Structure preserving model reduction for damped wave propagation on networks

Herbert Egger, **Thomas Kugler**

TECHNISCHE UNIVERSITÄT DARMSTADT

July 7, 2016

Damped wave propagation on network

TECHNISCHE UNIVERSITÄT DARMSTADT

$$
\begin{aligned}
\partial_t p^e + \partial_x \sigma^e &= 0, && \text{on } e \in \mathcal{E},\ t > 0,\\
\partial_t \sigma^e + \partial_x p^e + d^e \sigma^e &= 0, && \text{on } e \in \mathcal{E},\ t > 0,
\end{aligned}
$$

with coupling conditions

$$
\begin{aligned}
p^e(v) &= p^{e'}(v), && \text{for all } e, e' \in \mathcal{E}(v),\ v \in \mathcal{V}_0,\ t > 0,\\
\sum_{e \in \mathcal{E}(v)} n^e(v)\sigma^e(v) &= 0, && \text{for all } v \in \mathcal{V}_0,\ t > 0,
\end{aligned}
$$

boundary conditions for the pressure

$$
p^e(v) = u^e, \qquad \text{for } v \in \mathcal{V}_\partial,\ e \in \mathcal{E}(v),\ t > 0.
$$

$+$ initial conditions.

Galerkin discretization

$$
\begin{aligned}
(\partial_t p_h(t), q_h)_{\mathcal{E}} + (\partial_x' \sigma_h(t), q_h)_{\mathcal{E}} &= 0, && q_h \in Q_h, \\
(\partial_t \sigma_h(t), v_h)_{\mathcal{E}} - (p_h(t), \partial_x' v_h)_{\mathcal{E}} + (d\sigma_h(t), v_h)_{\mathcal{E}} &= (nu, v_h)_{\mathcal{V}_\partial}, && v_h \in V_h,
\end{aligned}
$$

Properties:

1. conservation of mass — (A1$_h$) $Q_h = \partial_x' V_h$
2. energy dissipation — (A2$_h$) $\{v : \partial_x' v = 0\} \subset V_h$
3. ∃! steady state — (A3$_h$) $1 \in Q_h$
4. exp. convergence if $u = 0$

Algebraic input-output systems

TECHNISCHE UNIVERSITÄT DARMSTADT

$$\begin{aligned} M_1\dot{x}_1 + Gx_2 \qquad\qquad &= 0, \\ M_2\dot{x}_2 - G^\top x_1 + Dx_2 &= B_2 u, \\ y &= B_2^\top x_2. \end{aligned}$$

Let V_1, V_2 be projection matrices and $\widehat{M}_i = V_i^\top M_i V_i$, $\widehat{G} = V_1^\top G V_2$, $\widehat{D} = V_2^\top D V_2$, $\widehat{B}_2 = V_2^\top B$. Then the Galerkin projection reads

$$\begin{aligned} \widehat{M}_1\dot{z}_1 + \widehat{G}z_2 \qquad\qquad &= 0, \\ \widehat{M}_2\dot{z}_2 - \widehat{G}^\top z_1 + \widehat{D}z_2 &= \widehat{B}_2 u, \\ y &= \widehat{B}_2^\top z_2, \end{aligned}$$

Structure preserving model reduction

$$
\begin{array}{ccccc}
PDE & \xrightarrow{(A1_h)-(A3_h)} & GD_h & & \\
 & & \big\downarrow & & \\
 & & Alg & \longrightarrow & \widehat{Alg}
\end{array}
$$

Structure preserving model reduction

$$\begin{array}{ccccc} PDE & \xrightarrow{(A1_h)-(A3_h)} & GD_h & \xrightarrow{(A1_H)-(A3_H)} & GD_H \\ & & \downarrow & & \downarrow \\ & & Alg & \xrightarrow{(A1')-(A3')} & \widehat{Alg} \end{array}$$

(A1$_H$) $Q_H = \partial'_x V_H$

(A2$_H$) $\{v \colon \partial'_x v = 0\} \subset V_H$

(A3$_H$) $1 \in Q_H$

(A1') $\mathcal{R}(M_1 V_1) = \mathcal{R}(G V_2)$

(A2') $\mathcal{N}(G) \subset \mathcal{R}(V_2)$

(A3') $o_1 \in \mathcal{R}(V_2)$

Reduction method: moment matching (1)

