FULLY RELIABLE A POSTERIORI ERROR ESTIMATES FOR MODELS IN POROUS MEDIA

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RELEVANCE OF MODELING PROCESSES IN POROELASTIC MEDIA

Environmental and petroleum engineering:

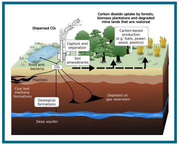
- surface subsidence due groundwater extraction and oil pumping,
- hydraulic and termal fracturing,
- oil and gas reservoirs simulation.

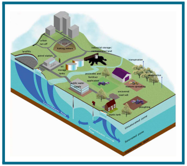
Biomechanics and chemical sciences:

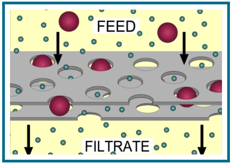
- filtering,
- effects cerebrospinal fluid flow on brain,
- poroelastic modeling of bone.

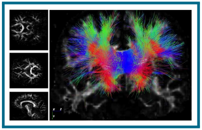
Earthquake engineering:

- liquefaction.









Motivation

Due to

1. simplifications in the model and
2. uncertainties of solvers,

we arrive at the numerical representation of the data, that contains **modeling and numerical errors**.

There is a clear need for

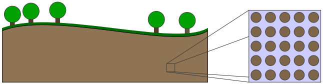
a posteriori error estimates

that allow engineers and mathematicians

1. to make conclusions on the studied model and
2. to detect the error of the numerical solvers.

BIOT CONSOLIDATION MODEL [BIOT (1941)]:

- linear elastic, isotropic, porous medium,
- linearized slightly compressible single-phase fluid,
- $\boldsymbol{u}$ is the solid's displacement, p is the fluid pressure.



MECHANICS (MOMENTUM CONSERVATION)

$$
\begin{aligned}
-\operatorname{div} \boldsymbol{\sigma}_{\text{por}}(\boldsymbol{u}, p) &= \boldsymbol{f} \quad \text{in } \Omega, \\
\boldsymbol{\sigma}_{\text{por}}(\boldsymbol{u}, p) &= \boldsymbol{\sigma}(\boldsymbol{u}) - \alpha p \mathbb{I}, \\
\boldsymbol{\sigma}(\boldsymbol{u}) &= 2\mu \boldsymbol{\varepsilon}(\boldsymbol{u}) + \lambda \operatorname{tr} \boldsymbol{\varepsilon}(\boldsymbol{u}),
\end{aligned}
$$

where $\boldsymbol{f}$ is a body force, $\mathbb{I}$ is the identity tensor,
λ, μ are the Lamé constants, and α is the dimensionless Biot-Willis coefficient.

FLOW (MASS CONSERVATION)

$$
\begin{aligned}
\frac{\partial}{\partial t}\Big(\beta p + \alpha \operatorname{div} \boldsymbol{u}\Big) - \operatorname{div} \boldsymbol{v}^D &= \tilde{q}, \\
\boldsymbol{v}^D &= -\frac{1}{\mu_f} \mathbb{K}\big(\nabla p - \rho_{f,r} \boldsymbol{g}\big),
\end{aligned}
$$

where $\tilde{q}$ is a source or sink term, μ_f is the fluid viscosity, $\mathbb{K}$ is the (symmetric, uniformly bounded, uniformly elliptic in space and constant in time) permeability tensor, i. e.
$\lambda_{\mathbb{K}} |\tau|^2 \leq \mathbb{K}\tau : \tau, \quad \tau \in \mathbb{R}^d$, $\boldsymbol{g}$ is the gravitation constant, and $\rho_{f,r}$ is the fluid phase density,
$\beta = \frac{1}{M} + c_f \varphi_0$ is a storage coefficient dependent on initial porosity φ_0, Biot constant M, and the fluid compressibility c_f.

BIOT SYSTEM [ZENISEK (1984)], [SHOWALTER (2000)], [PHILIPPS AND WHEELER (2007)]

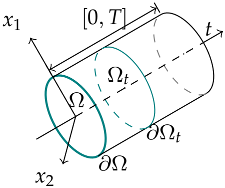
Find the pair $(\boldsymbol{u}, p) \in V \times \Theta$, where

$\Theta \equiv L^2(0,T;H^1_{p_D}(\Omega)) := \{\theta \in L^2(0,T;H^1(\Omega)) \mid \theta|_{\Gamma_D^p} = p_D\}$,

