Dual-Primal Isogeometric Tearing and Interconnecting Solvers for Continuous and Discontinuous Galerkin IgA Equations

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Doctoral Program
Computational Mathematics
Numerical Analysis and Symbolic Computation

- Introduction
- IETI-DP
- dG-IETI-DP
- Numerical examples
- Conclusion

Overview

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Motivation

What is the Dual Primal IsogEometric Tearing and Interconnecting (IETI-DP) method?

- adaption of Dual Primal Finite Element Tearing and Interconnecting (FETI-DP) for IgA
- non overlapping domain decomposition method

Why we need it?

- fast solvers
- parallelizeable solvers
- robust solvers w.r.t. jumping coefficients

State of the Art

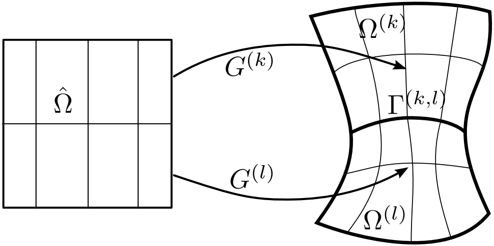
- Pechstein, C. (2012). *Finite and boundary element tearing and interconnecting solvers for multiscale problems* (Vol. 90). Springer Science & Business Media.
- Beirao da Veiga, L., Cho, D., Pavarino, L. F., Scacchi, S. (2013). *BDDC preconditioners for isogeometric analysis*. Mathematical Models and Methods in Applied Sciences, 23(06), 1099-1142.
- Dryja, M., Galvis, J., Sarkis, M. (2013). *A FETI-DP preconditioner for a composite finite element and discontinuous Galerkin method*. SIAM Journal on Numerical Analysis, 51(1), 400-422.
- Dryja, M., Sarkis, M. (2014). *3-D FETI-DP preconditioners for composite finite element-discontinuous Galerkin methods*. In Domain Decomposition Methods in Science and Engineering XXI (pp. 127-140). Springer International Publishing.

Isogeometric Discretization

The computational domain Ω can be represented as

$$\overline{\Omega} := \bigcup_{k=0}^{N} \overline{\Omega}^{(k)}, \quad \text{where } \Omega^{(k)} = G^{(k)}(\hat{\Omega}),\ \hat{\Omega} = (0,1)^d,$$

with $\Gamma^{(k,l)} := \overline{\Omega}^{(k)} \cap \overline{\Omega}^{(l)}$ and $G^{(k)}(\xi) = \sum_{i \in \mathcal{I}} \boldsymbol{P}_i^{(k)} N_{i,p}(\xi)$.



Examples

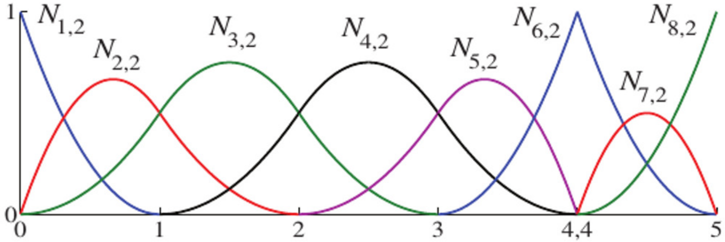


Abbildung : B-Splines with $p = 2$ and knot vector $\Xi = \{0, 0, 0, 1, 2, 3, 4, 4, 5, 5, 5\}$[1]

[1]Book: Isogeometric Analysis, Towards Integration of CAD and FEA, J. A. Cotrell and T. J. R. Hughes and Y. Bazilevs

Remarks

- $dim \geq 2$: $N_{i,p}$ is obtained via a tensorproduct
- using $\check{N}_{i,p}(x) := N_{i,p} \circ G^{-1}(x)$ as basis for discretization
- $\check{N}_{i,p}(x)$ is in general not interpolatory
- $dim = 1$: $\check{N}_{i,p}$ is interpolatory at $\partial G((0,1))$
- $dim = 2$: $\check{N}_{i,p}$ is interpolatory at the corner points, and it is possible to associate certain $N_{i,p}$ to $\partial G((0,1)^2)$

Problem formulation - cG setting

Find $u_h \in V_{D,h}$:

$$a(u_h, v_h) = \langle F, v_h \rangle \quad \forall v_h \in V_{D,h},$$

where $V_{D,h}$ is a conforming discrete subspaces of V_D, e.g.

