A discrete Korn's inequality and related finite elements

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AANMPDE-9-16, Strobl, Austria, 2016. 7. 7

Outline

Kernel of the Strain Tensor

Let $\Omega \subset R^d$ $(d = 2, 3)$ be bounded connected open and polyhedral, and $\boldsymbol{R}M(\Omega)$ be the space of rigid motions on Ω defined by $\boldsymbol{R}M(\Omega) = \{\boldsymbol{a} + \boldsymbol{A}\boldsymbol{x} : \boldsymbol{a} \in R^d$ and $\boldsymbol{A} \in R^{d\times d}$ is anti-symetric$\}$. Define $\varepsilon(\boldsymbol{u}) = \frac{\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T}{2}$ as the strain tensor, then for $\boldsymbol{u} \in H^1(\Omega)^d$,

$$\varepsilon(\boldsymbol{u}) = 0 \Longleftrightarrow \boldsymbol{u} \in \boldsymbol{R}M(\Omega).$$

Korn's Inequalities for H^1 functions

For any $\boldsymbol{u} \in H^1(\Omega)^d$, we have the following three inequalities:

$$|\boldsymbol{u}|_{H^1(\Omega)} \lesssim \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_0 + \|Q\boldsymbol{u}\|_0,$$

where $Q\boldsymbol{u} = \boldsymbol{u} - \frac{1}{|\Omega|}\int_\Omega \boldsymbol{u} dx$.

$$|\boldsymbol{u}|_{H^1(\Omega)} \lesssim \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_0 + \sup_{\boldsymbol{m}\in \boldsymbol{RM}_0(\Omega)} \int_{\partial\Omega} \boldsymbol{u}\cdot \boldsymbol{m} ds,$$

where $\boldsymbol{RM}_0(\Omega) = \{\boldsymbol{m} \in \boldsymbol{RM}(\Omega) : \ \|\boldsymbol{m}\|_{L^2(\partial\Omega)} = 1, \int_{\partial\Omega} \boldsymbol{m} ds = 0\}$.

$$|\boldsymbol{u}|_{H^1(\Omega)} \lesssim \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_0 + |\int_\Omega \nabla \times \boldsymbol{u} dx|.$$

Korn's Inequalities for H^1 functions

Hence the first Korn's inequality reads

$$|\boldsymbol{u}|_{H^1(\Omega)} \lesssim \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_0, \quad \forall \boldsymbol{u} \in H_0^1(\Omega)^d.$$

And the second Korn's ineqality reads

$$|\boldsymbol{u}|_{H^1(\Omega)} \lesssim \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_0, \quad \forall \boldsymbol{u} \in H^1(\Omega)^d, \ |\int_\Omega \nabla \times \boldsymbol{u} dx| = 0.$$

Notations

Let T_h be a shape-regular partition of Ω into simplexes $\{K\}$. We further denote by E_h^I the set of all interior edges (faces) of $\mathcal{T}_h$ and by E_h^B the set of all boundary edges (faces).
For any edge (or face F) e (or F) $\in E_h^I$ and generally $e = \partial K_1 \cap \partial K_2$, and vector $\mathbf{v}$, the jump

$$[\mathbf{v}]_e = \mathbf{v}|_{\partial K_1 \cap e} - \mathbf{v}|_{\partial K_2 \cap e}.$$

When $e \in E_h^B$ then the above quantity is defined as

$$[\mathbf{v}]_e = \mathbf{v}|_e.$$

The same defintion can be done for a face $F = \partial K_1 \cap \partial K_2$.

