On conservation laws of Navier-Stokes Galerkin discretizations

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The Navier–Stokes Equations

The Navier–Stokes equations for a viscous, incompressible Newtonian fluid in a domain $\Omega \subset \mathbb{R}^d$, d=2 or 3, and for $t > 0$,

$$\begin{aligned} \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \nu \Delta \mathbf{u} &= \mathbf{f}, \\ \operatorname{div} \mathbf{u} &= 0, \\ \mathbf{u}(0) &= \mathbf{u}_0, \end{aligned}$$

where

- $\mathbf{u} = (u, v, w)$ is the velocity field,
- p is the pressure,
- $\mathbf{f}$ is a known forcing term (e.g., gravity),
- $\nu > 0$ is the kinematic viscosity coefficient,
- $\mathbf{u}_0$ are given initial data.

Conserved quantities

With assumptions on ν, $\mathbf{f}$, b.c., conserved or properly balanced:

Kinetic energy	$\frac{1}{2}\int_\Omega \lvert\mathbf{u}\rvert^2 \mathrm{d}\mathbf{x};$
Linear momentum	$\int_\Omega \mathbf{u}\,\mathrm{d}\mathbf{x};$
Angular momentum	$\int_\Omega \mathbf{u}\times\mathbf{x}\,\mathrm{d}\mathbf{x}.$

and for $\mathbf{w} = \nabla \times \mathbf{u}$,

Helicity	$\int_\Omega \mathbf{u}\cdot\mathbf{w}\,\mathrm{d}\mathbf{x};$
2D Enstrophy	$\frac{1}{2}\int_\Omega \lvert\mathbf{w}\rvert^2\,\mathrm{d}\mathbf{x}$ (for a 2D flow);
Total vorticity	$\int_\Omega \mathbf{w}\,\mathrm{d}\mathbf{x}.$

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Bring back conservation laws to NSE, shallow water and other discretized fluid equations: The papers of A. Arakawa, G. Fix, V. Lamb, R. Temam, T. Hughes, A. Majda, L. Rebholz and others.

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Then $\operatorname{div}\mathbf{u} = 0$ yields to

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This implies $\operatorname{div}\mathbf{u}_h = 0$ if $\operatorname{div}\mathbf{X} \subset Q$.
However, $\mathbf{X}$ should be large enough to ensure the inf-sup stability:

$$\inf_{q\in Q} \sup_{\mathbf{u}_h\in\mathbf{X}} \frac{(\operatorname{div}\mathbf{u}_h, q)}{\|\nabla\mathbf{u}_h\|\|q\|} \geq c_0 > 0.$$

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Recent attempts to build pointwise mass preserving Galerkin methods: R. Falk and M. Neilan (SINUM 2013), J. Guzman and M. Neilan (Math Comp 2014, IMA Num Anal 2014), S. Zhang (Math Comp 2005), J. Evans and T. Hughes (JCP 2013).

Conservation properties with $\operatorname{div}\mathbf{u} \neq 0$

Galerkin method with a generic form of inertia term: find $\{\mathbf{u}, p\} \in \mathbf{X} \times Q$

$$\left(\frac{\partial \mathbf{u}}{\partial t} + N(\mathbf{u}), \mathbf{v}\right) - (p, \operatorname{div}\mathbf{v}) + (q, \operatorname{div}\mathbf{u}) + \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) = (\mathbf{f}, \mathbf{v})$$

for all $\mathbf{v} \in \mathbf{X}$, $q \in Q$. Conservation properties are largely dictated by $N(\mathbf{u})$.