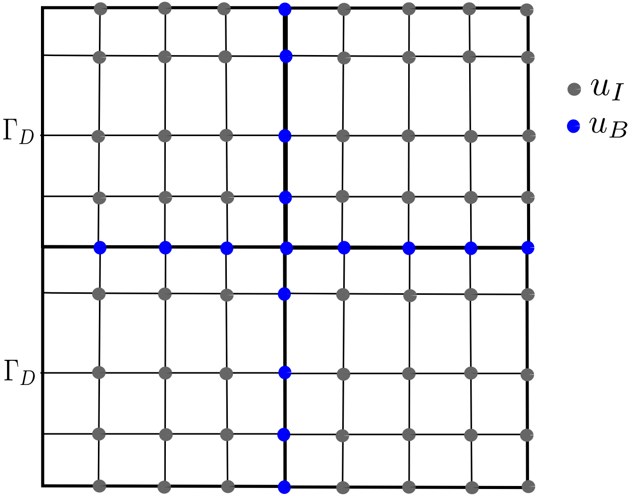
$$a(u,v) = \int_\Omega \alpha \nabla u \nabla v \, dx, \quad \langle F, v \rangle = \int_\Omega f v \, dx + \int_{\Gamma_N} g_N v \, ds$$

$$V_D = \{u \in H^1 : \gamma_0 u = 0 \text{ on } \Gamma_D\},$$

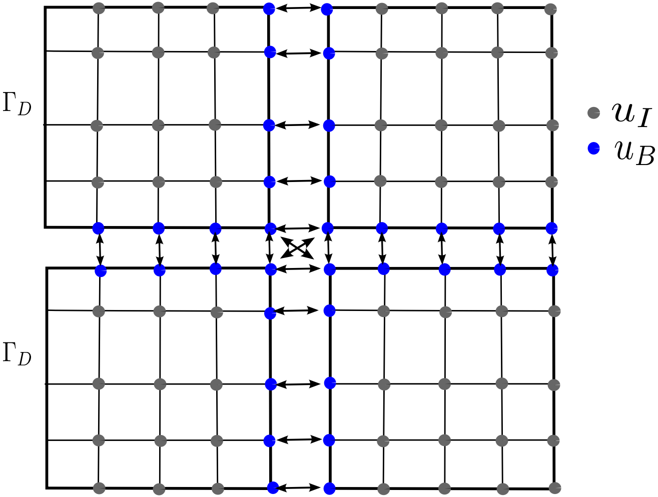
$$V_{D,h} = \prod_k \text{span}\{N_{i,p} \circ {G^{(k)}}^{-1}\} \cap H^1(\Omega).$$

The variational equation is equivalent to $\boldsymbol{K}\boldsymbol{u} = \boldsymbol{f}$.

Overview


Γ_D
Γ_D
u_I
u_B

Lagrange Multipliers



IETI-DP

Given $\boldsymbol{K}^{(k)}$ and $\boldsymbol{f}^{(k)}$, we can reformulate

$$\boldsymbol{K}\boldsymbol{u} = \boldsymbol{f} \quad \leftrightarrow \quad \begin{bmatrix} \boldsymbol{K}_e & \boldsymbol{B}^T \\ \boldsymbol{B} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_e \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_e \\ 0 \end{bmatrix},$$

where $\boldsymbol{K}_e = \text{diag}(\boldsymbol{K}^{(k)})$ and $\boldsymbol{f}_e = [\boldsymbol{f}^{(k)}]$.