Piecewise H^1 Space

The space $H^1(\Omega, \mathcal{T}_h)^d$ is defined by

$$H^1(\Omega, \mathcal{T}_h)^d = \{\boldsymbol{u} \in L^2(\Omega)^d : \ \boldsymbol{u}|_K \in H^1(K)^d, \forall K \in \mathcal{T}_h\},$$

and the seminorm $|\cdot|_{H^1(\Omega,\mathcal{T}_h)}$ is given by

$$|\boldsymbol{u}|_{H^1(\Omega,\mathcal{T}_h)} = \Big(\sum_{K \in \mathcal{T}_h} |\boldsymbol{u}|^2_{H^1(K)} \Big)^{1/2}$$

We also use the notation $\varepsilon_h(\boldsymbol{u})$ to denote the matrix function defined by

$$\varepsilon_h(\boldsymbol{u})|_K = \varepsilon(\boldsymbol{u}|_K) \ \ \forall K \in \mathcal{T}_h.$$

A Korn's Inequalities for Piecewise H^1 Functions

Let $\Phi(\boldsymbol{u})$ be $\|Q\boldsymbol{u}\|_0$, $\sup\limits_{\boldsymbol{m}\in \boldsymbol{RM}_0(\Omega)} \int_{\partial\Omega} \boldsymbol{u}\cdot\boldsymbol{m} ds$ or $|\sum\limits_{K\in\mathcal{T}_h} \int_K \nabla\times\boldsymbol{u} dx|$.

Theorem (C. Brenner 2003)

For any $\boldsymbol{u}\in H^1(\Omega,\mathcal{T}_h)^2$,the following inequality holds :

$$
\begin{aligned}
|\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{T}_h)} \quad &\lesssim \|\boldsymbol{\varepsilon}_h(\boldsymbol{u})\|_0^2 + \Phi^2(\boldsymbol{u}) \\
&+ \sum_{e\in E_h^I} (diam\ e)^{-1}\Big(\|\pi_1([\boldsymbol{u}]_e\cdot\boldsymbol{n}_e)\|^2_{0,e} + \|\pi_1([\boldsymbol{u}]_e\cdot\boldsymbol{t}_e)\|^2_{0,e},
\end{aligned}
$$

where π_1 is the L^2 projection to $\mathcal{P}^1(e)$, $\boldsymbol{n}_e$ is the normal unit to e and $\boldsymbol{t}_e$ is the tangential to e.

A similar result as the above theorem for $d = 3$ is also there.

A More Look Into $RM(\Omega)$

Lemma (H. and Lee 2015)

For any edge $e \subset K$ $(d = 2)$*, Then for any* $\boldsymbol{u} \in \boldsymbol{R}M(\Omega)$*,*

$$\boldsymbol{u} \cdot \boldsymbol{t}_e\big|_e = (\boldsymbol{a} + \boldsymbol{A}\boldsymbol{x}) \cdot \boldsymbol{t}_e\big|_e = \textit{constant}.$$

For any face $F \subset K$ $(d = 3)$*, let* $\boldsymbol{c_F}$ *be the barycenter of* F*. Then for any* $\boldsymbol{u} \in \boldsymbol{R}M(\Omega)$*, the following holds true that*

$$\boldsymbol{u} \times \boldsymbol{n}_F\big|_F = (\boldsymbol{a} + \boldsymbol{A}\boldsymbol{x}) \times \boldsymbol{n}_F\big|_F = \boldsymbol{a} \times \boldsymbol{n}_F + g(\boldsymbol{x} - \boldsymbol{c_F}),$$

where g *is a constant and* $(\boldsymbol{x} - \boldsymbol{c_F}) \cdot \boldsymbol{n}_F = 0$*. Therefore,*

$$\dim\,(\boldsymbol{a} + \boldsymbol{A}\boldsymbol{x}) \times \boldsymbol{n}_F\big|_F = 3.$$

Improved Korn's Inequalities

Theorem (H. and Lee 2015)

For any $\boldsymbol{u} \in H^1(\Omega, \mathcal{T}_h)^2$*, the following inequality holds:*

$$|\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{T}_h)} \lesssim \|\boldsymbol{\varepsilon}_h(\boldsymbol{u})\|_0^2 + \Phi^2(\boldsymbol{u}) + \sum_{e\in E_h^I} (diam\ e)^{-1} \Big(\|\pi_1([\boldsymbol{u}]_e \cdot \boldsymbol{n}_e)\|^2_{0,e} + \|\pi_0([\boldsymbol{u}]_e \cdot \boldsymbol{t}_e)\|^2_{0,e},$$

