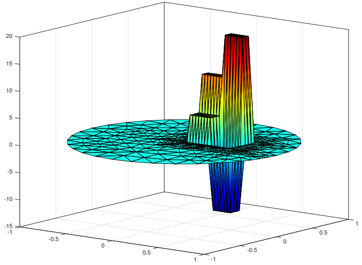


Figure : A linear combination of 4 piecewise-constant controllers.

Boundary controllers

In the figure below, we show two boundary controllers. Our boundary is $\Gamma = (0, 2\pi)$. We use control of the form $\Psi_i(\theta) = 1_{(\theta_0,\theta_1)} \sin\left(\frac{i(\theta-\theta_0)}{\theta_1-\theta_0}\right)$.

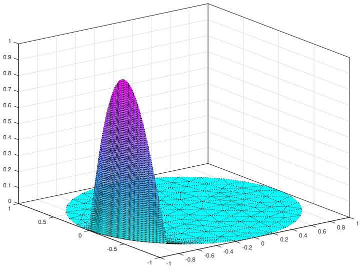


Figure : Ψ_1.

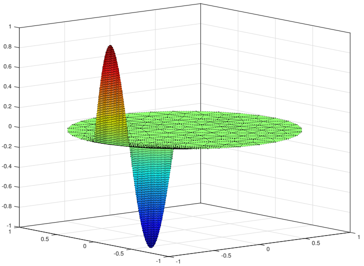


Figure : Ψ_2.

We can define the controllers in some separated sub-domains as in Figure below.

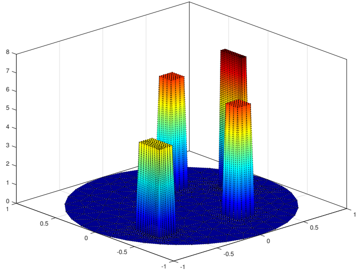


Figure : Internal Controllers.

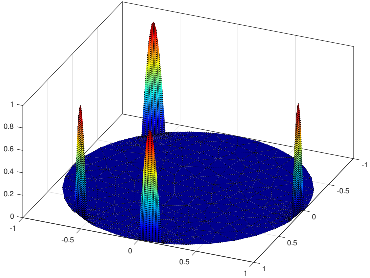


Figure : Boundary Controllers.

Goal

Let $\hat{y} = \hat{y}(t)$ be a **nonstationary** solution for the system

$$\partial_t \hat{y} - \nu\Delta\hat{y} + f(\hat{y}, \nabla\hat{y}) = 0, \qquad \hat{y}|_\Gamma = g, \qquad \hat{y}(0) = \hat{y}_0;$$

Given $\lambda > 0$, our goal is to find a control vector function u_i such that the solution of

$$\partial_t y - \nu\Delta y + f(y, \nabla y) + \sum_{i=1}^{M} u_i\Phi_i = 0, \qquad y|_\Gamma = g + \sum_{i=1}^{M} u_i\Psi_i,$$

with $y(0) = y_0 \neq \hat{y}_0$, goes to $\hat{y}$ exponentially with rate $\frac{\lambda}{2}$:

$$|y(t) - \hat{y}(t)|^2_{L^2(\Omega)} \leq C\mathrm{e}^{-\lambda t} |y(0) - \hat{y}(0)|^2_{L^2(\Omega)}.$$

Furthermore, we look for the control function u_i in feedback form $u_i(t) = K_i(t, y(t) - \hat{y}(t))$, for a suitable K_i.

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With $y(0) \neq \hat{y}(0)$, the behaviour of the uncontrolled solution may be quite different from the desired $\hat{y}$ (instability).