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\text{convective} \quad &: \quad N(\mathbf{u}) = \mathbf{u}\cdot\nabla\mathbf{u} \\
\text{skew-symmetric} \quad &: \quad N(\mathbf{u}) = \mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{2}(\operatorname{div}\mathbf{u})\mathbf{u} \\
\text{rotational} \quad &: \quad N(\mathbf{u}) = (\nabla\times\mathbf{u})\times\mathbf{u}, \\
\text{conservative} \quad &: \quad N(\mathbf{u}) = \operatorname{div}(\mathbf{u}\otimes\mathbf{u}) = \mathbf{u}\cdot\nabla\mathbf{u} + (\operatorname{div}\mathbf{u})\mathbf{u}.
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New *energy momentum and angular momentum (EMA) conserving form*:

$$\text{EMA conserving} \quad : \quad N(\mathbf{u}) = 2\mathbf{D}(\mathbf{u})\mathbf{u} + (\operatorname{div}\mathbf{u})\mathbf{u},$$

where $\mathbf{D}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla^T\mathbf{u})$ is the rate of strain tensor

Conservation properties with $\operatorname{div}\mathbf{u} \neq 0$

$\left(\frac{\partial \mathbf{u}}{\partial t} + N(\mathbf{u}), \mathbf{v}\right) - (p, \operatorname{div}\mathbf{v}) + (q, \operatorname{div}\mathbf{u}) + \nu(\nabla\mathbf{u}, \nabla\mathbf{v}) = (\mathbf{f}, \mathbf{v})$

name	$N(\mathbf{u})$	pressure
convective:	$\mathbf{u}\cdot\nabla\mathbf{u}$	p (kinematic)
skew-symmetric:	$\mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{2}(\operatorname{div}\mathbf{u})\mathbf{u}$	p (kinematic)
rotational:	$(\nabla\times\mathbf{u})\times\mathbf{u}$	$p + \frac{1}{2}\lvert\mathbf{u}\rvert^2$ (Bernoulli)
conservative:	$\operatorname{div}(\mathbf{u}\otimes\mathbf{u})$	p (kinematic)
EMAC:	$2\mathbf{D}(\mathbf{u})\mathbf{u} + (\operatorname{div}\mathbf{u})\mathbf{u}$	$p - \frac{1}{2}\lvert\mathbf{u}\rvert^2$ (no name)

EMAC is based on the following simple identity:

$$\mathbf{u}\cdot\nabla\mathbf{u} = 2\mathbf{D}(\mathbf{u})\mathbf{u} - \frac{1}{2}\nabla\lvert\mathbf{u}\rvert^2.$$

Conservation properties with $\operatorname{div}\mathbf{u} \neq 0$

$$\left(\frac{\partial \mathbf{u}}{\partial t} + N(\mathbf{u}), \mathbf{v}\right) - (p, \operatorname{div}\mathbf{v}) + (q, \operatorname{div}\mathbf{u}) + \nu(\nabla\mathbf{u}, \nabla\mathbf{v}) = (\mathbf{f}, \mathbf{v})$$

name	$N(\mathbf{u})$	Energy	Momentum	Ang. Moment.
convective:	$\mathbf{u}\cdot\nabla\mathbf{u}$			
skew-symmetric:	$\mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{2}(\operatorname{div}\mathbf{u})\mathbf{u}$	+		
rotational:	$(\nabla\times\mathbf{u})\times\mathbf{u}$	+		
conservative:	$\operatorname{div}(\mathbf{u}\otimes\mathbf{u})$		+	+
EMAC:	$2\mathbf{D}(\mathbf{u})\mathbf{u} + (\operatorname{div}\mathbf{u})\mathbf{u}$	+	+	+

Proposition

The *skew-symmetric*, *rotational*, and *EMAC* formulations conserve kinetic energy (for $\nu = 0$, $\mathbf{f} = \mathbf{0}$), and only the *EMAC* and *conservative* formulations conserve momentum (for $\mathbf{f}$ with zero linear momentum) and angular momentum (for $\mathbf{f}$ with zero angular momentum).

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$$\left(\frac{\partial \mathbf{u}}{\partial t} + N(\mathbf{u}), \mathbf{v}\right) - (p, \operatorname{div}\mathbf{v}) + (q, \operatorname{div}\mathbf{u}) + \nu(\nabla\mathbf{u}, \nabla\mathbf{v}) = (\mathbf{f}, \mathbf{v})$$

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$\mathbf{w}_h = \nabla\times\mathbf{u}_h$ conserves *nothing* of vorticity, helicity, 2D-enstrophy for any of the formulations. Even if $\operatorname{div}\mathbf{u}_h = 0$ pointwise.

