Damped wave propagation on networks: Exponential stability and uniform approximation

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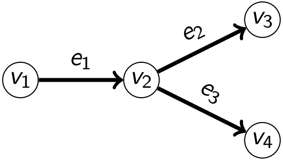
TRR 154 | Mathematische Modellierung, Simulation und Optimierung am Beispiel von Gasnetzwerken

AANMPDE 2016

Network



$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ a geom. graph

finite, connected, directed

$e = (v_i, v_j) \quad \sim \quad (0, \ell^e)$

$n^e(v_i) = -1$, $n^e(v_j) = +1$

AANM**PDE**: Damped wave propagtion on networks

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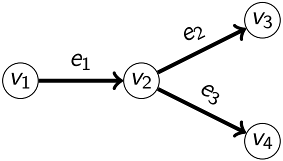

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Model equations

$$\partial_t p^e + \partial_x \sigma^e = 0 \quad e \in \mathcal{E} \qquad (1)$$

$$\partial_t \sigma^e + \partial_x p^e + d\sigma^e = 0 \quad e \in \mathcal{E} \qquad (2)$$

$$\sum\nolimits_e n^e(v)\sigma^e(v) = 0 \quad v \in \mathcal{V}_i \qquad (3)$$

$$p^e(v) - p^{e'}(v) = 0 \quad v \in \mathcal{V}_i \qquad (4)$$

$$p(v) = 0 \quad v \in \mathcal{V}_b \qquad (5)$$

plus initial cond. $p(0) = p_0$, $\sigma(0) = \sigma_0$

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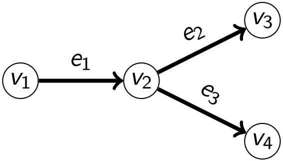

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Lemma 1: For all $p_0 \in H_0^1$ and $\sigma_0 \in H(\text{div})$ there exists a unique classical solution $(p, u) \in C([0, T]; H_0^1 \times H(\text{div})) \cap C^1([0, T]; L^2 \times L^2)$

AANMPDE: Main properties

Lemma 2 (Variational characterization)

$$\begin{aligned} (\partial_t p(t), q)_{\mathcal{E}} + (\partial_x' \sigma(t), q)_{\mathcal{E}} &= 0 \quad \forall q \in L^2 \\ (\partial_t \sigma(t), \tau)_{\mathcal{E}} - (p(t), \partial_x' \tau)_{\mathcal{E}} + (d\sigma(t), \tau)_{\mathcal{E}} &= 0 \quad \forall \tau \in H(\mathrm{div}) \end{aligned} \tag{VP}$$

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Lemma 6 (Exponential stability)

$$E(t) \leq Ce^{-\gamma(t-s)} E(s), \quad s \geq t \tag{P4}$$

Galerkin approximation

Find $p_h(t) \in Q_h$, $\sigma_h(t) \in V_h$ such that $p_h(0) = \pi_h p_0$, $\sigma_h(0) = \rho_h \sigma_0$ and

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Lemma 7: There exists a unique discrete solution (p_h, σ_h).

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Lemma 8 (Structure preserving approximations)

If

(A1$_h$) $\partial_x' V_h = Q_h$; **(A2$_h$)** $N(\partial_x') \subset V_h$; **(A3$_h$)** $1 \in Q_h$.

Then solutions (p_h, σ_h) of discrete problem satisfy (P1)–(P4) uniformly!

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Lemma 9 (FEM approximation)

Let $T_h(e)$ be mesh of e, define $T_h = \{T_h(e)\}$, and set

$$Q_h = P_k(T_h) \qquad \text{and} \qquad V_h = P_{k+1}(T_h) \cap H(\mathrm{div})$$

Then the compatibility conditions $(A1_h)$–$(A3_h)$ are satisfied.