Since $\boldsymbol{K}_e$ is not invertible, we need additional primal variables incorporated in $\boldsymbol{K}_e$ (partial assembly) $\rightsquigarrow \widetilde{\boldsymbol{K}}_e, \widetilde{\boldsymbol{B}}, \widetilde{\boldsymbol{f}}_e$:

- continuous vertex values
- continuous edge/face averages

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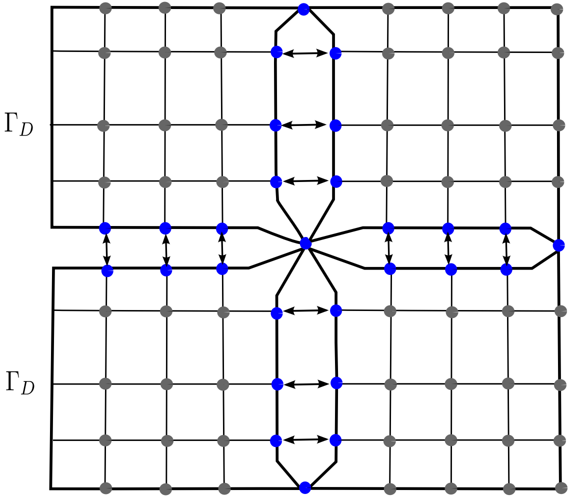
Introducing Primal Variables



Introducing Primal Variables

Find $(\boldsymbol{u}, \boldsymbol{\lambda}) \in \mathbb{R}^{\tilde{N}} \times U$:

$$\begin{bmatrix} \widetilde{\boldsymbol{K}}_e & \widetilde{\boldsymbol{B}}^T \\ \widetilde{\boldsymbol{B}} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \widetilde{\boldsymbol{f}}_e \\ 0 \end{bmatrix}.$$

$\widetilde{\boldsymbol{K}}_e$ is SPD, hence, we can define:

$$\boldsymbol{F} := \widetilde{\boldsymbol{B}} \widetilde{\boldsymbol{K}}_e^{-1} \widetilde{\boldsymbol{B}}^T \quad \boldsymbol{d} := \widetilde{\boldsymbol{B}} \widetilde{\boldsymbol{K}}_e^{-1} \widetilde{\boldsymbol{f}}_e$$

The saddle point system is equivalent to solving:

$$\text{find } \boldsymbol{\lambda} \in U : \quad \boldsymbol{F}\boldsymbol{\lambda} = \boldsymbol{d}.$$

Theorem

Let H_k be the diameter and h_k the local mesh size of $\Omega^{(k)}$. Let the Scaled Dirichlet preconditioner M_{sD}^{-1} be defined as:

$$M_{sD}^{-1} = \boldsymbol{B}_D \boldsymbol{S}_e \boldsymbol{B}_D^T$$

Then, under suitible assumption on the mesh, we have

$$\kappa(M_{sD}^{-1} \boldsymbol{F}_{|ker(\widetilde{\boldsymbol{B}}^T)}) \leq C \max_k \left(1 + \log\left(\frac{H_k}{h_k}\right)\right)^2,$$

where the constant C is independent of H_k, h_k.

- Hofer, C., Langer U.: *DP-IETI Solvers for large-scale systems of multipatch cG IgA equations.* RICAM-Report 2015-44.
- Beirao da Veiga, L., Cho, D., Pavarino, L. F., Scacchi, S. (2013). *BDDC preconditioners for isogeometric analysis*. Mathematical Models and Methods in Applied Sciences, 23(06), 1099-1142.