Vorticity equation

Even $\mathbf{u}_h$, s.t. $\operatorname{div}\mathbf{u}_h = 0$, does not conserve

$$\int_\Omega \mathbf{u}_h \cdot (\nabla \times \mathbf{u}_h)\mathbf{dx}, \quad \frac{1}{2}\int_\Omega |\nabla \times \mathbf{u}_h|^2 \mathbf{dx}, \quad \int_\Omega \nabla \times \mathbf{u}_h \mathbf{dx}.$$

To address this, consider vorticity equation:

$$\mathbf{w}_t + (\mathbf{u}\cdot\nabla)\mathbf{w} - (\mathbf{w}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{w} = \nabla \times \mathbf{f}.$$

O., Rebholz in JCP (2010); Evans, Hughes in JCP (2013)

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and use $\mathbf{u}_h$ instead of $\mathbf{u}$ in the Galerkin formulation:

$$\left(\frac{\partial \mathbf{w}_h}{\partial t} + (\mathbf{u}_h\cdot\nabla)\mathbf{w}_h - (\mathbf{w}_h\cdot\nabla)\mathbf{u}_h, \mathbf{v}\right) + \nu(\nabla\mathbf{w}_h, \nabla\mathbf{v}_h) + (\eta_h, \operatorname{div}\mathbf{v}) - (q, \operatorname{div}\mathbf{w}_h) = (\nabla\times\mathbf{f}, \mathbf{v})$$

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... or in the Galerkin formulation:

$$\left(\frac{\partial \tilde{\mathbf{w}}_h}{\partial t} + (\mathbf{u}_h\cdot\nabla)\tilde{\mathbf{w}}_h - (\tilde{\mathbf{w}}_h\cdot\nabla)\mathbf{u}_h, \mathbf{v}\right) + ((\operatorname{div}\mathbf{u}_h)\tilde{\mathbf{w}}_h, \mathbf{v}) - ((\operatorname{div}\tilde{\mathbf{w}}_h)\mathbf{u}_h, \mathbf{v}) + \nu(\nabla\tilde{\mathbf{w}}_h, \nabla\mathbf{v}) + (\eta_h, \operatorname{div}\mathbf{v}) - (q, \operatorname{div}\tilde{\mathbf{w}}_h) = (\nabla\times\mathbf{f}, \mathbf{v})$$

Similar for 2D vorticity!

O., Rebholz in JCP (2010); Evans, Hughes in JCP (2013)

Vorticity equation

Proposition

Let $\mathbf{u}_h$ Galerkin solutions with the *EMA-conserving* form, and $\mathbf{w}_h$, $\tilde{\mathbf{w}}_h$, w_h are finite element vorticity solutions. Then helicity (for $\mathbf{f} = 0$, $\nu = 0$), 2D enstrophy (for $\nabla \times \mathbf{f} = \mathbf{0}$, $\nu = 0$), and total vorticity are conserved in the sense of the quantities: $H = (\mathbf{u}_h, \mathbf{w}_h)$, $H_{2D} = \frac{1}{2}\|w_h\|^2$, and $W_i = (\tilde{\mathbf{w}}_h, \mathbf{e}_i)$.

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$$\left(\frac{\partial \mathbf{w}_h}{\partial t} + (\mathbf{u}_h \cdot \nabla)\mathbf{w}_h - (\mathbf{w}_h \cdot \nabla)\mathbf{u}_h, \mathbf{v}\right) + \nu(\nabla \mathbf{w}_h, \nabla \mathbf{v}_h) + (\eta_h, \operatorname{div} \mathbf{v}) - (q, \operatorname{div} \mathbf{w}_h) = (\nabla \times \mathbf{f}, \mathbf{v})$$

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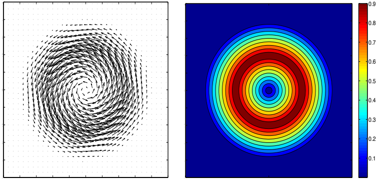
Numerical Example (Standing vortex)

Standing vortex equilibrium solution (solves $\mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = 0$):



True velocity solution as a vector plot (left) and speed contour plot (right)

Compute with different formulations, Crank-Nicolson time stepping, $\mathbf{f} = \mathbf{0}$, $\nu = 0$, (P_2, P_1) Taylor-Hood elements on a 32x32 uniform mesh, $\Delta t = 0.01$, and $u|_{t=0}=$ Interp(exact solution).