$V \equiv C(0,T;H^1_{u_D}(\Omega,\mathbb{R}^d)) := \{v \in C(0,T;H^1(\Omega,\mathbb{R}^d)) \mid \boldsymbol{v}|_{\Gamma_D^u} = \boldsymbol{u}_D\}$, such that:

$$
\begin{aligned}
-(\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{u}) - \mu\nabla^2\boldsymbol{u} - \alpha p\mathbb{I} &= \boldsymbol{f} \quad \text{in } Q,\\
-\frac{1}{\mu_f}\operatorname{div}\left(\mathbb{K}\nabla p\right) + \frac{\partial}{\partial t}(\beta p + \alpha \operatorname{div}\boldsymbol{u}) &= s_f \quad \text{in } Q,\\
p &= p_D \quad \text{on } \Gamma_D^p,\\
-\frac{1}{\mu_f}\mathbb{K}\nabla p\cdot\boldsymbol{n} &= p_N \quad \text{on } \Gamma_N^p,\\
\boldsymbol{u} &= \boldsymbol{u}_D \quad \text{on } \Gamma_D^u,\\
\boldsymbol{\sigma}_{\text{por}}\cdot\boldsymbol{n} &= \boldsymbol{t}_N \quad \text{on } \Gamma_N^u,\\
\boldsymbol{u}(x,0) = \boldsymbol{u}_0, \quad p(x,0) &= p_0 \quad \text{in } \Sigma\times\{0\},
\end{aligned}
$$

$x \in \Omega \subset \mathbb{R}^d, d = \{1,2,3\}$
Ω is Lipschitz bounded domain
$(x,t) \in Q := \Omega \times (0,T),\ T > 0$
$(x,t) \in \Sigma := \partial\Omega \times [0,T]$



where $\boldsymbol{f} \in C^1(0,T;H^{-1}(\Omega,\mathbb{R}^d))$, $s_f \in C(0,T;L^2(\Omega))$,

BCs: $p_D \in C(0,T;H^{1/2}(\Gamma_D^p))$, $p_N \in C(0,T;H^{-1/2}(\Gamma_N^p))$,

$\boldsymbol{u}_D \in C^1(0,T,H^{1/2}(\Gamma_D^u))$, $\boldsymbol{t}_N \in C^1(0,T;H^{-1/2}(\Gamma_N^u))$, and

IC: $p_0 \in H^1(\Omega)$, $\boldsymbol{u}(x,0) = \boldsymbol{u}_0 \in H^1(\Omega,\mathbb{R}^d)$.

NUMERICAL APPROACHES

There are 3 approaches for the coupling of fluid flow and mechanics:

- **Fully implicit coupling** (Lewis, Sukirman, Wan, Aziz):

  \+ stable and convergent;
  \- linear system is difficult to solve.

- **Loose or explicit coupling** (Park, Zienkiewicz, Armero, Yotov):

  \+ easy to solve and implementation;
  \- at best conditionally stable.

- **Iterative coupling** (Settari, Kim, Helmig, Ehlers, Juanes, Tchelepi, Nordbotten, Kumar, Wheeler, Mikelić):

  combines the advantages of above-mentioned approaches:
  \+ scalable,
  \+ useful in preconditioning for the fully implicit coupling, ect.

DISCRETIZATION

- Test the system with $\boldsymbol{v} = \boldsymbol{v}(x) \in V_0 := L^2(0,T;H_0^1(\Omega))$ and $\theta = \theta(x) \in \Theta_0 := C(0,T;H_0^1(\Omega,\mathbb{R}^d))$:

$$2\,\mu\,(\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})) + \lambda(\operatorname{div}\boldsymbol{u}, \operatorname{div}\boldsymbol{v}) + \alpha(\nabla p, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) + (\boldsymbol{t}_N, \boldsymbol{v})_{\Gamma_N^u}, \quad \forall\, \boldsymbol{v} \in V_0,$$

$$\tfrac{1}{\mu_f}(\mathbb{K}\nabla p, \nabla\theta) + (\tfrac{\partial}{\partial t}(\beta p + \alpha \operatorname{div}\boldsymbol{u}), \theta) = (s_f, \theta) + (p_N, \theta)_{\Gamma_N^p}, \quad \forall\, \theta \in \Theta_0.$$

