Fourier, Wavelet and Monte Carlo Methods in Computational Finance

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Agenda

- Derivatives pricing, Feynman-Kac Theorem
- Fourier methods
  - Basics of COS method;
  - Basics of SWIFT method;
  - Options with early-exercise features
    - COS method for Bermudan options
    - Monte Carlo method
  - BSDEs, BCOS method (very briefly)

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  - Options with early-exercise features
    - COS method for Bermudan options
    - Monte Carlo method
  - BSDEs, BCOS method (very briefly)
- Joint work with
  Fang Fang, Marjon Ruijter, Luis Ortiz, Shashi Jain, Alvaro Leitao, Fei Cong, Qian Feng

Feynman-Kac Theorem

- The linear partial differential equation:

$$\frac{\partial v(t,x)}{\partial t} + \mathcal{L}v(t,x) + g(t,x) = 0, \quad v(T,x) = h(x),$$

with operator

$$\mathcal{L}v(t,x) = \mu(x)Dv(t,x) + \frac{1}{2}\sigma^2(x)D^2v(t,x).$$

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Feynman-Kac theorem:

$$v(t,x) = \mathbb{E}\left[\int_t^T g(s,X_s)ds + h(X_T)\right],$$

where X_s is the solution to the FSDE

$$dX_s = \mu(X_s)ds + \sigma(X_s)d\omega_s,\ X_t = x.$$

HJB equation, dynamic programming

- Suppose we consider the Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned}\frac{\partial v(t,x)}{\partial t} &+ \sup_{a\in A}\{\mu'(x,a)Dv(t,x)+\frac{1}{2}\mathrm{Tr}[D^2v(t,x)\sigma\sigma'(x,a)]\\ &+ g(t,x,a)\}=0,\\ v(T,x) &= h(x).\end{aligned}$$

It is associated to a stochastic control problem with value function

$$v(t,x)=\sup_{\alpha}\mathbb{E}_t^x\left[\int_t^T g(s,X_s^\alpha,\alpha_s)ds+h(X_T^\alpha)\right],$$

where X_s is the solution to the controlled FSDE

$$dX_s^\alpha=\mu(X_s^\alpha,\alpha_s)ds+\sigma(X_s^\alpha,\alpha_s)d\omega_s,\; X_t^\alpha=x.$$

Semilinear PDE and BSDEs

- The semilinear partial differential equation:

$$\frac{\partial v(t,x)}{\partial t} + \mathcal{L}v(t,x) + g(t,x,v,\sigma(x)Dv(t,x)) = 0,\ v(T,x) = h(x).$$

We can solve this PDE by means of the FSDE:

$$dX_s = \mu(X_s)ds + \sigma(X_s)d\omega_s,\ X_t = x.$$

and the BSDE:

$$dY_s = -g(s,X_s,Y_s,Z_s)ds + Z_s d\omega_s,\ Y_T = h(X_T).$$

- Theorem:

$$Y_t = v(t,X_t),\ Z_t = \sigma(X_t)Dv(t,X_t).$$

is the solution to the BSDE.

Application

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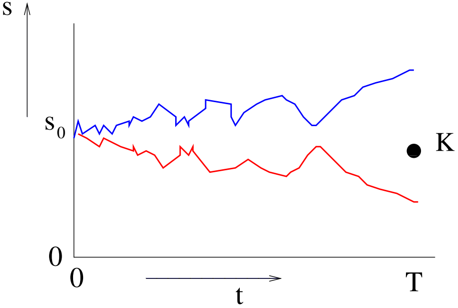
- You can buy insurance against falling stock prices.
- This is the standard put option (i.e. the right to sell stock at a future time point).
- The uncertainty is in the stock prices.

A call option gives the holder the right to trade in the future at a previously agreed strike price, K.



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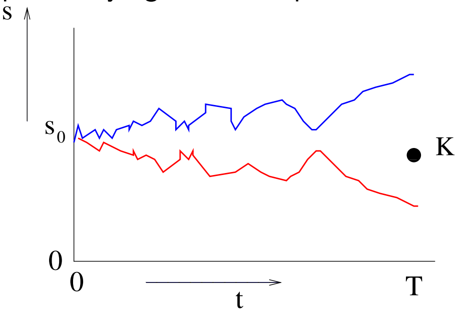

$v(T,S) = \max(K - S_T, 0) =: h(S_T)$

Feynman-Kac Theorem (option pricing context)

Given the final condition problem

$$\begin{cases} \dfrac{\partial v}{\partial t} & + \quad \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 v}{\partial S^2} + rS\dfrac{\partial v}{\partial S} - rv = 0, \\ v(T,S) & = \quad h(S_T) = \text{ given} \end{cases}$$

Then the value, $v(t,S)$, is the unique solution of

$$v(t,S) = e^{-r(T-t)}\mathbb{E}^Q\{v(T,S_T)|\mathcal{F}_t\}$$

with the sum of first derivatives square integrable, and $S = S_t$ satisfies the system of SDEs:

$$dS_t = rS_t dt + \sigma S_t d\omega_t^Q,$$

- Similar relations also hold for (multi-D) SDEs and PDEs!

A pricing approach; European options

$$v(t_0, S_0) = e^{-r(T-t_0)} \mathbb{E}^Q \{ v(T, S_T) | \mathcal{F}_0 \}$$

Quadrature:

$$v(t_0, S_0) = e^{-r(T-t_0)} \int_{\mathbb{R}} v(T, S_T) f(S_T, S_0) dS_T$$

- Trans. PDF, $f(S_T, S_0)$, typically not available, but the characteristic function, $\widehat{f}$, often is.

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- Banks sell insurance against changing FX rates. The option pays out in the best currency each year.
- Uncertain processes are the exchange rates, interest rate, but also the counterparty of the contract may go bankrupt!