where π_0 *is the* L^2 *projection to* $\mathcal{P}^0(e)$*.*
For any $\boldsymbol{u} \in H^1(\Omega, \mathcal{T}_h)^3$*, the following inequality holds:*

$$|\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{T}_h)} \lesssim \|\boldsymbol{\varepsilon}_h(\boldsymbol{u})\|_0^2 + \Phi^2(\boldsymbol{u}) + \sum_{F\in E_h^I} (diam\ F)^{-1} \Big(\|\pi_1([\boldsymbol{u}]_F \cdot \boldsymbol{n}_F)\|^2_{0,F} + \|\pi_r([\boldsymbol{u}]_F \times \boldsymbol{n}_F)\|^2_{0,F},$$

where π_r *is the* L^2 *projection onto* $\boldsymbol{RT}^0(F) = \mathcal{P}^0(F) + \mathcal{P}^0(F)\boldsymbol{x}$*.*

Improved Korn's Inequalities

Corollary (H. and Lee 2015)

For any $\boldsymbol{u} \in H^1(\Omega, \mathcal{T}_h)^2$, *for any* $e \in E_h^I$,
provided that $\int_e [\boldsymbol{u}]_e \cdot \boldsymbol{n}_e q ds = 0 \ \forall q \in \mathcal{P}^1(e)$ *and* $\int_e [\boldsymbol{u}]_e \cdot \boldsymbol{t}_e ds = 0$,
then the following inequality holds:

$$|\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{T}_h)} \lesssim \|\varepsilon_h(\boldsymbol{u})\|_0^2 + \Phi^2(\boldsymbol{u}).$$

For any $\boldsymbol{u} \in H^1(\Omega, \mathcal{T}_h)^3$, *for any* $F \in E_h^I$,
provided that $\int_F [\boldsymbol{u}]_F \cdot \boldsymbol{n}_F q dA = 0 \ \forall q \in \mathcal{P}^1(F)$ *and*
$\int_F ([\boldsymbol{u}]_e \times \boldsymbol{n}_F) \cdot \boldsymbol{r} dA = 0 \ \forall \boldsymbol{r} \in \boldsymbol{RT}^0(F)$, *then the following inequality holds:*

$$|\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{T}_h)} \lesssim \|\varepsilon_h(\boldsymbol{u})\|_0^2 + \Phi^2(\boldsymbol{u}).$$

2D case

***Example* 1**. [Xie, Xu, Xue 2008; Mardal, Tai, Winther 2002]
Degrees of Freedom (DOF): $(\boldsymbol{v}\cdot\boldsymbol{n}_e,\mu)_e, \forall\mu\in\mathcal{P}^1(e), (\boldsymbol{v}\cdot\boldsymbol{t}_e,1)_e$.

$$\boldsymbol{V}(K)=\boldsymbol{\mathcal{P}}^1(K)+\boldsymbol{curl}\left(b_K\mathcal{P}^1(K)\right);$$

$$\boldsymbol{V}(K)=\{\boldsymbol{v}\in\boldsymbol{\mathcal{P}}^3(K):\ div\,\boldsymbol{v}\in\mathcal{P}^0(K),\ (\boldsymbol{v}\cdot\boldsymbol{n}_e)|_e\in\mathcal{P}^1(e),\forall e\in\partial K\}.$$

Define

$$\boldsymbol{N}^{k-1}(K)=\boldsymbol{\mathcal{P}}^{k-2}(K)+\{\boldsymbol{v}\in\boldsymbol{\mathcal{P}}^{k-1}(K):\boldsymbol{v}\cdot\boldsymbol{x}=0\}.$$

$$Q_e^{k-1}(K) = \left\{q\in\mathcal{P}^{k-1}(K):(q,b_Kb_ew)_K=0,\ w\in\mathcal{P}^{k-2}(K)\right\};$$

$$Q^{k-1}(K)=\sum_e b_eQ_e^{k-1}(K)$$

***Example* 2**.[Guzmán, Neilan 2011]
Degrees of Freedom (DOF): $(\boldsymbol{v},\rho),\ \forall\rho\in\boldsymbol{N}^{k-1}(K)$,
$(\boldsymbol{v}\cdot\boldsymbol{n}_e,\mu)_e,\forall\mu\in\mathcal{P}^k(e),(\boldsymbol{v}\cdot\boldsymbol{t}_e,s)_e,\forall s\in\mathcal{P}^{k-1}(e)$.

