# Boundary Integral Representation for the Elastic Wave Equation

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5th July 2016, AANMPDE(JS)-9-16, Strobl

## Contents

## Introduction and Motivation

- We consider the 3 dimensional elastic wave equation given by:

$$\rho\partial_t^2 u = \mu\Delta u + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,u + \rho\dot{f}, \quad (x,t)\in\mathbb{R}^3\times[0,T],$$
$$u(x,0) = u_0 \quad \text{in } \mathbb{R}^3,$$
$$\partial_t u(x,0) = v_0 \quad \text{in } \mathbb{R}^3,$$

  where we assume that the Lame parameter $\lambda \geq 0$ and $\mu \geq 0$.
- We want to construct a stable numerical method, which couples the interior and exterior problem for a non-convex domain.

# Transmission Conditions between Interior and Exterior Space

- Problem in interior space:

$$\begin{aligned} \rho\partial_t^2 u^- &= \mu\Delta u^- + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\, u^- + \rho\dot{f}, \quad (x,t)\in\Omega\times[0,T], \\ u^-(x,0) &= u_0 \quad \text{in } \Omega, \\ \partial_t u^-(x,0) &= v_0 \quad \text{in } \Omega. \end{aligned}$$

- Problem in exterior space:

$$\begin{aligned} \rho\partial_t^2 u^+ &= \mu\Delta u^+ + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\, u^+, \quad (x,t)\in\Omega^+\times[0,T], \\ u^+(x,0) &= 0 \quad \text{in } \Omega^+, \\ \partial_t u^+(x,0) &= 0 \quad \text{in } \Omega^+, \end{aligned}$$

where $\Omega^+ = \mathbb{R}^3 \setminus \overline{\Omega}$.

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# Transmission Conditions between Interior and Exterior Space

- Transmission conditions:

$$\gamma^- u^- = \gamma^+ u^+,$$
$$T^- u^- = T^+ u^+,$$

where $\gamma^-$ and $\gamma^+$ represent the interior and exterior traces in the boundary $\Gamma$ and $T^-$ and $T^+$ denote the stress operator for the interior and exterior case, respectively.

- The stress operator is given by

$$Tu = \sigma(u) \cdot n = (\mu 2\varepsilon(u) + \lambda \nabla \cdot u) \cdot n.$$

## Background

In order to construct the Calderón operator, we need the

- potential layers,
- boundary integrals
- fundamental solution (see, e.g., [Costabel (2004)]) of the elastic wave equation:

$$G_{jk}(x,t) = \frac{1}{4\pi\rho|x|^3}\left(t^2\left(\frac{x_jx_k}{|x|^2}\delta\left(t-\frac{|x|}{c_p}\right)+\left(\delta_{jk}-\frac{x_jx_k}{|x|^2}\right)\delta\left(t-\frac{|x|}{c_s}\right)\right)\right.$$
$$\left.+\,t\left(3\frac{x_jx_k}{|x|^2}-\delta_{jk}\right)\left(\theta\left(t-\frac{|x|}{c_p}\right)-\theta\left(t-\frac{|x|}{c_s}\right)\right)\right),$$

$c_p = \sqrt{\frac{\lambda+2\mu}{\rho}}$, $c_s = \sqrt{\frac{\mu}{\rho}}$ and $\delta_{jk}$ is the Kronecker symbol, $\delta$ is the Dirac distribution and $\theta$ the Heaviside function,

- transmission conditions.

## Construction of the Calderón Operator

Start with the Calderón operator $B(s)$ for the Laplace transformed case. We have to take the transmission conditions into account:

$$\psi = -[\![\gamma u]\!] \quad \text{on } \Gamma,$$
$$\phi = \frac{1}{s}[\![Tu]\!] \quad \text{on } \Gamma,$$

where $[\![\gamma u]\!] = \gamma^- u - \gamma^+ u$ and $[\![Tu]\!] = T^- u - T^+ u$ are the jumps in the boundary traces.