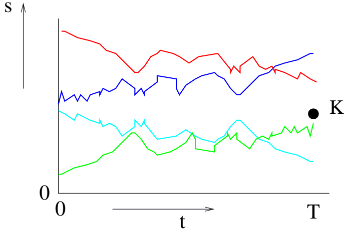
Multi-asset options

Multi-asset options belong to the class of exotic options.

$v(\mathbf{S},T) = \max(\max\{S_1,\ldots,S_d\}_T - K, 0)$ (max call)

$v(\mathbf{S},t_0) = e^{-r(T-t_0)}\int_{\mathbb{R}} v(\mathbf{S},T) f(\mathbf{S}_T|\mathbf{S}_0) d\mathbf{S}$

- $\Rightarrow$ High-dimensional integral or a high-D PDE.



Increasing dimensions: Multi-asset options

- The problem dimension increases if the option depends on more than one asset S_i (multiple sources of uncertainty).
- If each underlying follows a geometric (lognormal) diffusion process,
- Each additional asset is represented by an extra dimension in the problem:

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{d}[\sigma_i \sigma_j \rho_{i,j} S_i S_j \frac{\partial^2 v}{\partial S_i \partial S_j}] + \sum_{i=1}^{d}[r S_i \frac{\partial v}{\partial S_i}] - rv = 0 \ .$$

- Required information is the volatility of each asset σ_i and the correlation between each pair of assets $\rho_{i,j}$.

Numerical Pricing Approach

- One can apply several numerical techniques to calculate the option price:
    - Numerical integration,
    - Monte Carlo simulation,
    - Numerical solution of the partial-(integro) differential equation
- Each of these methods has its merits and demerits.
- Numerical challenges:
    - The problem's dimensionality
    - Speed of solution methods
    - Early exercise feature ($\rightarrow$ free boundary problem)

Financial engineering; Work at Banks

- Financial engineering, pricing approach:
  1. Start with some financial product
  2. Model asset prices involved (SDEs)
  3. Calibrate the model to market data (Numerics, Opt.)
  4. Model product price correspondingly (PDE, Integral)
  5. Price the product of interest (Numerics, MC)

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  5a. Price the risk related to default (SDE, Opt.)

  6. Understand and remove risk (Stoch., Opt., Numer.)

Motivation Fourier Methods

- A characteristic function of a continuous random variable X, equals the Fourier transform of the density of X.
- Derive pricing methods that
  - are computationally fast
  - are not restricted to Gaussian-based models
  - should work as long as we have a characteristic function,

$$\widehat{f}(u;x) = \int_{-\infty}^{\infty} e^{iux} f(x) dx;$$

(available for Lévy processes and also for SDE systems).

Class of Affine Jump Diffusion (AJD) processes

Suppose we have given a following system of SDEs:

$$d\mathbf{X}_t = \mu(\mathbf{X}_t)dt + \sigma(\mathbf{X}_t)d\omega_t + d\mathbf{Z}_t,$$

For processes in the AJD class drift, volatility, jump intensities and interest rate components are of the affine form, i.e.

$$\begin{aligned}
\mu(\mathbf{X}_t) &= a_0 + a_1\mathbf{X}_t \text{ for } (a_0, a_1) \in \mathbb{R}^n \times \mathbb{R}^{n\times n}, \\
\lambda(\mathbf{X}_t) &= b_0 + b_1^T\mathbf{X}_t, \text{ for } (b_0, b_1) \in \mathbb{R} \times \mathbb{R}^n, \\
\sigma(\mathbf{X}_t)\sigma(\mathbf{X}_t)^T &= (c_0)_{ij} + (c_1)_{ij}^T\mathbf{X}_t, (c_0, c_1) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n\times n}, \\
r(\mathbf{X}_t) &= r_0 + r_1^T\mathbf{X}_t, \text{ for } (r_0, r_1) \in \mathbb{R} \times \mathbb{R}^n.
\end{aligned}$$

Duffie, Pan, Singleton (2000): For affine jump diffusion processes the discounted characteristic function can be derived!

Exponential Lévy Processes

- Lévy process $\{X_t\}_{t\geq 0}$: process with stationary, independent increments.
- Brownian motion and Poisson processes belong to this class, as well jump processes with either finite or infinite activity
- Asset prices can be modeled by exponential Lévy processes
  - small jumps describe the day-to-day "noise";
  - big jumps describe large stock price movements.
- The characteristic function of a Lévy process is known:

$$\begin{aligned}\widehat{f}(u; X_t) &= \mathbb{E}[\exp(iuX_t)] = \\ &\exp\left(t(i\mu u - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}}(e^{iuy} - 1 - iuy\mathbf{1}_{[|y|<1]}\nu(dy))\right),\end{aligned}$$

the celebrated Lévy-Khinchine formula.

Fourier-Cosine Expansions, COS Method (with Fang Fang)

- The COS method:
  - Exponential convergence;
  - Greeks (derivatives) are obtained at no additional cost.
- All based on the availability of a characteristic function.
- The basic idea:
  - Replace the density by its Fourier-cosine series expansion;
  - Coefficients have simple relation to characteristic function.