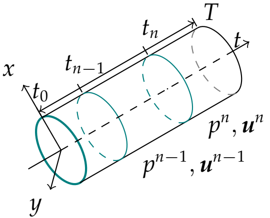
- Assume that $\boldsymbol{t}_N = (0,0)^{\mathrm{T}}$ and $p_N = 0$.
- Consider semi-discrete scheme at the n-th time-step (t_{n-1}, t_n):

  p^n and $\boldsymbol{u}^n$ are unknowns, and $\tau_n = t_n - t_{n-1}$,

$$2\,\mu\,(\boldsymbol{\varepsilon}(\boldsymbol{u}^n) : \boldsymbol{\varepsilon}(\boldsymbol{v})) + \lambda(\operatorname{div}\boldsymbol{u}^n, \operatorname{div}\boldsymbol{v}) + \alpha(\nabla p^n, \boldsymbol{v}) = (\boldsymbol{f}^n, \boldsymbol{v}),$$

$$\tfrac{\tau_n}{\mu_f}(\mathbb{K}\nabla p^n, \nabla\theta) + \beta(p^n, \theta) + \alpha(\operatorname{div}\boldsymbol{u}^n, \theta) = (\tilde{s}_f^n, \theta),$$

where $\tilde{s}_f^n = \tau_n s_f^n + \beta p^{n-1} + \alpha \operatorname{div}\boldsymbol{u}^{n-1}$.



Iterative algorithm on each (t_{n-1}, t_n)

- Initial value:
  $p^{n,0} = p(x,0)$, where $p(x,0)$ follows from $\nabla p(x,0) = \varrho_f \boldsymbol{g}$,
  $\boldsymbol{u}^{n,0} = \boldsymbol{u}(x,0)$, where $\boldsymbol{u}(x,0)$ follows from mechanics.
- k-th iterative: $p^{n,k}$ and $\boldsymbol{u}^{n,k}$
  (a) solve the flow equation for $p^{n,k}$ using $\boldsymbol{u}^{n,k-1}$:

$$\frac{1}{\mu_f}(\mathbb{K}\nabla p^{n,k}, \nabla\theta) + \beta(p^{n,k}, \theta) + \alpha(\operatorname{div} \boldsymbol{u}^{n,k-1}, \theta) = (\widetilde{s}_f^{n,k}, \theta).$$

  (b) solve the mechanics equation for $\boldsymbol{u}^{n,k}$ using $p^{n,k}$:

$$2\,\mu\,(\boldsymbol{\varepsilon}(\boldsymbol{u}^{n,k}) : \boldsymbol{\varepsilon}(\boldsymbol{v})) + \lambda(\operatorname{div} \boldsymbol{u}^{n,k}, \operatorname{div} \boldsymbol{v}) + \alpha(\nabla p^{n,k}, \boldsymbol{v}) = (\boldsymbol{f}^{n,k}, \boldsymbol{v}).$$

- On each k-th iteration, solve (a) and (b) to obtain:

  approximation $p_h^{n,k}$ and $\boldsymbol{u}_h^{n,k}$.
- Stability [Kim, Tchelepi, and Juanes (2011)],
  contraction [Mikelić and Wheeler (2013)],
  convergence [Mikelić, Wang, and Wheeler (2014)].

GOAL: TO STUDY A POSTERIORI ERROR ESTIMATES

- $(\boldsymbol{u}_h^{n,k}, p_h^{n,k})$ is the conforming approximation of the pair $(\boldsymbol{u}, p) \in V \times \Theta$,
- $\mathcal{D}$ is the problem data.

FUNCTIONAL A POSTERIORI ERROR ESTIMATES

$$\left|\left[(p - p_h^{n,k}, \boldsymbol{u} - \boldsymbol{u}_h^{n,k})\right]\right| := \| p - p_h^{n,k} \|_P^2 + \| \boldsymbol{u} - \boldsymbol{u}_h^{n,k} \|_U^2 \leq \overline{\mathcal{M}}(p_h^{n,k}, \boldsymbol{u}_h^{n,k}, \mathcal{D}),$$

where

$$\| p - p_h^{n,k} \|_P^2 := \nu \| \nabla (p - p_h^{n,k}) \|_{\mathbb{K}}^2 + \varrho \| p - p_h^{n,k} \|^2,$$
$$\| \boldsymbol{u} - \boldsymbol{u}_h^{n,k} \|_U^2 := \kappa \| \boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}_h^{n,k}) \|^2 + \xi \| \operatorname{div} (\boldsymbol{u} - \boldsymbol{u}_h^{n,k}) \|^2$$

with parameters $\nu = \frac{1}{\mu_f}$, $\varrho = \frac{\beta}{\tau_n}$, $\kappa = 2\,\mu$, $\xi = \lambda$, and norm
$\| \boldsymbol{w} \|_{\mathbb{K}}^2 := \int_\Omega \mathbb{K} \boldsymbol{w} \cdot \boldsymbol{w} \, \mathrm{d}x, \quad \boldsymbol{w} \in L^2(\Omega, \mathbb{R}^d).$

DERIVATION OF $\overline{\mathcal{M}}$ IS BASED ON COMBINATION OF:

- Ostrowski's estimates for the contractive iterations [Ostrowski (1971)] and
- functional error estimates [Repin (2000)].