Notice that, in terms of the difference $v := y - \hat{y}$ our goal

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Linearized system

Thus. we want the difference $z = y - \hat{y}$ to go to zero exponentially. We find

$$\partial_t z - \nu \Delta z + f(y, \nabla y) - f(\hat{y}, \nabla \hat{y}) + \sum_{i=1}^{M} u_i \Phi_i = 0, \qquad z\big|_{\Gamma} = 0.$$

By writing $f: \ (\xi^1, \xi^2) \in \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}$, $(\xi^1, \xi^2) \in \mathbb{R} \times \mathbb{R}^d$, we denote $\partial_1 f := \frac{\partial f}{\partial_{\xi^1}}$ and $\partial_2 f := \frac{\partial f}{\partial_{\xi^2}}$. Formally, we can rewrite

$$\partial_t z - \nu \Delta z + az + \nabla \cdot (bz) - \mathcal{N}(z) + \sum_{i=1}^{M} u_i \Phi_i = 0.$$

where $a := \partial_1 f\big|_{(\hat{y}, \nabla \hat{y})} - \nabla \cdot \partial_2 f\big|_{(\hat{y}, \nabla \hat{y})}$, $\ b := \partial_2 f\big|_{(\hat{y}, \nabla \hat{y})}$ and $\mathcal{N}(z)$ is a nonlinear term.

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Family of internal controllers

Following the ideas in [BRS11, KR15b, KR15a, BKR15], we consider a family $\widehat{\mathcal{C}}_\omega = \{\widehat{\Phi}_i \in H \mid i \in \{1, 2, \dots, M\}\} \subset H$ and denote by P_M the orthogonal projection in H onto $\operatorname{span}\widehat{\mathcal{C}}_\omega$.

By defining

$$\overline{\Theta}(\xi_1, \xi_2, \xi_3, d, \nu) := \widehat{D}\left(1 + \xi_1^2 + d\xi_2^2\right) + 2(\widehat{D})^{\frac{1}{2}}\left(D_{\mathrm{rc}}\xi_3^2 + \widehat{D}\left(\xi_1 + d\xi_2^2\right)\right)^{\frac{1}{2}}$$

The following result gives us a sufficient condition on the family $\widehat{\mathcal{C}}_\omega$ for the existence of a stabilizing control.

Theorem

Let us be given $\chi \in C^\infty(\overline{\Omega})$ satisfying $\emptyset \neq \omega \cap \operatorname{supp}\chi$. If

$$|1_\omega \chi (1 - P_M) 1_\omega|^2_{\mathcal{L}(H, V')} \leq \Upsilon^{-1},$$

with $\Upsilon = Ce^{\overline{\Theta}\left(\left|a - \frac{\lambda}{2}\right|_{L^\infty(\mathbb{R}_0, L^d)}, |b|_{L^\infty(\mathbb{R}_0, L^\infty)}, \left|(a - \frac{\lambda}{2}, b)\right|_{\mathcal{W}}, d\right)}$, then the system is

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Combining some of the arguments in [Bad09, Rod15], again we obtain

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Let us be given a nonzero $\chi_\Gamma \in C^\infty(\overline{\Omega})$ satisfying $\operatorname{supp}\chi_\Gamma \subseteq \overline{\Gamma_c}$. If

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The dimension of the internal controller

Some estimates for the dimension of controllers in 1D were presented in [KR15b]. For some type of controllers like when we have suitable eigenfunctions of the Laplacian in $L^2(\omega)$ or piecewise constant controllers, the estimates above allow us to derive some estimates on the number of needed controls like

$$M \geq (C\Upsilon)^{\frac{d}{2}}\,. \tag{4}$$

For internal controls without no restriction on the support of the controllers ($\omega = \Omega$) we can derive the estimate

$$M \geq D_d^{-\frac{d}{2}} \mathrm{e}^{\frac{d}{2}} \left(4|a|_{\mathcal{W}} + \tfrac{3}{2}|b|^2_{\mathcal{W}} + 2\lambda\right)^{\frac{d}{2}}\,. \tag{5}$$

That is why we decided to perform some numerical simulations. The results of those simulations suggest that an estimate like (5) might also hold true in the general case $\omega \neq \Omega$.