Series Coefficients of the Density and the ChF

- Fourier-Cosine expansion of density function on interval $[a, b]$:

$$f(x) = {\sum_{n=0}^{\infty}}' F_n \cos\left(n\pi \frac{x-a}{b-a}\right),$$

with $x \in [a, b] \subset \mathbb{R}$ and the coefficients defined as

$$F_n := \frac{2}{b-a} \int_a^b f(x) \cos\left(n\pi \frac{x-a}{b-a}\right) dx.$$

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$$F_n := \frac{2}{b-a} \int_a^b f(x) \cos\left(n\pi \frac{x-a}{b-a}\right) dx.$$

- F_n has direct relation to the ChF, $\widehat{f}(u) := \int_{\mathbb{R}} f(x) e^{iux} dx$ ($\int_{\mathbb{R}\setminus[a,b]} f(x) \approx 0$),

$$\begin{aligned} F_n \approx P_n \quad &:= \quad \frac{2}{b-a} \int_{\mathbb{R}} f(x) \cos\left(n\pi \frac{x-a}{b-a}\right) dx \\ &= \quad \frac{2}{b-a} \Re\left\{ \widehat{f}\left(\frac{n\pi}{b-a}\right) \exp\left(-i \frac{na\pi}{b-a}\right)\right\}. \end{aligned}$$

Recovering Densities

- Replace F_n by P_n, and truncate the summation:

$$f(x) \approx \frac{2}{b-a} {\sum_{n=0}^{N-1}}{}' \Re\left\{ \widehat{f}\left(\frac{n\pi}{b-a}\right) \exp\left(in\pi \frac{-a}{b-a}\right) \right\} \cos\left(n\pi \frac{x-a}{b-a}\right).$$

- Example: $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$, $[a,b] = [-10, 10]$ and $x = \{-5, -4, \cdots, 4, 5\}$.

N	4	8	16	32	64
error	0.2538	0.1075	0.0072	4.04e-07	3.33e-16
cpu time (sec.)	0.0025	0.0028	0.0025	0.0031	0.0032

- Exponential error convergence in N.

Pricing European Options

- Start from the risk-neutral valuation formula:

$$v(t_0, x) = e^{-r\Delta t}\mathbb{E}^{\mathbb{Q}}\left[v(T, y)|\mathcal{F}_0\right] = e^{-r\Delta t}\int_{\mathbb{R}} v(T, y)f(y, x)dy.$$

- Truncate the integration range:

$$v(t_0, x) = e^{-r\Delta t}\int_{[a,b]} v(T, y)f(y, x)dy + \varepsilon.$$

- Replace the density by the COS approximation, and interchange summation and integration:

$$\hat{v}(t_0, x) = e^{-r\Delta t}{\sum_{n=0}^{N-1}}' \Re\left\{\widehat{f}\left(\frac{n\pi}{b-a}; x\right)e^{-in\pi\frac{a}{b-a}}\right\}\mathcal{H}_n,$$

where the series coefficients of the payoff, $\mathcal{H}_n$, are analytic.

Pricing European Options

- Log-asset prices: $x := \log(S_0/K)$ and $y := \log(S_T/K)$.
- The payoff for European call options reads

$$v(T,y) \equiv \max\left(K(e^y - 1), 0\right).$$

- For a call option, we obtain

$$\begin{aligned}\mathcal{H}_k^{call} &= \frac{2}{b-a}\int_0^b K(e^y - 1)\cos\left(k\pi\frac{y-a}{b-a}\right)dy \\ &= \frac{2}{b-a}K\left(\chi_k(0,b) - \psi_k(0,b)\right).\end{aligned}$$

- For a vanilla put, we find

$$\mathcal{H}_k^{put} = \frac{2}{b-a}K\left(-\chi_k(a,0) + \psi_k(a,0)\right).$$

Results, Heston stochastic volatility PDE

$$\frac{\partial v}{\partial t} = \frac{1}{2}S^2 y \frac{\partial^2 v}{\partial S^2} + \rho\gamma S y \frac{\partial^2 v}{\partial S \partial y} + \frac{1}{2}\gamma^2 y \frac{\partial v}{\partial y^2} + rS\frac{\partial v}{\partial S} + \kappa(\overline{\sigma} - y)\frac{\partial v}{\partial y} - rv.$$

- GPU computing: Multiple strikes for parallelism, 21 IC's.

Heston model				
	N	64	128	256
MATLAB	msec	3.850890	7.703350	15.556240
	max.abs.err	6.0991e-04	2.7601e-08	$< 10^{-14}$
GPU	msec	0.177860	0.209093	0.333786

Table 1: Maximum absolute error when pricing a vector of 21 strikes.

- Exponential convergence, Error analysis in our papers.
- Also work with wavelets instead of cosines.

SWIFT Wavelet Option Pricing (with Luis Ortiz)

Approximate density f by a finite combination of Shannon scaling functions and recovering the coefficients from its Fourier transform.

$$\varphi_{m,k}(x) = 2^{m/2}\frac{\sin(\pi(2^m x - k))}{\pi(2^m x - k)}, \quad k \in \mathbb{Z}. \tag{1}$$

For $m = k = 0$, we have the father wavelet,

$$\varphi(x) = \text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}.$$

Wavelets can be moved (by k), stretched or compressed (by m) to accurately represent local properties of a function.
Shannon wavelets have a slow decay in the time domain but a very sharp compact support in the frequency (Fourier) domain.

Using the classical *Vieta formula*, the cardinal sine can be approximated:

$$\text{sinc}(t) \approx \prod_{j=1}^{J} \cos\left(\frac{\pi t}{2^j}\right) = \frac{1}{2^{J-1}} \sum_{j=1}^{2^{J-1}} \cos\left(\frac{2j-1}{2^J}\pi t\right). \tag{2}$$

Following the wavelets theory, function f can be approximated at a level of resolution m, i.e.,

$$f(x) \approx \mathcal{P}_m f(x) = \sum_{k \in \mathbb{Z}} c_{m,k}\varphi_{m,k}(x) \approx f_m(x) := \sum_{k=k_1}^{k_2} c_{m,k}\varphi_{m,k}(x), \tag{3}$$

for certain accurately chosen values k_1 and k_2.