CONTRACTION IN 'FIXED STRESS' SPLIT SCHEME

$$\text{'Fixed stress' split:} \quad \gamma\sigma^{n,k} = \alpha \text{div}\, \boldsymbol{u}^{n,k} - L\, p^{n,k}, \quad \gamma, L > 0.$$

Consider $\delta p^{n,k} = p^{n,k} - p^{n,k-1}$ and $\delta \boldsymbol{u}^{n,k} = \boldsymbol{u}^{n,k} - \boldsymbol{u}^{n,k-1}$:

$$(1): \quad (\beta + L)\,(\delta p^{n,k}, \theta) + \tfrac{\tau_n}{\mu_f}(\mathbb{K}\nabla \delta p^{n,k}, \nabla\theta) = (\overbrace{L\, \delta p^{n,k-1} - \alpha \text{div}\, \delta \boldsymbol{u}^{n,k-1}}^{-\gamma\delta\sigma^{n,k-1}}, \theta),$$

$$(2): \quad 2\,\mu\,(\boldsymbol{\varepsilon}(\delta \boldsymbol{u}^{n,k}) : \boldsymbol{\varepsilon}(\boldsymbol{v})) + \lambda(\text{div}\, \delta \boldsymbol{u}^{n,k}, \text{div}\, \boldsymbol{v}) - \alpha(\delta p^{n,k}, \text{div}\, \boldsymbol{v}) = 0.$$

THEOREM (ON CONTRACTION IN 'FIXED STRESS' SPLIT SCHEME)

With $\gamma^2 = 2\,L$ and $L = \frac{\alpha^2}{2\,\lambda}$, the 'fixed stress' split iterative scheme, represented by (1) and (2), is a contraction given by

$$2\,\mu\,\|\boldsymbol{\varepsilon}(\delta \boldsymbol{u}^{n,k})\|^2 + \tfrac{\tau_n}{\mu_f} q\,\|\nabla \delta p^{n,k}\|^2_{\mathbb{K}} + \|\delta\sigma^{n,k}\|^2 \le q^2 \|\delta\sigma^{n,k-1}\|^2,$$

where $q = \frac{L}{\beta + L}$.

ESTIMATES OF $\|\!| p^{n,k} - p_h^{n,k} \|\!|_P^2$

$$\tfrac{1}{\mu_f}(\mathbb{K}\nabla p^{n,k}, \nabla\theta) + \tfrac{\beta}{\tau_n}(p^{n,k}, \theta) = (F_2^k, \theta), \quad \forall\theta \in \Theta_0,$$

$$F_2^k = \widetilde{f}_2^{n,k} - \tfrac{\alpha}{\tau_n}\operatorname{div} \boldsymbol{u}^{n,k-1}.$$

LEMMA 1 [REPIN (2008)]

For $\forall p_h^{n,k} \in \Theta$, $\forall \boldsymbol{y} \in H(\Omega, \operatorname{div}) := \{\, \boldsymbol{y} \in L(\Omega, \mathbb{R}^d) \mid \operatorname{div} \boldsymbol{y} \in L^2(\Omega) \,\}$, and $\zeta \geq 0$, we have

$$\|\!| p^{n,k} - p_h^{n,k} \|\!|_P^2 := \tfrac{1}{\mu_f} \| \nabla (p^{n,k} - p_h^{n,k}) \|_{\mathbb{K}}^2 + \tfrac{\beta}{\tau_n} \| p^{n,k} - p_h^{n,k} \|^2 \leq \overline{\mathrm{M}}_P(p_h^{n,k}, \boldsymbol{y}, \zeta),$$

where

$$\overline{\mathrm{M}}_P(p_h^{n,k}, \boldsymbol{y}, \zeta) := \int_\Omega \mu_f \, (1 + \tfrac{1}{\zeta}) \, \mathbb{K}^{-1} \mathbf{r}_{\mathrm{dl}} \cdot \mathbf{r}_{\mathrm{dl}} \, \mathrm{d}x + \int_\Omega \frac{C\,(1+\zeta)}{C\,(1+\zeta)^{\beta}/_{\tau_n} + 1} |\mathbf{r}_{\mathrm{eq}}|^2 \, \mathrm{d}x,$$