TECHNISCHE UNIVERSITÄT DARMSTADT

- Descriptor system: $E\dot{x} + Ax = Bu, \quad y = B^\top x$
- Use Laplace-trafo: $sE\tilde{x} + A\tilde{x} = B\tilde{u}, \quad \tilde{y} = B^\top \tilde{x}$
- Input-output relation: $\tilde{y} = B^\top (sE + A)^{-1} B\tilde{u}$
- Transfer function:
  $H(s) := B^\top (sE + A)^{-1} B = \sum_{l=0}^{\infty} m_l (s - s_0)^l$, with moments m_l defined by $m_l = B^\top r_l$ and the recursion

$$r_0 = (sE + A)^{-1} B, \quad r_l = -(sE + A)^{-1} r_{l-1}$$

Reduction method: moment matching (2)

V projection matrix, then

- Reduced descriptor system: $\widehat{E}\dot{z} + \widehat{A}z = \widehat{B}u, \quad y = \widehat{B}^\top z$
- Reduced transfer function: $\widehat{H}(s) := \sum_{l=0}^{\infty} \widehat{m}_l (s - s_0)^l$

Lemma (moment matching)

Let $\mathbb{K}_L := span\{r_0, \ldots, r_{L-1}\} \subset \mathcal{R}(V)$, then $m_l = \widehat{m}_l$, for $l = 0, \ldots, 2L-1$.

Remark

Let $\tilde{V} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ basis of $\mathbb{K}_L$, then $V := \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}$ fulfills assumption of the lemma.

Construction of projection matrices

- Construct subspaces $\mathbb{V}_1, \mathbb{V}_2$; then compute orthonormal matrices V_1, V_2, such that $\mathcal{R}(V_i) = \mathbb{V}_i$.
- Splitting: $\mathbb{V}_1 = \mathbb{W}_1 + \mathbb{Z}_1$ and $\mathbb{V}_2 = \mathbb{W}_2 + \mathbb{Z}_2$
- Choose $\mathbb{W}_i := (\mathbb{K}_L)_i$, $\mathbb{Z}_1 := \{o_1\}$ and $\mathbb{Z}_2 = \{o_2\} + \mathcal{N}(G)$, where $o_2 = M_2^{-1} G^\top (G M_2^{-1} G^\top)^{-1} M_1 o_1$.

Lemma

Let $s_0 > 0$. Then $M_1 \mathbb{W}_1 = G \mathbb{W}_2$.

Lemma

Let $s_0 > 0$. Then $\mathbb{V}_1, \mathbb{V}_2$ satisfy $(A1'') - (A3'')$.

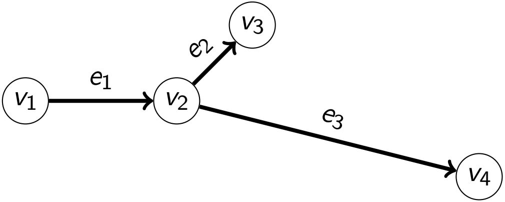
(A1") $M_1 \mathbb{V}_1 = G \mathbb{V}_2$

(A2") $\mathcal{N}(G) \subset \mathbb{V}_2$

(A3") $o_1 \in \mathbb{V}_2$

Illustration of the method

TECHNISCHE UNIVERSITÄT DARMSTADT



- a=1; number of moments=4; movie1
- a=1; number of moments=8; movie2
- a=0.1; number of moments=8; movie3

- Dimension of truth approximation pressure space is 420.
- Dimension of 8 moments pressure space is 28.
- Dimension of 4 moments pressure space is 16.

Thank you for your attention!

Appendix

The Krylov sequences are defined by

$$\begin{array}{rcrcrcl} s_0 M_1 x_1^0 & & & + & G x_2^0 & = & 0, \\ s_0 M_2 x_2^0 & - & G^\top x_1^0 & + & D x_2^0 & = & B, \end{array}$$

and

$$\begin{array}{rcrcrcl} s_0 M_1 x_1^l & & & + & G x_2^l & = & M_1 x_1^{l-1}, \\ s_0 M_2 x_2^l & - & G^\top x_1^l & + & D x_2^l & = & M_2 x_2^{l-1}. \end{array}$$

Hence $\mathbb{W}_1$ and $\mathbb{W}_2$ satisfy $(A1')$ for $s_0 > 0$. For $s_0 = 0$ one needs to add x_2^L to satsfy the condition.