A Posteriori Error Bounds for Approximations of the Stokes Problem with Friction Type Boundary Conditions

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The Stokes problem

The classical Stokes problem consists of finding a velocity field $u \in S_0(\Omega, \mathbb{R}^d) + u_D$ and a pressure field $p \in \widetilde{L}_2(\Omega)$ which satisfy the relations

$$\begin{aligned} -\operatorname{Div}(\nu \nabla u) + \nabla p &= f && \text{in } \Omega, \\ \operatorname{div} u &= 0 && \text{in } \Omega, \\ u &= u_D && \text{on } \Gamma_D, \end{aligned}$$

where $f \in L_2(\Omega, \mathbb{R}^d)$, u_D is a given divergence free function, and $\sigma = \nu \nabla u$ is the stress tensor.

Nonlinear boundary condition

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Since

$$\partial|z| = \begin{cases} \frac{z}{|z|} & \text{if } z \neq 0,\ z \in \mathbb{R}^d, \\ \zeta \in \mathbb{R}^d, \quad |\zeta| \leq 1 & \text{if } z = 0,\ z \in \mathbb{R}^d, \end{cases}$$

the condition is equivalent to

$$|\sigma_n| \leq g,\ \sigma_n u_n + g|u_n| = 0 \quad \text{on } \Gamma.$$

The generalized solution $u \in S_0(\Omega, \mathbb{R}^d) + u_D$ is defined by the variational inequality

$$a(u, v-u) + \int_\Gamma (j(v_t) - j(u_t))\mathrm{d}S \geq (f, v-u) \quad \forall v \in S_0(\Omega, \mathbb{R}^d),$$

where

$$a(u, v-u) := \int\limits_\Omega \nu \nabla u : \nabla(v-u)\mathrm{d}x,$$

$j(\zeta) := g|\zeta|$ for $\zeta \in H^{1/2}(\Gamma)$. Hence, if $|\zeta^*| \leq g$, then

$$D_j(\zeta, \zeta^*) = \int\limits_\Gamma (j(\zeta) + j*(\zeta^*) - \zeta^* \cdot \zeta)\mathrm{d}S = \int\limits_\Gamma (g|\zeta| - \zeta^* \cdot \zeta)\mathrm{d}S.$$

For any $v \in S_0(\Omega, \mathbb{R}^d) + u_D$, $\tau \in \Sigma$, $q \in L_2$, and $\alpha < 2$

$$\frac{1}{2} \| u - v \|^2 \leq M_1 := \frac{1}{2-\alpha} D_j(v_t, \eta) + \frac{1}{2\alpha(2-\alpha)} \left(\| d(v, q, \tau) \|_* + \frac{1}{\sqrt{\underline{\nu}}} \| \mathcal{L}_{\eta,\tau} \| \right)^2,$$

where

$$D_j(v_t, \eta) := \int_\Gamma (j(v_t) + j^*(\eta) - \eta \cdot v_t) \, \mathrm{d}S,$$

$$d(v, q, \tau) := \nu \nabla v - \tau - \mathrm{q} \,,$$

$$\mathcal{L}_{\eta,\tau}(w) := \int_\Omega (\tau : \nabla w - f \cdot w) dx + \int_\Gamma \eta \cdot w_t \mathrm{d}S$$

$$\| \mathcal{L}_{\eta,\tau} \| := \sup_{w \in V_0} \frac{|\mathcal{L}_{\eta,\tau}(w)|}{\|\nabla w\|}.$$

In order to deduce a fully computable error majorant, the sets of admissible η and τ are narrowed, namely,

$$\tau \in H_\Gamma(\mathrm{Div}, \Omega) := \left\{ \tau \in H(\mathrm{Div}, \Omega),\ \tau_t \in \widetilde{L}^2(\Omega, \mathbb{R}^d) \text{ on } \Gamma \right\},$$

$$\int_\Gamma (\eta - \tau_t) \mathrm{d}S = 0.$$

It is clear that for $\tau = \sigma$ and $\eta = \sigma_t$ this condition holds, so that these restrictions do not exclude physically meaningful functions.

In this case,

$$\mathcal{L}_{\eta,\tau}(w) = \int\limits_{\Omega} (f + \operatorname{Div}\tau) \cdot w \mathrm{d}x + \int\limits_{\Gamma} (\eta - \tau_t) \cdot w_t \, \mathrm{d}S.$$

Notice that $w_n = 0$ on Γ, so that the last integral is estimated as follows:

$$\int_{\Gamma} (\eta - \tau_t) \cdot w_t \mathrm{d}S = \int_{\Gamma} (\eta - \tau_t) \cdot (w_t - \{\, w_t \,\}_{\Gamma}) \mathrm{d}S$$
$$\leq \|\eta - \tau_t\|_{\Gamma} \, \|w - \{\, w \,\}_{\Gamma}\| \leq \widetilde{C}_{\Gamma} \|\eta - \tau_t\|_{\Gamma} \, \|\nabla w\|.$$

where $\widetilde{C}_{\Gamma}$ is a constant in the inequality

$$\|v\|_{\Gamma}^2 = \widetilde{C}_{\Gamma}^2(\Omega)\|\nabla v\|^2,$$

$w \in H^1(\Omega)$ such that $\{w\} = 0$.