$$\boldsymbol{V}(K)=\boldsymbol{\mathcal{P}}^k(K)+\boldsymbol{curl}\left(b_KQ^{k-1}(K)\right).$$

3D case

***Example* 3**. [Xie, Xu, Xue 2008]
Degrees of Freedom (DOF):
$(\boldsymbol{v} \cdot \boldsymbol{n}_F, \mu)_F, \forall \mu \in \mathcal{P}^1(F), (\boldsymbol{v} \times \boldsymbol{n}_F, \boldsymbol{r})_F, \forall \boldsymbol{r} \in \boldsymbol{RT}^0(F)$;

$$\boldsymbol{V}(K) = \boldsymbol{\mathcal{P}}^1(K) + \boldsymbol{curl}\left(b_K \boldsymbol{\mathcal{P}}^1(K)\right).$$

Define

$$\begin{aligned} \boldsymbol{Q}_F^{k-1}(K) &= \Big\{\boldsymbol{q} \times \boldsymbol{n}_F \in \boldsymbol{\mathcal{P}}^{k-1}(K) \times \boldsymbol{n}_F : \\ &\qquad (\boldsymbol{q} \times \boldsymbol{n}_F, b_K b_F(\boldsymbol{w} \times \boldsymbol{n}_F))_K = 0,\ \boldsymbol{w} \in \boldsymbol{\mathcal{P}}^{k-2}(K)\Big\}, \end{aligned}$$

$\boldsymbol{Q}^{k-1}(K) = \sum_F b_F \boldsymbol{Q}_F^{k-1}(K)$.
***Example* 4**.[Guzmán, Neilan 2011]
Degrees of Freedom (DOF): $(\boldsymbol{v}, \rho),\ \forall \rho \in \boldsymbol{N}^{k-1}(K)$,
$(\boldsymbol{v} \cdot \boldsymbol{n}_F, \mu)_F, \forall \mu \in \mathcal{P}^k(F), (\boldsymbol{v} \times \boldsymbol{n}_F, \boldsymbol{r})_F, \forall \boldsymbol{r} \in \boldsymbol{\mathcal{P}}^{k-1}(F)$;

$$\boldsymbol{V}(K) = \boldsymbol{\mathcal{P}}^k(K) + \boldsymbol{curl}\left(b_K \boldsymbol{Q}^{k-1}(K)\right).$$

We should note that for Example 4 we need $k \geq 2$.

A Remedy for $k = 1$

Define

$$\begin{aligned}\boldsymbol{Q}_F^*(K) &= \{\boldsymbol{q}\times\boldsymbol{n}_F \in \boldsymbol{RM}(K)\times\boldsymbol{n}_F : \\ &\quad (\boldsymbol{q}\times\boldsymbol{n}_F, b_K b_F(\boldsymbol{w}\times\boldsymbol{n}_F))_K = 0,\ \boldsymbol{w}\in\boldsymbol{\mathcal{P}}^0(K)\},\end{aligned}$$

$\boldsymbol{Q}^*(K) = \sum_F b_F \boldsymbol{Q}_F^*(K)$.

***Example* 5**.[H. and Lee 2016]

Degrees of Freedom (DOF):

$(\boldsymbol{v}\cdot\boldsymbol{n}_F, \mu)_F, \forall\mu\in\mathcal{P}^1(F), (\boldsymbol{v}\times\boldsymbol{n}_F, \boldsymbol{r})_F, \forall\boldsymbol{r}\in\boldsymbol{RT}^0(F)$;

$$\boldsymbol{V}^*(K) = \boldsymbol{\mathcal{P}}^1(K) + \boldsymbol{curl}\,(b_K\boldsymbol{Q}^*(K))\,.$$

Theorem (H. and Lee 2016)

The element $\boldsymbol{V}^*(K)$ is unisolvent.