Overview

Problem formulation - dG setting

Find $u \in V_h := \prod_k \mathrm{span}\{N_{i,p} \circ {G^{(k)}}^{-1}\}$, such that

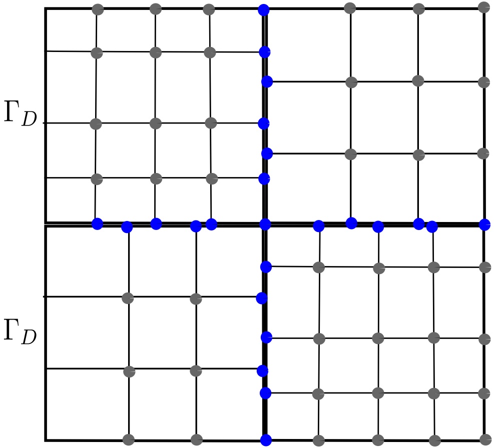
$$a_{dG}(u, v) = \langle F, v \rangle \quad \forall v \in V_h,$$

where

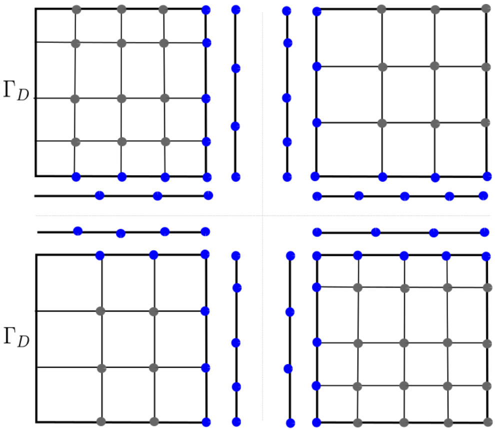
$$a_{dG}(u,v) := \sum_{k=1}^{N} a_k(u,v), \quad \langle F, v \rangle := \sum_{k=1}^{N} \int_{\Omega_k} f v_k dx,$$

$$\begin{aligned} a_k(u,v) = &\int_{\Omega_k} \alpha_k \nabla u_k \cdot \nabla v_k dx \\ &+ \sum_{l \in \mathcal{E}_k} \int_{\Gamma^{(k,l)}} \frac{\alpha_k}{2} \left(\frac{\partial u_k}{\partial n}(v_l - v_k) + \frac{\partial v_k}{\partial n}(u_l - u_k) \right) ds \\ &+ \sum_{l \in \mathcal{E}_k} \int_{\Gamma^{(k,l)}} \frac{\mu \alpha_k}{h}(u_l - u_k)(v_l - v_k) ds, \end{aligned}$$

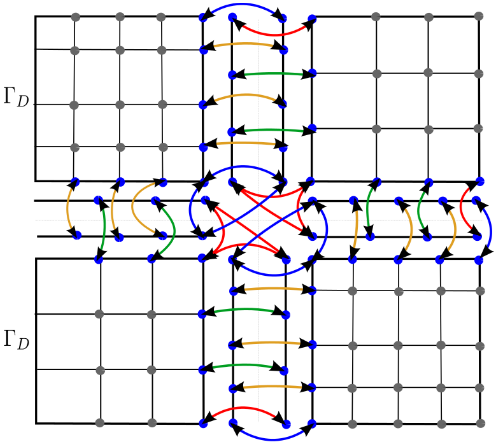
Illustration of multipatch dG



Decoupling of interface dofs



Lagrange multipliers for decoupled dofs



Remarks

- the dG-IETI-DP method can be seen as a cG-IETI-DP method on an extended grid
- principle can be extended to 3D: $\mathcal{V} + \mathcal{E} + \mathcal{F}$
- *scaled Dirichlet preconditioner:* numerical examples also show quasi-optimal behavior w.r.t. $H/h = \max\{H_i/h_i\}$ and robustness w.r.t. jumps in α.

$$\kappa(M_{sD}^{-1}F) \leq C(1 + \log(H/h))^2$$

Hofer, C., Langer, U. (2016). *Dual-primal isogeometric tearing and interconnecting solvers for multipatch dG-IgA equations*, Computer Methods in Applied Mechanics and Engineering.