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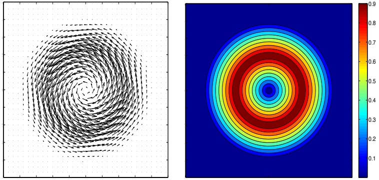

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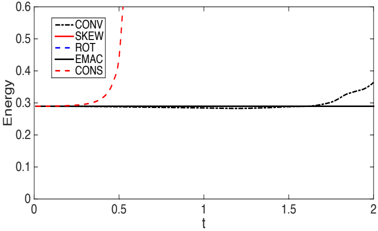

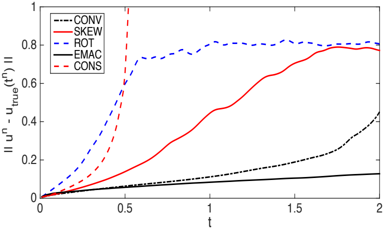

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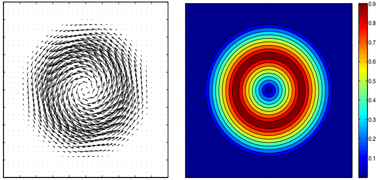

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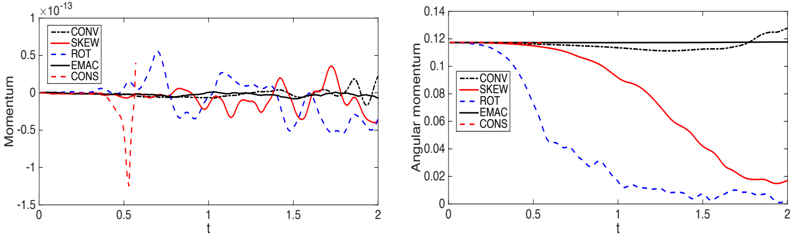
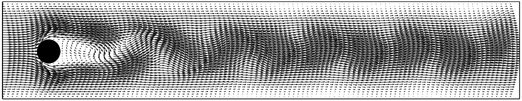


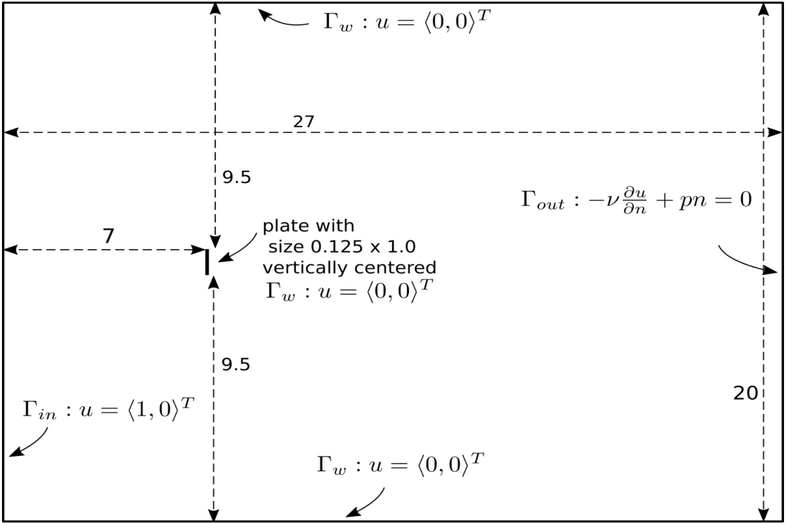
Numerical Example: Channel flow around a cylinder

Benchmark from Schafer & Turek with variable Re (from 0 to 100)