Summary

$$v(x,t) = e^{-r(T-t)} \int_{\mathbb{R}} v(y,T) f(y|x)\, dy,$$

$$f(y|x) \approx \sum_{k=k_1}^{k_2} c_{m,k}(x) \varphi_{m,k}(y),$$

With: $c_{m,k} = \langle f, \varphi_{m,k} \rangle = \int_{\mathbb{R}} f(x) \varphi(2^m x - k) dx.$

$$c_{m,k} \approx c^*_{m,k} := \frac{2^{m/2}}{2^{J-1}} \sum_{j=1}^{2^{J-1}} \int_{\mathbb{R}} f(x) \cos\left(\frac{2j-1}{2^J} \pi (2^m x - k) \right) dx. \qquad (4)$$

or:

$$c_{m,k} \approx c^*_{m,k} = \frac{2^{m/2}}{2^{J-1}} \sum_{j=1}^{2^{J-1}} \Re \left\{ \hat{f}\left(\frac{(2j-1)\pi 2^m}{2^J} \right) e^{\frac{ik\pi(2j-1)}{2^J}} \right\}. \qquad (5)$$

Example: Fat-tailed densities:

We can control density values and the mass as a byproduct.

$$\mathbf{f}\left(\frac{\mathbf{h}}{\mathbf{2^m}}\right) \approx \mathcal{P}_m f\left(\frac{h}{2^m}\right) = 2^{\frac{m}{2}} \sum_{k\in\mathbb{Z}} c_{m,k}\delta_{k,h} = \mathbf{2^{\frac{m}{2}} c_{m,h}}, \tag{6}$$

...plus...

$$\mathcal{A} = \frac{1}{2^{m/2}}\left(\frac{c_{m,k_1}}{2} + \sum_{k_1<k<k_2} c_{m,k} + \frac{c_{m,k_2}}{2}\right), \tag{7}$$

$[a,b]$	$\lvert f\left(\frac{k_1}{2^m}\right)\rvert$	$\lvert f\left(\frac{k_2}{2^m}\right)\rvert$	Error (area)
$[-1,1]$	5.97e-02	4.20e-02	9.82e-01
$[-5,5]$	1.30e-01	1.06e-03	3.40e-01
$[-20,20]$	1.10e-08	1.82e-15	7.05e-09

$\Rightarrow$ Difficult to accurately determine $[a,b]$ by cumulants.

- Replace f by its approximation f_m:

$$v(x,t) \approx e^{-r(T-t)} \sum_{k=k_1}^{k_2} c_{m,k}(x) \cdot \mathcal{H}_{m,k},$$

with the pay-off coefficients,

$$\mathcal{H}_{m,k} := \int_{\mathcal{I}_k} v(y,T) \varphi_{m,k}(y)\, dy.$$

- With $\bar{k}_1 := \max(k_1, 0)$, pay-off coefficients, $\mathcal{H}_{m,k}$, for a European call option are approximated by,

$$\mathcal{H}^*_{m,k} := \begin{cases} \frac{K 2^{m/2}}{2^{J-1}} \sum_{j=1}^{2^{J-1}} \left[I_1\left(\frac{\bar{k}_1}{2^m}, \frac{k_2}{2^m}\right) - I_2\left(\frac{\bar{k}_1}{2^m}, \frac{k_2}{2^m}\right) \right], & \text{if } k_2 > 0, \\ 0, & \text{if } k_2 \leq 0, \end{cases}$$

where I_1 and I_2 are closed formulae. **FFT** can be applied to compute the pay-off coefficients.

Example: Multiple strikes valuation:

21 strikes	scale m	4	5	6
	CPU time (milli-seconds)	3.04	3.84	6.62
	Max. absolute error	$2.04e-02$	$5.63e-05$	$3.63e-06$
1 strike	scale m	4	5	6
	CPU time (milli-seconds)	0.32	0.52	1.00
	Absolute error	$4.78e-03$	$1.61e-05$	$6.56e-07$

Table 2: Simultaneous valuation of 21 call options with K from 50 to 150, and of only one European call option under the Heston dynamics with parameters $S_0 = 100, \mu = 0, \lambda = 1.5768, \eta = 0.5751, \bar{u} = 0.0398, u_0 = 0.0175, \rho = -0.5711$.

Optimal stopping, dynamic programming

- The controller only has control over his terminal time.

$$dX_t = \mu(X_t)dt + \sigma(X_t)d\omega_t,$$

- With $t \in [0, T]$, and stopping times $\mathcal{T}_{t,T}$, the finite horizon optimal stopping problem is formulated as

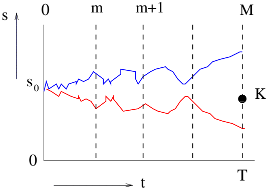
$$v(t,x) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\left[\int_t^\tau e^{-r(T-t)} g(s, X_s)ds + e^{-r(\tau-t)} h(X_\tau)\right].$$

- The value function v is related to the HJB variational inequality:

$$\min[-\frac{\partial v}{\partial t} - \mathcal{L}v - g, v - h] = 0,$$

- The problem is called a free boundary problem.
- C is the continuation region, the complement set is the stopping or exercise region (receive the reward g).

Pricing Options with Early-Exercise



- The pricing formulas for a Bermudan option with M exercise dates reads, for $m = M-1, \ldots, 1$:

$$\begin{cases} c(S,t_m) &= e^{-r\Delta t_m}\mathbb{E}\left[v(S,t_{m+1})|S_{t_m}\right], \\ v(S,t_m) &= \max\left(h(S,t_m), c(S,t_m)\right) \end{cases}$$

and $v(S,t_0) = e^{-r\Delta t_0}.\mathbb{E}\left[v(S,t_1)|S_{t_0}\right]$

  - Use Newton's method to determine early exercise point x^*_{m+1}, i.e. the root of $h(x,t_{m+1}) - c(x,t_{m+1}) = 0$.
  - Use the COS formula for $v(x,t_0)$.