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Linear parabolic system with feedback control

Here, we focus on the approximation of the linearized closed-loop systems with feedback control in internal-control case

$$\partial_t z - \nu\Delta z + (a - \tfrac{\lambda}{2})z + \nabla\cdot(bz) + \mathcal{F}^{\mathrm{in}} z = 0, \qquad z\big|_\Gamma = 0$$
$$\tilde{z}(s_0) = z_0.$$

with $\mathcal{F}^{\mathrm{in}} z := B_M B_M^* \Pi_\lambda z$; and in boundary-control case

$$\partial_t z - \nu\Delta z + (a - \tfrac{\lambda}{2})z + \nabla\cdot(bz) = 0, \qquad z\big|_\Gamma = B_\Psi^\Gamma \kappa,$$
$$\partial_t \kappa + (\varsigma - \tfrac{\lambda}{2})\kappa + \mathcal{F}^{\mathrm{bo}}(z - B_\Psi\kappa, \kappa) = 0,$$
$$z(s_0) = y_0 + B_\Psi \kappa_0, \qquad \kappa(s_0) = \kappa_0,$$

with .

Π_λ and Π_λ^Γ are the solutions of a suitable differential Riccati equation.

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Π_λ and Π_λ^Γ are the solutions of a suitable differential Riccati equation.

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Here, we focus on the approximation of the linearized closed-loop systems with feedback control in internal-control case

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Discretization of feedback control

We look for an approximation of Π_D by solving a suitable matrix Riccati equation

$$\partial_t \Pi_D + \Pi_D \mathbf{X} + \mathbf{X}^\top \Pi_D - \Pi_D \mathbf{R}\mathbf{R}^\top \Pi_D + \lambda \Pi_D + \mathbf{C}^\top \mathbf{C} = 0, \qquad t > 0.$$

where the matrices $\mathbf{X}$, $\mathbf{R}$ and $\mathbf{C}$ come from the discretization of the differential Riccati equation.

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Feedback control for a family of trajectories

We work with the linear system

$$\partial_t v - \nu\Delta v + av + \nabla\cdot(bv) = 0, \quad v|_\Gamma = 0, \quad v(x,0) = v_0;$$

where

$$\begin{aligned} \nu &= 0.25, \quad v_0 = \sin(2x)\cos(y), \\ a &= -\sin(t)\cos(\mathbf{i}x) + \sin(5t)\sin(\mathbf{j}y) - 3, \\ b_1 &= \cos(t)\sin(-\mathbf{k}x) - \cos(3t)\cos(\mathbf{l}y), \\ b_2 &= \sin(-t)\sin(\mathbf{m}x) - \cos(2t)\sin(\mathbf{n}y). \end{aligned}$$

with $(i,j,k,l,m,n) \in \mathcal{T}$ where $\mathcal{T}$ is presented below

$$\begin{aligned} \mathcal{T} = \{&(1,1,1,1,1,1),(1,2,2,1,1,1),(2,-1,1,-3,5,1), \\ &(-1,5,3,1,1,5),(1,2,3,4,5,6),(6,-2,5,3,4,1)\}. \end{aligned}$$

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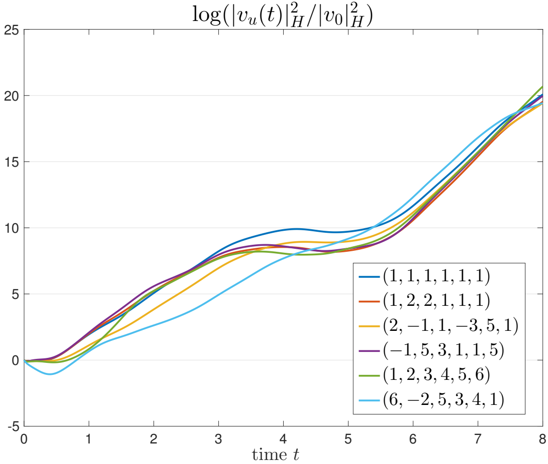