with residual functionals

$$\mathbf{r}_{\mathrm{dl}} := \tfrac{1}{\mu_f} \mathbb{K}\nabla p_h^{n,k} - \boldsymbol{y}, \quad \mathbf{r}_{\mathrm{eq}} := F_2^k - \tfrac{\beta}{\tau_n}\, p_h^{n,k} + \operatorname{div} \boldsymbol{y},$$

and $C = \frac{C_{\mathrm{F}\Omega}^2}{\lambda_{\mathbb{K}}}$, represented with help of Friedrichs' constants $C_{\mathrm{F}\Omega}$ and minimal eigenvalue of the tensor $\mathbb{K}$.

ESTIMATE OF $\|p^{n,k} - p_h^{n,k}\|^2$

COROLLARY 1

For $\forall p_h^{n,k} \in \Theta$, $\forall \boldsymbol{y} \in H(\Omega, \mathrm{div}\,)$, and $\zeta > 0$, we have

$$\|p^{n,k} - p_h^{n,k}\|^2 \leq \overline{\mathrm{M}}_p(p_h^{n,k}, \boldsymbol{y}, \zeta) := \frac{C_{\mathrm{F}\Omega}^2}{\nu\,\lambda_{\mathbb{K}} + C_{\mathrm{F}\Omega}^2\,\varrho} \overline{\mathrm{M}}_P(p_h^{n,k}, \boldsymbol{y}, \zeta),$$

where $\overline{\mathrm{M}}_P(p_h^{n,k}, \boldsymbol{y}, \zeta)$ is defined in LEMMA 1, $\nu = \frac{1}{\mu_f}$, and $\varrho = \frac{\beta}{\tau_n}$, and $\lambda_{\mathbb{K}}$.

Estimate of $\|\boldsymbol{u}^{n,k} - \boldsymbol{u}_h^{n,k}\|_U^2$

$$2\,\mu\left(\boldsymbol{\varepsilon}(\boldsymbol{u}^{n,k}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\right) + \lambda(\operatorname{div}\boldsymbol{u}^{n,k}, \operatorname{div}\boldsymbol{v}) = (\boldsymbol{f}_1^{n,k} - \alpha\nabla p^{n,k}, \boldsymbol{v}), \quad \forall v \in V_0.$$

Lemma 2

For $\forall(\boldsymbol{u}_h^{n,k}, p_h^{n,k}) \in V \times \Theta$, $\forall \boldsymbol{y} \in H(\Omega, \operatorname{div})$, $\zeta \geq 0$, and $\eta \in \left[\frac{1}{2\,\lambda}, +\infty\right)$, the estimate

$$\kappa\|\,\boldsymbol{\varepsilon}(\boldsymbol{u}^{n,k} - \boldsymbol{u}_h^{n,k})\,\|^2 + \xi\|\operatorname{div}(\boldsymbol{u}^{n,k} - \boldsymbol{u}_h^{n,k})\,\|^2 =: \|\boldsymbol{u}^{n,k} - \boldsymbol{u}_h^{n,k}\|_U^2$$
$$\leq \overline{\mathrm{M}}_U(p_h^{n,k}, \boldsymbol{y}, \zeta) := \max\left\{1, \frac{\lambda}{\lambda - \frac{1}{2\eta}}\right\}\frac{\eta\,\alpha^2}{2}\overline{\mathrm{M}}_p(p_h^{n,k}, \boldsymbol{y}, \zeta)$$

holds. Here, $\overline{\mathrm{M}}_p(p_h^{n,k}, \boldsymbol{y}, \zeta)$ is defined in Corollary 1, and α and λ are Biot constant and bulk modulus, respectively.