COS pricing method for early–exercise options

The pricing formulas, for $m = M-1, \ldots, 1$:

$$\begin{cases} c(x, t_m) &= e^{-r\Delta t} \int_{\mathbb{R}} v(y, t_{m+1}) f(y|x) dy, \\ v(x, t_m) &= \max\left(h(x, t_m), c(x, t_m)\right), \end{cases}$$

followed by

$$v(x, t_0) = e^{-r\Delta t} \int_{\mathbb{R}} v(y, t_1) f(y|x) dy.$$

Here x, y are state variables of consecutive exercise dates. In order to get the option value, a backward recursion procedure is performed on the Fourier–Cosine coefficient $\mathcal{H}_k(t_{m+1})$:

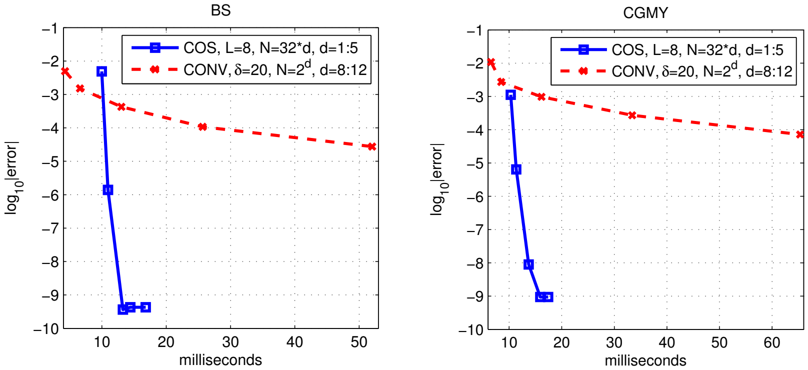
$$\mathcal{H}_k(t_{m+1}) = \frac{2}{b-a} \int_a^b \max(c(x, t_{m+1}), h(x, t_{m+1})) \cos\left(k\pi \frac{y-a}{b-a}\right) dy,$$

and then use $\mathcal{H}_k(t_1)$ in the COS formula.

Bermudan puts with 10 early-exercise dates

Table 3: Test parameters for pricing Bermudan options

Test No.	Model	S_0	K	T	r	σ	Other Parameters
2	BS	100	110	1	0.1	0.2	–
3	CGMY	100	80	1	0.1	0	$C=1, G=5, M=5, Y=1.5$



Towards higher dimensions: Monte Carlo Formulation

- Dynamic Programming:

- COS approximation is fine for up to 3, 4 dimensions! SWIFT works as well for these options.
  Higher D: Monte Carlo simulation techniques, like SGBM.
- The Bermudan option at time t_m and state $\mathbf{S}_{t_m}$ is given by

$$v_{t_m}(\mathbf{S}_{t_m}) = \max(h(\mathbf{S}_{t_m}), c_{t_m}(\mathbf{S}_{t_m})). \quad (8)$$

- The continuation value c_{t_m}, is :

$$c_{t_m}(\mathbf{S}_{t_m}) = e^{-r\Delta t_m}\mathbb{E}\left[v_{t_{m+1}}(\mathbf{S}_{t_{m+1}})|\mathbf{S}_{t_m}\right].$$

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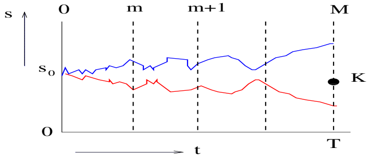

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Step 1: The grid points in SGBM are generated by simulation, $\{\mathbf{S}_{t_0}(n), \ldots, \mathbf{S}_{t_M}(n)\},\ n = 1, \ldots, N,$

Step 2: Compute the option value at terminal time.

Step 3: Bundle the grid points at t_m into $\mathcal{B}_{t_m}(1), \ldots, \mathcal{B}_{t_m}(\nu)$ non-overlapping bundles.

Step 4: For $\mathcal{B}_{t_m}(\beta),\ \beta = 1, \ldots, \nu,$ compute $Z(\mathbf{S}_{t_{m+1}}, \alpha^{\beta}_{t_{m+1}})$.
$Z : \mathbb{R}^d \times \mathbb{R}^K \mapsto \mathbb{R},$ is a parametrized function which assigns values to states $\mathbf{S}_{t_{m+1}}$.

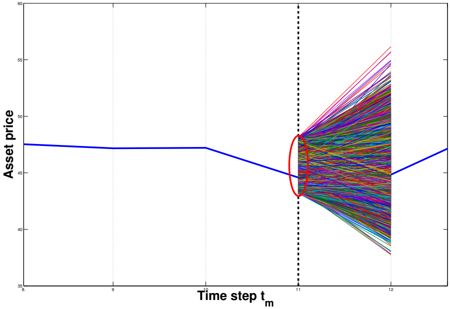
Step 5: The continuation values for grid points in $\mathcal{B}_{t_m}(\beta),\ \beta = 1, \ldots, \nu,$ are approximated by

$$\widehat{c}_{t_m}(\mathbf{S}_{t_m}(n)) = \mathbb{E}[Z(\mathbf{S}_{t_{m+1}}, \alpha^{\beta}_{t_{m+1}})|\mathbf{S}_{t_m}(n)]$$

Intuition

- The objective is to choose, corresponding to each bundle β at t_m, a parameter vector $\alpha^{\beta}_{t_{m+1}}$ so that, $V_{t_{m+1}}(\mathbf{S}_{t_{m+1}}) \approx Z(\mathbf{S}_{t_{m+1}}, \alpha^{\beta}_{t_{m+1}})$.
- We use OLS, to define

$$Z(\mathbf{S}_{t_{m+1}}, \widehat{\alpha}^{\beta}_{t_{m+1}}) = \sum_{k=1}^{K} \widehat{\alpha}^{\beta}_{t_{m+1}}(k)\phi_k(\mathbf{S}_{t_{m+1}}). \tag{9}$$



- In the limiting case, of number of bundles ν, and paths N, the

Computing the continuation value

- The continuation value is computed as:

$$\hat{c}_{t_m}(\mathbf{S}_{t_m}(n)) = \mathbb{E}[Z(\mathbf{S}_{t_{m+1}}, \hat{\alpha}^{\beta}_{t_{m+1}})|\mathbf{S}_{t_m} = \mathbf{S}_{t_m}(n)], \qquad (10)$$

  where $\mathbf{S}_{t_m}(n) \in \mathcal{B}_{t_m}(\beta)$.