A Remedy for linear CR element in 2D

It is well known that the linear CR element does not satisfy Korn's inequality [S. Falk 1991].
Noting that $\frac{1}{|e_{ij}|}\int_{e_{ij}} \lambda_i\lambda_j ds = 1/6$, we define a function
$d_K = \lambda_i\lambda_j + \lambda_j\lambda_k + \lambda_k\lambda_i - \frac{1}{6}$.
We define the enriched CR element on element K by

$$\boldsymbol{ECR}(K) = \mathcal{P}^1(K) + \boldsymbol{VEC}(K)$$

where $\boldsymbol{VEC}(K)$ is spanned by the following functions

$$\psi_{ij} = d_K(\lambda_i - \lambda_j)\boldsymbol{n}_{ij}, \quad \psi_{jk} = d_K(\lambda_j - \lambda_k)\boldsymbol{n}_{jk}, \quad \psi_{ki} = d_K(\lambda_k - \lambda_i)\boldsymbol{n}_{ki}.$$

The degree of freedom are defined as following

$$\int_e \boldsymbol{v} ds \text{ and } \int_e \boldsymbol{v}\cdot\boldsymbol{n} q ds, \forall q \in \mathcal{P}^1(e)/R.$$

A Remedy for linear CR element in 2D

Theorem (H. and Lee 2016)

The element $\boldsymbol{ECR}(K)$ is unisolvent.

Proof.

Firstly, for any $\boldsymbol{v_1} \in \boldsymbol{VEC}(K)$, we have $\int_e \boldsymbol{v_1} ds = 0 \ \forall e \in \partial K$ since $\int_e d_K q ds = 0, \forall q \in \mathcal{P}^1(e)$.
Next, we set $\boldsymbol{v} = \boldsymbol{v_0} + \boldsymbol{v_1}$, where $\boldsymbol{v_0} \in \boldsymbol{\mathcal{P}}^1(K)$ and $\boldsymbol{v_1} \in \boldsymbol{VEC}(K)$.
From $\int_e \boldsymbol{v} ds = 0$ and the above inclusion, we obtain that $\boldsymbol{v_0} = 0$.
Now for $\boldsymbol{v_1} = c_1\psi_{ij} + c_2\psi_{jk} + c_3\psi_{ki}$. From $\int_e \boldsymbol{v} \cdot \boldsymbol{n} q ds, \forall q \in \mathcal{P}^1(e)/R$, we have the following linear system:

$$
\begin{aligned}
-2c_1 + c_2 \boldsymbol{n}_{jk} \cdot \boldsymbol{n}_{ij} + c_3 \boldsymbol{n}_{ki} \cdot \boldsymbol{n}_{ij} &= 0, \\
c_1 \boldsymbol{n}_{jk} \cdot \boldsymbol{n}_{ij} - 2c_2 + c_3 \boldsymbol{n}_{ki} \cdot \boldsymbol{n}_{jk} &= 0, \\
c_1 \boldsymbol{n}_{ij} \cdot \boldsymbol{n}_{ki} + c_2 \boldsymbol{n}_{jk} \cdot \boldsymbol{n}_{ki} - 2c_3 &= 0.
\end{aligned}
$$

Since $|\boldsymbol{n}_{ij} \cdot \boldsymbol{n}_{ki}| + |\boldsymbol{n}_{jk} \cdot \boldsymbol{n}_{ki}| < 2$, then the coefficient matrix of the unknowns c_1, c_2, c_3 is diagonal dominated, which implies$c_1, c_2, c_3 = 0$. □

Concluding Remarks and Future Work

We presented

- A discrete Korn's inequality which shows the relation to the definition of DOF for finite elements,
- Construction of some finite elements satisfying the Korn's inequality.

Future work will address

- a necessary condition for piecewise H^1 functions to satisfy classic Korn's inequality,
- Remedy for linear CR element in 3D,
- Applications to elasticity and Stokes problems.

Thank you for your attention!