Method	$dim(X_h)$	Δt	c_d^{max}	$\|error\|$	c_l^{max}	$\|error\|$	$\Delta p(8)$	$\|error\|$
ROT	34,762	0.005	2.94442	6.48E-3	0.412069	6.59E-2	-0.11168	8.20E-5
CONV	34,762	0.005	2.94672	4.18E-3	0.470062	**7.94E-3**	-0.11176	1.62E-4
SKEW	34,762	0.005	2.94678	4.12E-3	0.467538	1.05E-2	-0.11177	1.70E-4
CONS	34,762	0.005	2.94667	4.25E-3	0.450239	2.77E-2	-0.11179	1.90E-4
EMAC	34,762	0.005	2.94819	**2.71E-3**	0.525675	4.77E-2	-0.11166	**5.68E-5**
ROT	61,694	0.005	2.94638	4.52E-3	0.484486	6.49E-3	-0.11139	2.10E-4
CONV	61,694	0.005	2.94893	1.97E-3	0.478282	**2.82E-4**	-0.11159	**1.13E-5**
SKEW	61,694	0.005	2.94892	1.98E-3	0.477249	7.51E-4	-0.11158	2.15E-5
CONS	61,694	0.005	2.94891	1.99E-3	0.477013	9.37E-4	-0.11149	1.10E-4
EMAC	61,694	0.005	2.94961	**1.29E-3**	0.490655	1.27E-2	-0.11119	4.06E-4
ROT	95,738	0.005	2.94919	1.71E-3	0.480021	2.02E-3	-0.11186	2.64E-4
CONV	95,738	0.005	2.94966	1.24E-3	0.478567	5.67E-4	-0.11155	**5.00E-5**
SKEW	95,738	0.005	2.94966	1.24E-3	0.478106	**1.06E-4**	-0.11154	6.04E-5
CONS	95,738	0.005	2.94966	1.24E-3	0.477831	1.19E-4	-0.11155	**5.00E-5**
EMAC	95,738	0.005	2.94986	**1.04E-3**	0.484425	6.43E-3	-0.11141	1.93E-4

Numerical Example: Flow past a flat plate, Re=100

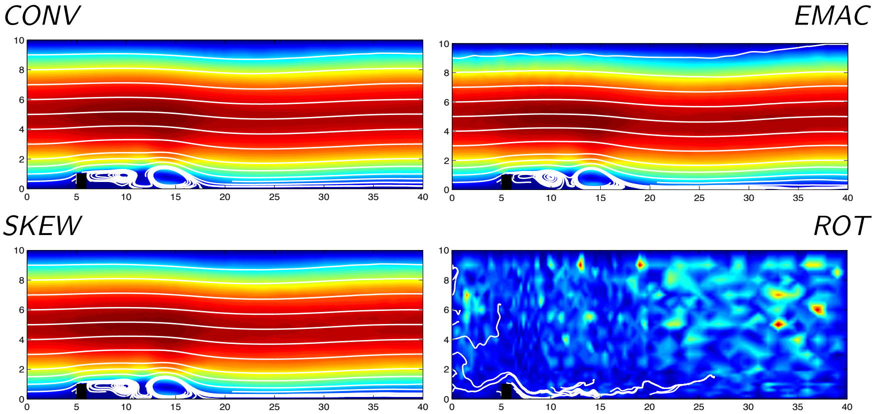


Numerical Example: Flow past a flat plate, Re=100

Formulation	Re	Average C_d	Recirculation point
CONV	100	2.5434	1.1577
EMAC	100	2.6598	1.1648
SKEW	100	2.5903	1.1565
ROT	100	failed: energy blows up at T=25	
CONS	100	failed: energy blows up at T=78	
Very fine discretization	100	2.6454	1.1373

Numerical example: Channel flow past a forward-backward facing step

Finer grid:

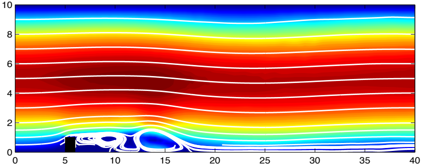


CONS formulation FE solution blows up.