Sketch of the proof:

- $2\,\mu\,(\boldsymbol{\varepsilon}(\boldsymbol{u}^{n,k} - \boldsymbol{u}_h^{n,k}) : \boldsymbol{\varepsilon}(\boldsymbol{v})) + \lambda(\operatorname{div}(\boldsymbol{u}^{n,k} - \boldsymbol{u}_h^{n,k}), \operatorname{div}\boldsymbol{v}) = -\alpha(p^{n,k} - p_h^{n,k}, \operatorname{div}\boldsymbol{v})$,
- by setting $\boldsymbol{v} = \boldsymbol{u}^{n,k} - \boldsymbol{u}_h^{n,k}$, we obtain

$$2\,\mu\,\|\boldsymbol{\varepsilon}(\boldsymbol{u}^{n,k} - \boldsymbol{u}_h^{n,k})\|^2 + \lambda\|\operatorname{div}(\boldsymbol{u}^{n,k} - \boldsymbol{u}_h^{n,k})\,\|^2 = -\alpha(p^{n,k} - p_h^{n,k}, \operatorname{div}(\boldsymbol{u}^{n,k} - \boldsymbol{u}_h^{n,k})).$$

- by applying Cauchy and Young inequality with $\eta > 0$, we arrive at

$$2\,\mu\,\|\boldsymbol{\varepsilon}(\boldsymbol{u}^{n,k} - \boldsymbol{u}_h^{n,k})\|^2 + (\lambda - \tfrac{1}{2\,\eta})\,\|\operatorname{div}(\boldsymbol{u}^{n,k} - \boldsymbol{u}_h^{n,k})\,\|^2 \leq \tfrac{\eta}{2}\alpha^2\|p^{n,k} - p_h^{n,k}\|^2.$$

Estimate of $\| \operatorname{div} (\boldsymbol{u}^{n,k} - \boldsymbol{u}_h^{n,k}) \|^2$

Corollary 2

For $\forall (\boldsymbol{u}_h^{n,k}, p_h^{n,k}) \in V \times \Theta$, $\boldsymbol{y} \in H(\Omega, \operatorname{div})$, $\zeta \geq 0$, and $\eta \in [\frac{1}{2\lambda}, +\infty)$, we have

$$\| \operatorname{div} (\boldsymbol{u}^{n,k} - \boldsymbol{u}_h^{n,k}) \|^2 \leq \overline{\mathrm{M}}_u(p_h^{n,k}, \boldsymbol{y}, \zeta) := \frac{2\,\eta\,\alpha^2}{2(2\,\mu + \lambda - \frac{1}{2\eta})} \overline{\mathrm{M}}_p(p_h^{n,k}, \boldsymbol{y}, \zeta),$$

where $\overline{\mathrm{M}}_p(p_h^{n,k}, \boldsymbol{y}, \zeta)$ is defined in Corollary 1, and α is the Biot constant, and μ and λ are Lame coefficients.

ESTIMATE OF $\| p - p_h^{n,k} \|_P^2$

THEOREM 2 (ON GENERAL ESTIMATE FOR PRESSURE)

For $\forall(\boldsymbol{u}_h^{n,k}, p_h^{n,k}), (\boldsymbol{u}_h^{n,k-1}, p_h^{n,k-1}) \in V \times \Theta$, $\zeta \geq 0$, we have

$$\| p - p_h^{n,k} \|_P^2 \leq \overline{\mathrm{M}}_P^{\oplus}(p_h^{n,k}, p_h^{n,k-1}, \boldsymbol{u}_h^{n,k}, \boldsymbol{u}_h^{n,k-1}, \boldsymbol{y}, \zeta)$$
$$:= \overline{\mathrm{M}}_P(p_h^{n,k}, \boldsymbol{y}, \zeta) + \overline{\mathrm{M}}_P^{\mathrm{it}}(\boldsymbol{u}_h^{n,k}, \boldsymbol{u}_h^{n,k-1}, p_h^{n,k}, p_h^{n,k-1}, \boldsymbol{y}, \zeta),$$

where $\overline{\mathrm{M}}_P(p_h^{n,k}, \boldsymbol{y}, \zeta)$ is defined in LEMMA 1 and

$$\overline{\mathrm{M}}_P^{\mathrm{it}}(\boldsymbol{u}_h^{n,k}, \boldsymbol{u}_h^{n,k-1}, p_h^{n,k}, p_h^{n,k-1}, \boldsymbol{y}, \zeta) := \frac{q}{1-q^2} \frac{\mu_f}{\tau_n} \left(\varrho \frac{C_{\mathrm{F}\Omega}}{\lambda_K} + \nu\right) \Big(\|\sigma_h^{n,k} - \sigma_h^{n,k-1}\|^2 + \frac{\alpha^2}{2L} \left(\overline{\mathrm{M}}_u(p_h^{n,k}) + \overline{\mathrm{M}}_u(p_h^{n,k-1})\right) + L\left(\overline{\mathrm{M}}_p(p_h^{n,k}) + \overline{\mathrm{M}}_p(p_h^{n,k-1})\right) \Big)$$