$$\int_\Omega (f + \operatorname{Div} \tau) \cdot w \mathrm{d}x \leq C_F \|f + \operatorname{Div} \tau\| \|\nabla w\|,$$

where C_F is a constant in the Friedrichs type inequality

$$\|w\| \leq C_F \|\nabla w\| \quad \forall w \in V_0.$$

As a result, we find that

$$\| \mathcal{L}_{\eta,\tau} \| \ \leq \ C_F \|f + \operatorname{Div} \tau\| + \widetilde{C}_\Gamma \|\eta - \tau \cdot n\|_\Gamma$$

Let $v \in V_0 + u_D$. Then, for any $\beta > 0$, we have

$$\begin{aligned} ||| u - v |||^2 &\leq (||| u - v_0 ||| + ||| v - v_0 |||)^2 \\ &\leq (1 + \beta) ||| u - v_0 |||^2 + (1 + \frac{1}{\beta}) ||| v - v_0 |||^2, \end{aligned}$$

where $v_0 \in V$ is a divergence free field vanishing on Γ.

Consider the respective parts of the majorant.

$$\| d(v_0, q, \tau) \|_* \leq \| \nu \nabla v - \tau - \mathrm{q} \ | \|_* + \sqrt{\overline{\nu}} \| \nabla (v - v_0) \|.$$

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Since v_0 vanishes on the boundary

$$\int_\Gamma (j(v_{0t}) + j^*(\eta) - \eta \cdot v_{0t}) \, \mathrm{d}S = D_j(v_t, \eta) + \int_\Gamma (\eta \cdot v_t - j(v_t)) \mathrm{d}S$$

$$\leq D_j(v_t, \eta) + \int_\Gamma ((\eta - \tau_t) \cdot v_t + j^*(\tau_t)) \mathrm{d}S.$$

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$$\leq D_j(v_t, \eta) + \int_\Gamma ((\eta - \tau_t) \cdot v_t + j^*(\tau_t)) \mathrm{d}S.$$

Notice that $j^*(\tau_t) = 0$ if $|\tau_t| \leq g$ and

$$\int_\Gamma (\eta - \tau_t) \cdot v_t \mathrm{d}S = \int_\Gamma (\eta - \tau_t) \cdot (v_t - v_{0t})\, \mathrm{d}S$$

$$\leq \widetilde{C}_\Gamma \|\eta - \tau_t\| \|\nabla (v_t - v_{0t})\| \leq \widetilde{C}_\Gamma \|\eta - \tau_t\| \|\nabla (v - v_0)\|.$$

We find that

$$M_1 \leq \frac{1}{2-\alpha} D_j(v_t, \eta) + \widetilde{C}_\Gamma \|\eta - \tau_t\| \|\nabla(v - v_0)\| + \frac{1}{2\alpha(2-\alpha)} \Big(R(v, q, \tau, \eta) + \sqrt{\overline{\nu}} \|\nabla(v - v_0)\| \Big)^2,$$

where

$$R(v, q, \tau, \eta) := \| d(v, q, \tau) \|_* + \frac{1}{\sqrt{\underline{\nu}}} \Big(C_F \|f + \operatorname{Div} \tau\| + \widetilde{C}_\Gamma \|\eta - \tau_t\|_\Gamma \Big).$$

There exists v_0 such that

$$\|\nabla(v - v_0)\| \leq \mathbb{C}_\Omega \| \operatorname{div} v\|.$$

$$\left\| u - v \right\|^2 \le \frac{1+\beta}{2-\alpha}\Bigg(D_j(v_t,\eta) + \frac{1}{2\alpha} R^2(v,q,\tau,\eta)$$

$$+ \left(\frac{\sqrt{\overline{\nu}}}{\alpha} R(v,q,\tau,\eta) + (2-\alpha)\widetilde{C}_\Gamma \|\eta - \tau_t\| \right) \mathbb{C}_\Omega \| \operatorname{div} v \|$$

$$+ \frac{4\alpha - 2\alpha^2 + \beta}{2\alpha} \overline{\nu} \mathbb{C}_\Omega^2 \| \operatorname{div} v \|^2 \Bigg).$$

where

$$R(v,q,\tau,\eta) := \left\| d(v,q,\tau) \right\|_* + \frac{1}{\sqrt{\underline{\nu}}} \left(C_F \|f + \operatorname{Div}\tau\| + \widetilde{C}_\Gamma \|\eta - \tau_t\|_\Gamma \right),$$

$\alpha \in (0,2)$, $\beta > 0$, and τ, η satisfy conditions

$$|\tau_t| \le g, \quad |\eta| \le g.$$

Thank you for your attention!

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