If we have a closer look at the single layer and double layer potentials, we observe the following jump relations,

$$[\![\gamma S(s)\phi]\!] = 0,$$
$$\frac{1}{s}[\![TS(s)\phi]\!] = \phi,$$
$$-[\![\gamma D(s)\psi]\!] = \psi,$$
$$[\![TD(s)\psi]\!] = 0.$$

## Construction of the Calderón Operator

Similar, we introduce the averages:

$$
\begin{aligned}
J(s)\phi &= \{\{\gamma S(s)\phi\}\} = \gamma^{\pm} S(s)\phi, \\
K(s)\phi &= \{\{TS(s)\phi\}\}, \\
K^T(s)\psi &= \{\{\gamma D(s)\psi\}\}, \\
W(s)\psi &= -\{\{TD(s)\psi\}\} = -T^{\pm} D(s)\psi.
\end{aligned}
$$

Finally, we formulate the representation theorem:

$$
u = sS(s) \underbrace{\frac{1}{s}[\![Tu]\!]}_{:=\phi} + D(s) \underbrace{[\![\gamma u]\!]}_{:=-\psi}.
$$

## Construction of the Calderón Operator

In order to construct our Calderón operator, we introduce the corresponding boundary integral operators in the Laplace domain (in a similar way as in [Kielhorn and Schanz (2008)]):

$$J(s)\phi(x) = \int_\Gamma G(x-y,s)\phi(y)d\Gamma_y,$$

$$K(s)\phi(x) = \int_\Gamma (TG(x-y,s))\,\phi(y)d\Gamma_y,$$

$$K^T(s)\phi(x) = T\int_\Gamma G(x-y,s)\phi(y)d\Gamma_y,$$

$$W(s)\phi(x) = -T\int_\Gamma (TG(x-y,s))\,\phi(y)d\Gamma_y \quad \text{for } x \in \Gamma.$$

Then the Calderón operator $B(s)$ (see, e.g., [Banjai,Lubich and Sayas (2015)]) is given by

$$B(s) = \begin{pmatrix} sJ(s) & K^T(s) \\ -K(s) & \frac{1}{s}W(s) \end{pmatrix},$$

# Positivity Results for the Calderón Operator

**Lemma**

*There exists $\tilde{\beta} > 0$ such that the Calderón operator $B(s)$ satisfies*

$$\mathrm{Re}\left\langle \begin{pmatrix}\phi\\ \psi\end{pmatrix}, B(s)\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right\rangle_{\Gamma} \geq \mathrm{Re}\, s\, \tilde{\beta}\frac{1}{|s|^2}\min\left(1,|s|^2\right)\left(\|\phi\|^2_{H^{-\frac{1}{2}}(\Gamma)} + \|\psi\|^2_{H^{\frac{1}{2}}(\Gamma)}\right)$$

*for $Re\, s > 0$ and for all $\phi \in H^{-\frac{1}{2}}(\Gamma)$ and $\psi \in H^{\frac{1}{2}}(\Gamma)$.*

# Positivity Results for the Calderón Operator

Proof:
We consider:

$$\begin{aligned}
&\mathrm{Re}\left\langle \begin{pmatrix}\phi\\ \psi\end{pmatrix}, B(s)\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right\rangle_\Gamma \\
&= \mathrm{Re}\left\langle \begin{pmatrix}[\![\frac{1}{s}Tu]\!]\\ -[\![\gamma u]\!]\end{pmatrix}, \begin{pmatrix}\{\{\gamma u\}\}\\ \{\{-\frac{1}{s}Tu\}\}\end{pmatrix}\right\rangle_\Gamma \\
&= \mathrm{Re}\,\langle \frac{1}{s}T^- u, \gamma^- u\rangle_\Gamma + \mathrm{Re}\,\langle -\gamma^+ u, -\frac{1}{s}T^+ u\rangle_\Gamma \\
&= \mathrm{Re}\left(\frac{1}{s}\int_\Gamma T^- u\cdot\gamma^- u\, dy_\Gamma\right) - \mathrm{Re}\left(\frac{1}{s}\int_\Gamma T^+ u\cdot\gamma^+ u\, dy_\Gamma\right).
\end{aligned}$$