with $\overline{\mathrm{M}}_u(p_h^{n,k})$ and $\overline{\mathrm{M}}_p(p_h^{n,k})$ defined in COROLLARY 1 AND 2, $\sigma^{n,k} = \frac{1}{\gamma}(\alpha \,\mathrm{div}\, \boldsymbol{u}^{n,k} - L p^{n,k})$, Friedrichs' constant $C_{\mathrm{F}\Omega}$, and eigenvalue λ_K of tensor $\mathbb{K}$. Parameters $\nu = \frac{1}{\mu_f}$, and $\varrho = \frac{\beta}{\tau_n}$, $q = \frac{L}{\beta + L}$, $\gamma^2 = 2L$ and $L = \frac{\alpha^2}{2\lambda}$, α, μ_f, τ_n depend on the problem data.

SKETCH OF THE PROOF I

- By applying the triangle inequality

$$\| p - p_h^{n,k} \|_P^2 \leq \underbrace{\| p(t_n) - p^n \|_P^2}_{\leq C\tau_n^m \| p \|^2} + \| p^n - p^{n,k} \|_P^2 + \underbrace{\| p^{n,k} - p_h^{n,k} \|_P^2}_{\leq \overline{\mathrm{M}}_P(p_h^{n,k}, \boldsymbol{y}, \zeta)(lemma1)} .$$

- By using the AUXILIARY LEMMA

$$\begin{aligned} \| p^n - p^{n,k} \|_P^2 &= \varrho \, \| p^n - p^{n,k} \|^2 + \nu \, \| \nabla (p^n - p^{n,k}) \|_{\mathbb{K}}^2 \\ &\leq \left(\varrho \frac{C_{\mathrm{F}\Omega}^2}{\lambda_K} + \nu \right) \| \nabla (p^n - p^{n,k}) \|_{\mathbb{K}}^2 \\ &\leq \left(\varrho \frac{C_{\mathrm{F}\Omega}^2}{\lambda_K} + \nu \right) \frac{\mu_f}{\tau_n} \frac{q}{1-q^2} \|\sigma^{n,k} - \sigma^{n,k-1}\|^2. \end{aligned}$$

AUXILIARY LEMMA

Assume that

$$\|\sigma^{n,k}\|^2 \leq q^2 \|\sigma^{n,k-1}\|^2 \quad \text{and} \quad \|\nabla(p^{n,k+1} - p^{n,k})\|_{\mathbb{K}}^2 \leq q \frac{\mu_f}{\tau_n} \|\sigma^{n,k} - \sigma^{n,k-1}\|^2$$

holds. Then,

$$\|\nabla(p^n - p^{n,k})\|_{\mathbb{K}}^2 \leq \frac{\mu_f}{\tau_n} \frac{q}{1-q^2} \|\sigma^{n,k} - \sigma^{n,k-1}\|^2.$$

SKETCH OF THE PROOF II

- By adding and deducting the disctretized approximations $\sigma_h^{n,k-1}$ and $\sigma_h^{n,k}$, we obtain

$$\|\sigma^{n,k} - \sigma^{n,k-1}\|^2 \leq \underbrace{\|\sigma_h^{n,k} - \sigma_h^{n,k-1}\|^2}_{\text{explicitly known}} + \|\sigma^{n,k} - \sigma_h^{n,k}\|^2 + \|\sigma^{n,k-1} - \sigma_h^{n,k-1}\|^2.$$

- By using the relation $\sigma^{n,k} = \frac{1}{\gamma}(\alpha \operatorname{div} \boldsymbol{u}^{n,k} - Lp^{n,k})$, we obtained

$$\begin{aligned}\|\sigma^{n,k} - \sigma_h^{n,k}\|^2 &\leq \frac{1}{\gamma^2}(\alpha^2\|\operatorname{div}(\boldsymbol{u}^{n,k} - \boldsymbol{u}_h^{n,k})\|^2 + L^2\|p^{n,k} - p_h^{n,k}\|^2 \\ &\leq \frac{1}{\gamma^2}(\alpha^2\, \overline{\mathrm{M}}_u(p_h^{n,k}) + L^2\, \overline{\mathrm{M}}_p(p_h^{n,k})).\end{aligned}$$

ESTIMATES OF $\| \boldsymbol{u} - \boldsymbol{u}_h^{n,k} \|_U^2$

THEOREM 3 (ON GENERAL ESTIMATE FOR DISPLACEMENT)