Figure : Without any controls

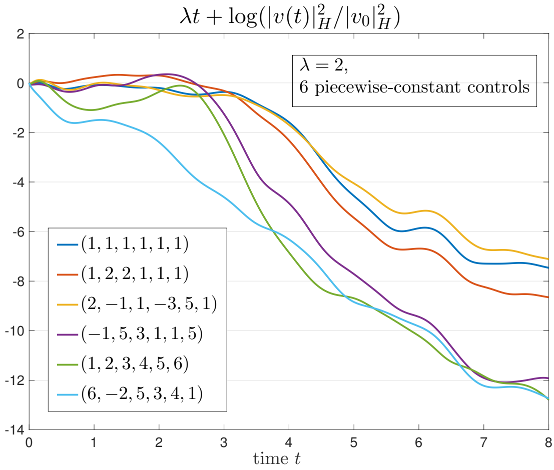

Figure : With 6 piecewise-constant internal controllers.

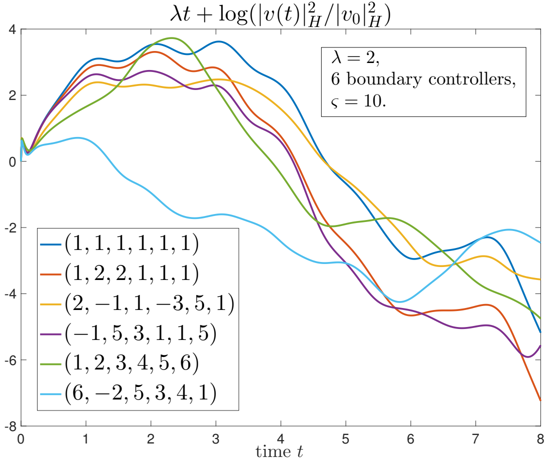

Figure : With 6 boundary controllers.

Increasing the number of internal controllers

We recall the trajectories presented above with $(i, j, k, l, m, n) = (2, -1, 1, -3, 5, 1)$ and show the controlled solutions with 1, 2 and 4 internal controller(s).

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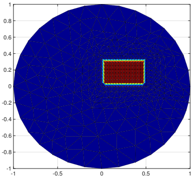

Figure : 1 internal controller.

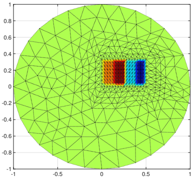

Figure : 4 internal controllers (4,1).

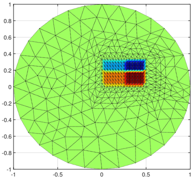

Figure : 4 internal controllers (2,2).

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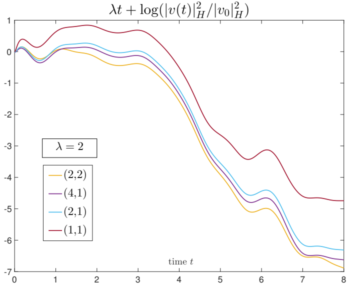

Figure : With 1,2 and 4 piecewise-constant internal controller(s).

Increasing the number of boundary controllers

We recall the trajectories presented above with $(i, j, k, l, m, n) = (6, -2, 5, 3, 4, 1)$ and show the controlled solutions with 1, 2, 4 and 6 boundary controller(s).

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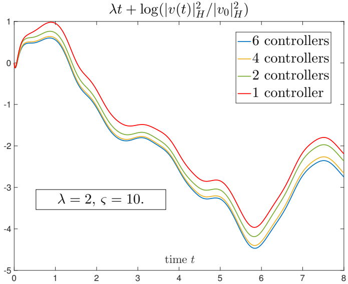

Figure : With 1, 2, 4 and 6 boundary controller(s).

The transient bound for linearized system

We recall that we want $|z|_H^2 \le C_\lambda \mathrm{e}^{-\lambda t}|z_0|_H^2$ for the solution of linear system.