- for finite elements: proven for 2D and 3D
  - Dryja, M., Galvis, J., Sarkis, M. (2013). *A FETI-DP preconditioner for a composite finite element and discontinuous Galerkin method*
  - Dryja, M., Sarkis, M. (2014). *3-D FETI-DP preconditioners for composite finite element-discontinuous Galerkin methods*

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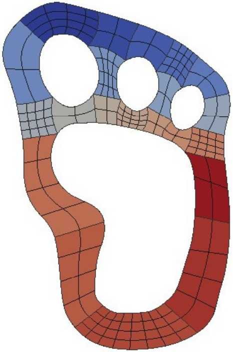
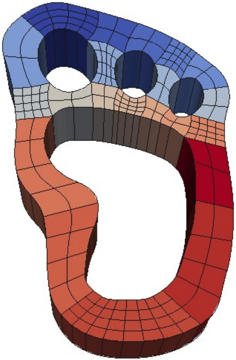
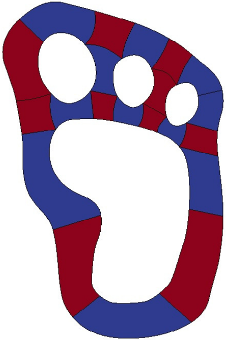
Numerical example



(a) Initial mesh of 2D computational domain



(b) Initial mesh of 3D computational domain

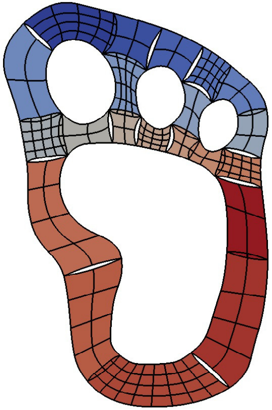


(c) Jumping coefficient pattern

2D/3D domain, $\alpha \in \{10^{-4}, 10^{4}\}$, $p = 2$

2D				3D			
$\mathcal{V}$				$\mathcal{V}$			
#dofs	H/h	κ	It.	#dofs	H/h	κ	It.
1610	8	3.82	12	2800	3	50.3	28
4706	16	5.11	13	9478	6	72.2	29
15602	32	6.58	15	42922	12	176	43
56210	64	8.23	15	244594	25	400	52
212690	128	10.1	17				
$\mathcal{V}+\mathcal{E}$				$\mathcal{V}+\mathcal{E}+\mathcal{F}$			
1610	8	1.4	7	2800	3	2.11	11
4706	16	1.7	7	9478	6	12.6	17
15602	32	2.06	8	42922	12	15.7	22
56210	64	2.46	8	244594	25	18.9	28
212690	128	2.9	9				

Gaps & Overlaps with dG-IETI-DP, $\alpha = 1$



$d_g \sim h^1$						
2D				3D		
		$\mathcal{V}+\mathcal{E}$			$\mathcal{V}+\mathcal{E}+\mathcal{F}$	
H/h	#dofs	κ	It.	#dofs	κ	It.
4	468	2.17	8	1488	1.19	7
8	1316	2.25	8	5508	4.45	18
16	4188	1.48	10	25908	6.31	23
32	14636	1.78	11	149748	8.17	27
64	54348	2.14	13	998388	10.2	31
128	209036	2.56	14			
256	819468	3.04	15			

- gap/overlap $\to 0$ if $h \to 0$
- C. Hofer, U. Langer, and I. Toulopoulos (2015): *Discontinuous Galerkin isogeometric analysis of elliptic problems on segmentations with gaps.* RICAM-Report 2015-40
- C. Hofer, and I. Toulopoulos (2015): *Discontinuous Galerkin Isogeometric Analysis of Elliptic Problems on Segmentations with Non-Matching Interfaces*. RICAM-Report 2015-46.
- C. Hofer, U. Langer, and I. Toulopoulos (2016): *Discontinuous Galerkin Isogeometric Analysis on non matching segmentations: Error estimates and efficient solvers (under preparation)*

Overview

Conclusion and further work

- quasi-optimal condition number estimate for IETI-DP
- adapted and implemented the IETI-DP algorithm for dG-IgA formulation
- dG-IETI-DP for non-matching interface parametrizations (gaps & overlaps between patches)
- MPI is used to parallelize the method.