- This can be written as:

$$\begin{aligned}\hat{c}_{t_m}(\mathbf{S}_{t_m}(n)) &= \mathbb{E}\left[\left(\sum_{k=1}^{K} \hat{\alpha}^{\beta}_{t_{m+1}}(k)\phi_k(\mathbf{S}_{t_{m+1}})\right)|\mathbf{S}_{t_m} = \mathbf{S}_{t_m}(n)\right] \\ &= \sum_{k=1}^{K} \hat{\alpha}^{\beta}_{t_{m+1}}(k)\mathbb{E}\left[\phi_k(\mathbf{S}_{t_{m+1}})|\mathbf{S}_{t_m} = \mathbf{S}_{t_m}(n)\right].\end{aligned}$$

- Choose basis functions ϕ so that $\mathbb{E}\left[\phi_k(\mathbf{S}_{t_{m+1}})|\mathbf{S}_{t_m} = X\right]$ has an analytic solution.

Arithmetic Basket Option on 15 assets

- Convergence and computational time:

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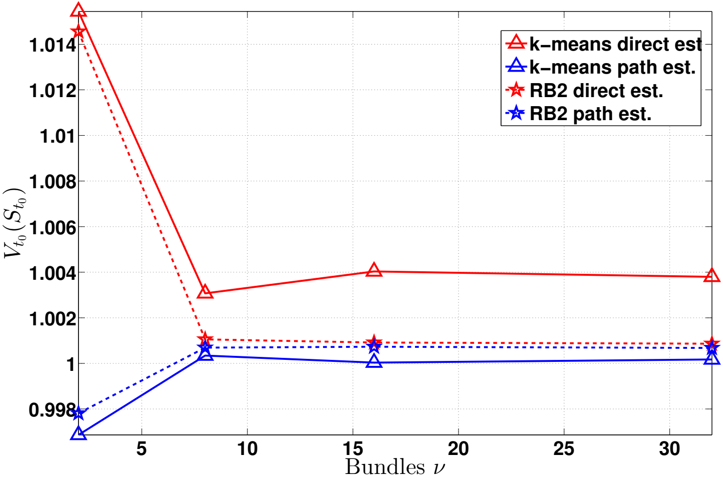

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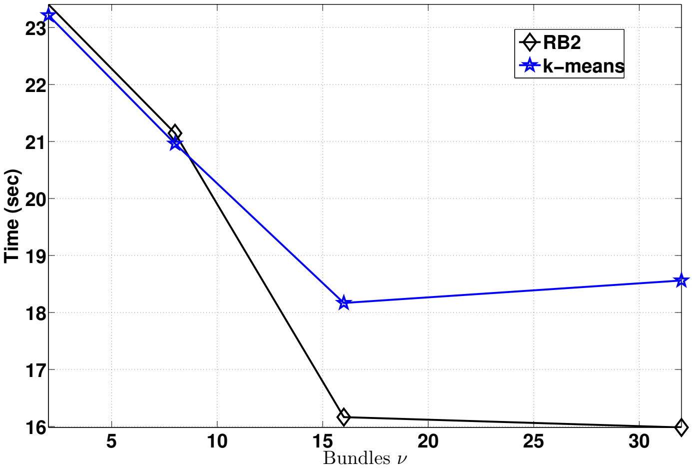
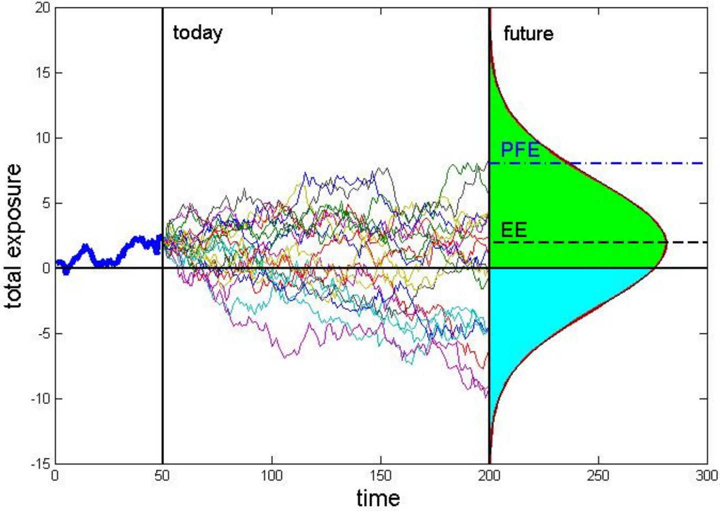


Expected exposure in CVA (joint work with Qian Feng)



Exposure and Counterparty risk

- Exposure at a path at time t_m with stock $\mathbf{S}_{t_m}$

$$E(\mathbf{S}_{t_m}, t_m) = \begin{cases} c_{t_m}(\mathbf{S}_{t_m}), & \text{not exercised} \\ 0, & \text{exercised} \end{cases} \tag{11}$$

- Expected Exposure (EE) at time t_m

$$\begin{aligned} \mathrm{EE}(t_m) &= \mathbb{E}\left[E(\mathbf{S}_{t_m}, t_m)\right] \\ &\approx \frac{1}{N}\sum_{n=1}^{N} E(\mathbf{S}_{t_m}(n), t_m) \end{aligned} \tag{12}$$

$\Rightarrow c_{t_m}(\mathbf{S}_{t_m})$ is calculated at each time step, along the path, by SGBM.