Theorem

There are nonnegative constants $\mathcal{T}_0$, and $\mathcal{T}_{\frac{1}{2}}$, such that for a suitable control function $\eta \in L^2\left((0,+\infty), L^2(\omega)\right)$, the solution of the linear system $z_t - \nu\Delta z + az + \nabla\cdot(bz) + 1_\omega\eta = 0$ satisfies for all $z_0 \in H$

$$|z(t)|_H^2 \le \mathrm{e}^{\mathcal{T}_0 + \mathcal{T}_{\frac{1}{2}}\lambda^{\frac{1}{2}}}\mathrm{e}^{-\lambda(t-s_0)}|z_0|_H^2, \text{ for } t \ge s_0.$$

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Transcient Bound

Here, we work with linear system where

$$\nu = 0.5,\ a = -10 + 2x + \cos y,\ b_1 = -x^2,\ b_2 = -\sin y$$

with no restriction number of controllers.

For every value of λ, we receive a convergence rate function $r_\lambda(t) := \lambda t + \log\left(\frac{|v(t)|_H^2}{|v_0|_H^2}\right)$ depending on time. We define a new parameter m_λ as follows

$$m_\lambda = \max_{t_0 \geq 0} \max_{t \geq t_0} |r_\lambda(t) - r_\lambda(t_0)|.$$

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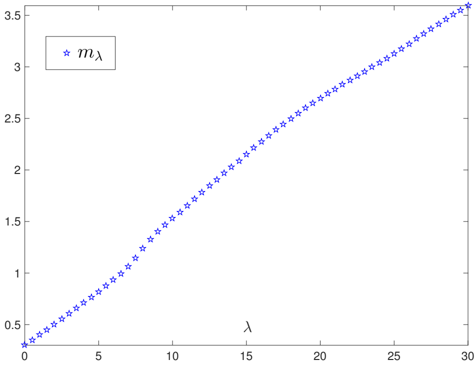

Figure : m_λ with $\lambda \in [0, 30]$.

One nonlinear example

$$\partial_t y - \nu \Delta y - 3y^3 - y^2 - 2y + \tfrac{1}{2}\nabla \cdot \left(y^2, y^2\right) + f_0 = 0, \qquad y|_\Gamma = g,$$
$$y(0,x) = \hat{y}(0,x) + \delta v_0,$$

where f_0 and v_0 are (appropriate) functions.

We will set $\nu = 0.2, \quad \lambda = 1$, and $\boxed{\hat{y}(t,x) = (2x^3 + y^2)\sin t}$ as our targetted trajectory.

Remark. To make $\hat{y}$ a solution we (must) just take

$$f_0 = f_0(\hat{y}) = -\left(\partial_t \hat{y} - \nu \Delta \hat{y} + c_3 \hat{y}^3 + c_2 \hat{y}^2 + c_1 \hat{y} + \frac{1}{2}\nabla \cdot \left(\hat{y}^2, \hat{y}^2\right)\right),$$
$$g = g(\hat{y}) = \hat{y}|_\Gamma .$$

Compatibility conditions

Some boundary compatibility conditions must be satisfied. We consider a perturbation $y_0 = \hat{y}_0 + \delta v_0$, with δ small (the result is "local" for nonlinear system), and v_0 the (numerical) solution of

$$-0.5\Delta v_0 + r v_0 + \nabla \cdot (c v_0) + h = 0, \qquad v_0|_\Gamma = \gamma$$

$$r = \sin(x_1) + x_2,\ c = \begin{bmatrix} 2x_1x_2 \\ -2\sin(x_2) \end{bmatrix},\ \ h = \cos^2(3x_2) + \sin(x_1) + 2.$$

With 4 internal piecewise constants controllers (defined in one sub-rectangle).
$\delta = 0.1$, $\gamma = 0$. Play

With 4 internal piecewise constants controllers (defined in 4 separated sub-rectangles).
$\delta = 0.35$, $\gamma = 0$. Play

With 6 boundary sinus-like controllers.
$\delta = 0.035$, $\gamma = \sum_{i=1}^{6} \kappa_i \Psi_i$. $\kappa = (1, 1, 0, 0.5, 0, 0)$. Play

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THANK FOR YOUR ATTENTION