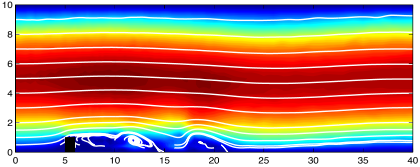
Numerical example: Channel flow past a forward-backward facing step

Coarser grid

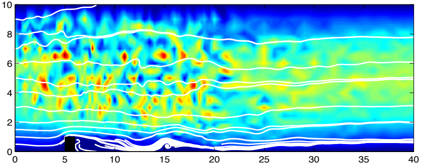
CONV



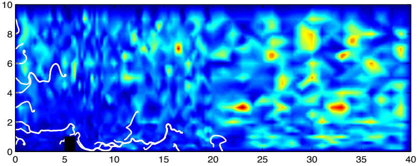
EMAC



SKEW



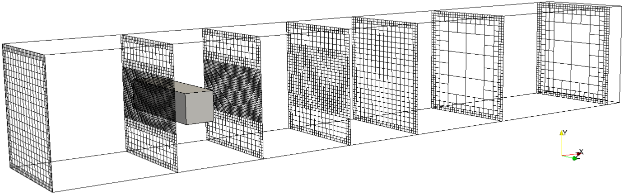
ROT



CONS formulation FE solution blows up.

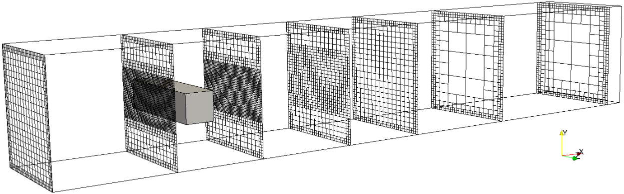
Numerical example: 3D Flow around a Square Cylinder

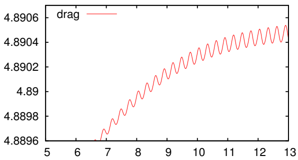
Re=100 is close to the critical one, where the transition from equilibrium to unsteady periodic solution takes place. ⇒ Right balance of inertia and viscous diffusion is crucial for numerical method to produce stable periodic solutions.

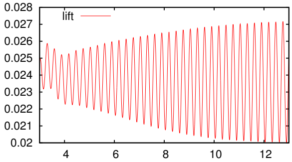


Numerical example: 3D Flow around a Square Cylinder

Re=100 is close to the critical one, where the transition from equilibrium to unsteady periodic solution takes place. $\Rightarrow$ Right balance of inertia and viscous diffusion is crucial for numerical method to produce stable periodic solutions.



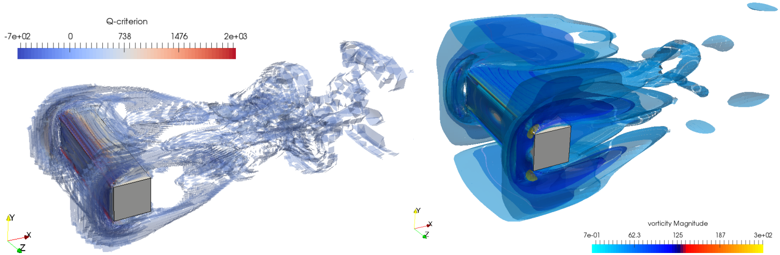




Lift and drag coefficients plotted over time.

Numerical example: 3D Flow around a Square Cylinder

	max drag	max lift	Strouhal	Dofs
EMAC results	4.890	0.0271	0.351	6.4 mill
OTV (2013)	4.484	0.0316	0.307	17 mill
Schafer et al (1996)	4.32-4.67	0.015-0.05	0.27-0.35	Up to 6 mill



Q-criterion and vorticity magnitude contours at t=12.0

Conclusions

- New (?!) EMAC formulation, $2(\mathbf{D}\mathbf{u})\mathbf{u} + (\operatorname{div}\mathbf{u})\mathbf{u} - \frac{1}{2}\nabla|\mathbf{u}|^2$, conserves all of energy, momentum, angular momentum of Galerkin solutions;
- It also conserves appropriately defined vorticity, helicity, and enstrophy;
- None of *convective*, *conservative*, *rotational*, and *skew-symmetric* formulations conserve each of these quantities;
- In a few numerical experiments EMAC performs at least as good, or better, than the commonly used formulations;
- Things to do: More testing, efficient solvers, higher Re numbers and turbulent flows, preservation of coherent flow structures, etc.

Alternative to div-free CFD?!