For $\forall(\boldsymbol{u}_h^{n,k}, p_h^{n,k}), (\boldsymbol{u}_h^{n,k-1}, p_h^{n,k-1}) \in V \times \Theta$, and $\boldsymbol{y} \in H(\Omega, \mathrm{div}\,)$, $\zeta > 0$, we have

$$\| \boldsymbol{u} - \boldsymbol{u}_h^{n,k} \|_U^2 \leq \overline{\mathrm{M}}_U^{\oplus}(p_h^{n,k}, p_h^{n,k-1}, \boldsymbol{u}_h^{n,k}, \boldsymbol{u}_h^{n,k-1}, \boldsymbol{y}, \zeta)$$
$$:= \overline{\mathrm{M}}_U(p_h^{n,k}, \boldsymbol{y}, \zeta) + \overline{\mathrm{M}}_U^{\mathrm{it}}(\boldsymbol{u}_h^{n,k}, \boldsymbol{u}_h^{n,k-1}, p_h^{n,k}, p_h^{n,k-1}, \boldsymbol{y}, \zeta),$$

where $\overline{\mathrm{M}}_U(p_h^{n,k}, \boldsymbol{y}, \zeta)$ is defined in LEMMA 2 and

$$\overline{\mathrm{M}}_U^{\mathrm{it}}(\boldsymbol{u}_h^{n,k-1}, \boldsymbol{u}_h^{n,k}, p_h^{n,k-1}, p_h^{n,k-1}, \boldsymbol{y}, \zeta) := (1 + \tfrac{\lambda}{2\,\mu}) \tfrac{q^2}{1-q^2} \Big(\|\sigma_h^{n,k} - \sigma_h^{n,k-1}\|$$
$$+ \tfrac{1}{2\,L} (\alpha^2 (\overline{\mathrm{M}}_u(p_h^{n,k}) + \overline{\mathrm{M}}_u(p_h^{n,k-1}))$$
$$+ L^2 (\overline{\mathrm{M}}_p(p_h^{n,k}) + \overline{\mathrm{M}}_p(p_h^{n,k-1}))) \Big),$$

with $\overline{\mathrm{M}}_u(p_h^{n,k})$ and $\overline{\mathrm{M}}_p(p_h^{n,k})$ defined in COROLLARY 1 AND 2, $\sigma^{n,k} = \frac{1}{\gamma}(\alpha\, \mathrm{div}\, \boldsymbol{u}^{n,k} - L p^{n,k})$, where $q = \frac{L}{\beta + L}$ with $L = \frac{\alpha^2}{2\,\lambda}$, and parameters λ, μ are the Lame coefficients, and α is the Biot contant.

ESTIMATE OF $\left\|\left[(p - p_h^{n,k}, \boldsymbol{u} - \boldsymbol{u}_h^{n,k})\right]\right\|$

THEOREM 4

For any $(\boldsymbol{u}_h^{n,k}, p_h^{n,k}), (\boldsymbol{u}_h^{n,k-1}, p_h^{n,k-1}) \in V \times \Theta$, $\boldsymbol{y} \in H(\Omega, \mathrm{div}\,)$, $\zeta \geq 0$, and

$$\left\|\left[(p - p_h^{n,k}, \boldsymbol{u} - \boldsymbol{u}_h^{n,k})\right]\right\| \leq \overline{\mathrm{M}}_P^{\oplus}(p_h^{n,k}, p_h^{n,k-1}, \boldsymbol{u}_h^{n,k}, \boldsymbol{u}_h^{n,k-1}, \boldsymbol{y}, \zeta) + \overline{\mathrm{M}}_U^{\oplus}(p_h^{n,k}, p_h^{n,k-1}, \boldsymbol{u}_h^{n,k}, \boldsymbol{u}_h^{n,k-1}, \boldsymbol{y}, \zeta),$$

where $\overline{\mathrm{M}}_P^{\oplus}$ and $\overline{\mathrm{M}}_U^{\oplus}$ are define in THEOREMS 2 AND 3, respectively.

Conclusions and outline for the further work

We have derived a posteriori error estimates for poroelastic Biot model

- for iterative coupling, which are based on Ostrowski estimates and functional error for static problems, and
- for fully implicit coupling, which are based on a poteriori error estimates for parabolic problems.

Outline for the further work:

- numerical tests supporting the theoretical results,
- testing sensitivity of the estimates with respect to different problem parameters (effecting q),
- derivation estimates for localized bulk modulus and Biot constant (effecting q),
- testing how initial data effects the contractive iterates.

THANK YOU FOR YOUR TIME!

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