- The semilinear partial differential equation:

$$\frac{\partial v(t,x)}{\partial t} + \mathcal{L}v(t,x) + g(t,x,v,\sigma(x)Dv(t,x)) = 0,\ v(T,x) = h(x),$$

We can solve this PDE by means of the FSDE:

$$dX_s = \mu(X_s)ds + \sigma(X_s)d\omega_s,\ X_t = x.$$

and the BSDE:

$$dY_s = -g(s, X_s, Y_s, Z_s)ds + Z_s d\omega_s,\ Y_T = h(X_T).$$

- Theorem:

$$Y_t = v(t, X_t),\ Z_t = \sigma(X_t)Dv(t, X_t).$$

is the solution to the decoupled FBSDE.

Backward SDE

$$dY_t = -g(t, Y_t, Z_t)dt + Z_t d\omega_t, \quad Y_T = \xi.$$

g is the *driver* function. ξ is $\mathcal{F}_T$-measurable random variable.
A solution is a *pair* adapted processes (Y, Z) satisfying

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s d\omega_s.$$

Y is adapted if and only if, for every realization and every t, Y_t is known at time t. Adapted process cannot "see into the future".

A BSDE is *not a time-revered* FSDE: at time t (Y_t, Z_t) is $\mathcal{F}_t$-measurable and the process does not "know" the terminal condition yet.

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Y component

$$Y_m = Y_{m+1} + \int_{t_m}^{t_{m+1}} g(s, \mathcal{X}_s)ds - \int_{t_m}^{t_{m+1}} Z_s d\omega_s. \tag{13}$$

$\mathcal{X}_t := (X_t, Y_t, Z_t)$.

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Taking conditional expectation $\mathbb{E}_m[.] = \mathbb{E}[.|\mathcal{F}_{t_m}]$:

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Z component

$$Y_m = Y_{m+1} + \int_{t_m}^{t_{m+1}} g(s, \mathcal{X}_s) ds - \int_{t_m}^{t_{m+1}} Z_s d\omega_s. \qquad (15)$$

Multiplying by $\Delta\omega_{m+1}$, taking the conditional expectation gives

Z component

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Multiplying by $\Delta\omega_{m+1}$, taking the conditional expectation gives

$$Z_m \approx \frac{1}{\Delta t} \mathbb{E}_m[Y_{m+1} \Delta\omega_{m+1}].$$

Discretization scheme FBSDE

$$
\begin{aligned}
X_0^\Delta &= x_0, \\
&\text{for } m = 0, \dots, M-1: \\
X_{m+1}^\Delta &= X_m^\Delta + \mu(X_m^\Delta)\Delta t + \sigma(X_m^\Delta)\Delta\omega_{m+1}, \\
Y_M^\Delta &= h(X_M^\Delta), \\
&\text{for } m = M-1, \dots, 0: \\
Z_m^\Delta &= \frac{1}{\Delta t}\mathbb{E}_m[Y_{m+1}^\Delta \Delta\omega_{m+1}], \\
Y_m^\Delta &= \mathbb{E}_m[Y_{m+1}^\Delta] + g(t_m, \mathcal{X}_m^\Delta)\Delta t.
\end{aligned}
$$

Ingredients:

- Characteristic function X_{m+1}^Δ, given $X_m^\Delta = x$
- $\mathbb{E}_m^x[h(t_{m+1}, X_{m+1}^\Delta)]$
- $\mathbb{E}_m^x[h(t_{m+1}, X_{m+1}^\Delta)\Delta\omega_{m+1}]$
- Recover Fourier cosine coefficients backward in time
- FFT algorithm

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COS formula

$$I := \mathbb{E}_m^x[h(t_{m+1}, X_{m+1}^\Delta)] = \int_{\mathbb{R}} h(t_{m+1}, \zeta) f(\zeta|x) d\zeta$$
$$\approx \int_a^b h(t_{m+1}, \zeta) f(\zeta|x) d\zeta. \quad (16)$$

Replace the density function and function h by their Fourier cosine series expansions.

$$I \approx \frac{b-a}{2} {\sum_{k=0}^{N-1}}' \mathcal{H}_k(t_{m+1}) P_k(x). \quad (17)$$

The COS formula:

$$\mathbb{E}_m^x[h(t_{m+1}, X_{m+1}^\Delta)] \approx {\sum_{k=0}^{N-1}}' \mathcal{H}_k(t_{m+1}) \Re \left(\widehat{f}_{X_{m+1}^\Delta} \left(\frac{k\pi}{b-a} | X_m^\Delta = x \right) e^{ik\pi \frac{-a}{b-a}} \right).$$

Example: European Call Option - GBM - $\mathbb{P}$-Measure

Asset price: $dS_t = \bar{\mu} S_t dt + \bar{\sigma} S_t d\omega_t.$

Hedge portfolio Y_t with: a_t assets S_t and $Y_t - a_t S_t$ bonds

$$dY_t = r(Y_t - a_t S_t)dt + a_t dS_t$$

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$$\begin{aligned} dY_t &= r(Y_t - a_t S_t)dt + a_t dS_t \\ &= \left(rY_t + \frac{\bar{\mu} - r}{\bar{\sigma}} \bar{\sigma} a_t S_t \right) dt + \bar{\sigma} a_t S_t d\omega_t, \\ Y_T &= h(S_T) = \max(S_T - K, 0). \end{aligned}$$

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If we set $Z_t = \bar{\sigma} a_t S_t$, then (Y, Z) solves a BSDE with driver,

$$g(t, x, y, z) = -ry - \frac{\bar{\mu} - r}{\bar{\sigma}} z.$$

Y_t corresponds to the value of the option and Z_t is related to the hedging strategy. The option value is given by $v(t, S_t) = Y_t$ and

Results European call option

Exact solutions $Y_0 = v(0, S_0) = 3.66$ and $Z_0 = \bar{\sigma} S_0 v_S(0, S_0) = 14.15$.