- proof for condition number bound of dG-IETI-DP is ongoing work.
- using inexact solvers for local problems (e.g. multigrid)
- dG-IETI-DP for space-time discretization

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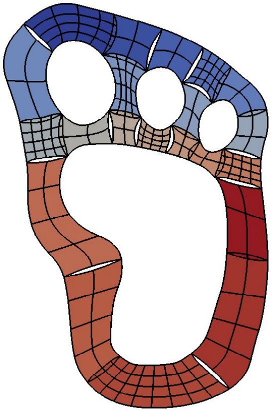
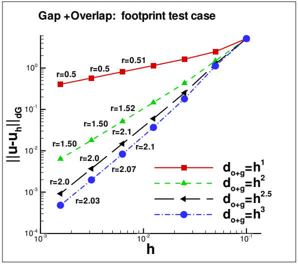
Conclusion and further work



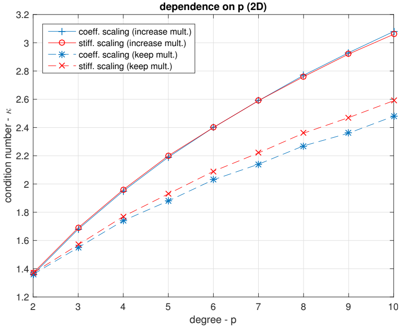


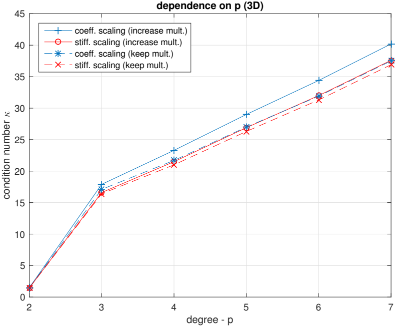
$$\|u-u_h\|_{dG} \leq Ch^{\min(s,\beta)} \left(\sum \|u\|_{W^{l,2}} + \kappa_2\right),$$

where we assume that $d_g \leq h^\lambda$, and $s = \min(p+1,l) - 1$ and $\beta = \lambda - \frac{1}{2}$

p dependence: 2D + 3D & different multiplicity

- keeping multiplicity & increasing smoothness (- - - -)
  $\rightsquigarrow C^{p-1}$ elements
- increasing multiplicity & keeping smoothness (——)
  $\rightsquigarrow C^{1}$ elements





Conclusion and further work

dim = 2					dim = 3				
$\mathcal{V}$			coeff.		$\mathcal{V}+\mathcal{E}$			coeff.	
#dofs	h_i/h_j	H/h	κ	It.	#dofs	h_i/h_j	H/h	κ	It.
1816	1	2	4.92	13	9362	1	1	14.8	21
2134	2	4	4.93	13	11902	2	3	17.7	23
2962	4	8	4.93	13	20426	4	6	29.2	24
5386	8	16	4.93	13	56626	8	12	52.2	26
13306	16	32	4.93	13	345268	16	25	98.4	27
41434	32	64	4.92	13	1758004	32	50	191	28
$\mathcal{V}+\mathcal{E}$					$\mathcal{V}+\mathcal{E}+\mathcal{F}$				
1816	1	2	1.67	7	9362	1	1	14.8	15
2134	2	4	1.67	7	11902	2	3	19.5	15
2962	4	8	1.67	7	20426	4	6	29.2	16
5386	8	16	1.67	7	56626	8	12	52.2	17
13306	16	32	1.67	7	345268	16	25	98.4	17
41434	32	64	1.67	7	1758004	32	50	308	20