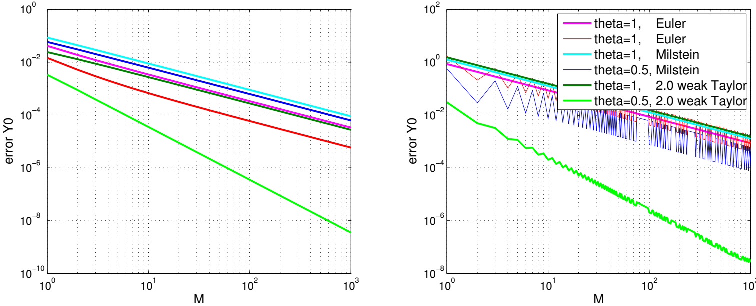

Figure 1: Euler, Milstein and order 2.0 weak Taylor scheme, $\theta = 1$ and $\theta = 0.5$.

Summary and conclusion

- COS and SWIFT methods for pricing European options
- BCOS method for nonlinear PDEs
- Second-order convergence met 2.0 weak Taylor scheme and $\theta = 1/2$ discretization of integrals!
- Higher dimensions: Generalization to Monte Carlo method SGBM
- Challenges: Higher dimensions (MC, SGBM), stochastic control, portfolio selection

$\Rightarrow$ Towards accurate, efficient and robust solution methods.

Computational finance

- Efficient valuation of financial options
  - → COS method is used at many (financial) institutions world-wide;
  - works efficient for a variety of options (barrier, Asian, multi-asset, . . .)
  - High-D American options, Monte Carlo simulation, SGBM
  - Interpolation methods, Stochastic Collocation Monte Carlo methods (SCMC sampler)
  - Under recent asset price model dynamics, for different asset classes
- Risk management
  - Accurate hedge parameters;
  - Numerical estimation of tail probabilities and expected shortfall, Value-at-Risk;
  - (Counterparty) Credit risk and other types of risk;
  - Inflation options for pension funds.
- Portfolio optimization (2010-...)
  - Energy portfolio, real options analysis
  - Dynamic portfolios for pensions (target based vs time-consistent mean-variance strategy)

⇒ Challenge: Incorporate (big) financial data

Characteristic function

We can write the Euler, Milstein, and 2.0 weak Taylor discretization schemes in the following general form

$$X_{m+1}^{\Delta} = x + m(x)\Delta t + s(x)\Delta\omega_{m+1} + \kappa(x)(\Delta\omega_{m+1})^2, \quad X_m^{\Delta} = x.$$

For the Euler scheme:

$$m(x) = \mu(x), \quad s(x) = \sigma(x), \quad \kappa(x) = 0.$$

For the Milstein scheme:

$$m(x) = \mu(x) - \frac{1}{2}\sigma\sigma_x(x), \quad s(x) = \sigma(x), \quad \kappa(x) = \frac{1}{2}\sigma\sigma_x(x).$$

For the order 2.0 weak Taylor scheme:

$$\begin{aligned} m(x) &= \mu(x) - \frac{1}{2}\sigma\sigma_x(x) + \frac{1}{2}\left(\mu\mu_x(x) + \frac{1}{2}\mu_{xx}\sigma^2(x)\right)\Delta t, \\ s(x) &= \sigma(x) + \frac{1}{2}\left(\mu_x\sigma(x) + \mu\sigma_x(x) + \frac{1}{2}\sigma_{xx}\sigma^2(x)\right)\Delta t, \\ \kappa(x) &= \frac{1}{2}\sigma\sigma_x(x). \end{aligned}$$

Characteristic function

$$
\begin{aligned}
X^{\Delta}_{m+1} &= x + m(x)\Delta t + \kappa(x)\left(\Delta\omega_{m+1} + \frac{1}{2}\frac{s(x)}{\kappa(x)}\right)^2 - \frac{1}{4}\frac{s^2(x)}{\kappa(x)} \\
&\stackrel{d}{=} x + m(x)\Delta t - \frac{1}{4}\frac{s^2(x)}{\kappa(x)} + \kappa(x)\Delta t\left(U_{m+1} + \sqrt{\lambda(x)}\right)^2,
\end{aligned}
$$

with $\lambda(x) := \frac{1}{4}\frac{s^2(x)}{\kappa^2(x)\Delta t}$, $U_{m+1} \sim \mathcal{N}(0,1)$. $(U_{m+1} + \sqrt{\lambda(x)})^2 \sim \chi_1'^2(\lambda(x))$ non-central chi-squared distributed.

The characteristic function of X^{Δ}_{m+1}, given $X^{\Delta}_m = x$

$$
\begin{aligned}
\widehat{f}_{X^{\Delta}_{m+1}}(u|X^{\Delta}_m = x) &= \mathbb{E}\left[\exp\left(iuX^{\Delta}_{m+1}\right)\Big|X^{\Delta}_m = x\right] \\
&= \exp\left(iux + ium(x)\Delta t - \frac{\frac{1}{2}u^2 s^2(x)\Delta t}{1-2iu\kappa(x)\Delta t}\right)(1 - 2iu\kappa(x)\Delta t)^{-1/2}.
\end{aligned}
$$