$$\begin{aligned}
&\mathrm{Re}\left\langle \begin{pmatrix}\phi\\ \psi\end{pmatrix} B(s)\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right\rangle_\Gamma \\
&\geq \mathrm{Re}\, s\left(\lambda\|\frac{1}{s}\mathrm{div} u\|^2_{L^2(\mathbb{R}^3\setminus\Gamma)} + \rho\|u\|^2_{L^2(\mathbb{R}^3\setminus\Gamma)} + \mu\|\frac{1}{s}\varepsilon(u)\|^2_{L^2(\mathbb{R}^3\setminus\Gamma)^3}\right)
\end{aligned}$$

## Positivity Results for the Calderón Operator

In order to finish this estimate, we consider $\|\phi\|_{H^{-\frac{1}{2}}(\Gamma)}$ and $\|\psi\|_{H^{\frac{1}{2}}(\Gamma)}$ separately:

$$\begin{aligned}
\|\phi\|^2_{H^{-\frac{1}{2}}(\Gamma)} &= \|[\![\frac{1}{s}Tu]\!]\|^2_{H^{-\frac{1}{2}}(\Gamma)} \\
&= \|\frac{1}{s}\left[(2\mu\varepsilon(u)+\lambda(\operatorname{div}(u))I)\,n\right]\|^2_{H^{-\frac{1}{2}}(\Gamma)} \\
&\leq C_1\frac{1}{|s|^2}\|\operatorname{div}\left(2\mu\varepsilon(u)+\lambda(\operatorname{div}(u))I\right)\|^2_{L^2(\mathbb{R}^3\setminus\Gamma)} \\
&= C_1|\rho|^2\|su\|^2_{L^2(\mathbb{R}^3\setminus\Gamma)} \\
&= C_2|s|^2\|u\|^2_{L^2(\mathbb{R}^3\setminus\Gamma)}.
\end{aligned}$$

$$\begin{aligned}
\|\psi\|^2_{H^{\frac{1}{2}}(\Gamma)} &= \|[\![\gamma u]\!]\|^2_{H^{\frac{1}{2}}(\Gamma)} \\
&= \|u|_\Gamma n\|^2_{H^{\frac{1}{2}}(\Gamma)} \\
&\leq \tilde{C}_1|s|^2\|\frac{1}{s}\operatorname{div}u\|^2_{L_2(\mathbb{R}\setminus\Gamma)^3}.
\end{aligned}$$

# Positivity Results for the Calderón Operator

Thus, we get

$$
\begin{aligned}
&\|\phi\|^2_{H^{-\frac{1}{2}}(\Gamma)} + \|\psi\|^2_{H^{\frac{1}{2}}(\Gamma)} \\
&\leq C_2|s|^2\|u\|^2_{L^2(\mathbb{R}^3\setminus\Gamma)} + \tilde{C}_1|s|^2\|\frac{1}{s}\mathrm{div}u\|^2_{L_2(\mathbb{R}^3\setminus\Gamma)} \\
&\leq \tilde{C}_1|s|^2\|\frac{1}{s}\mathrm{div}u\|^2_{L_2(\mathbb{R}^3\setminus\Gamma)} + C_2|s|^2\|u\|^2_{L^2(\mathbb{R}^3\setminus\Gamma)} + \mu\|\frac{1}{s}\varepsilon(u)\|^2_{L^2(\mathbb{R}^3\setminus\Gamma)^3} \\
&\leq \beta|s|^2\max\left(1,\frac{1}{|s|^2}\right)\left(\|\frac{1}{s}\mathrm{div}u\|^2_{L_2(\mathbb{R}^3\setminus\Gamma)^3} + \|u\|^2_{L^2(\mathbb{R}^3\setminus\Gamma)^3} + \mu\|\frac{1}{s}\varepsilon(u)\|^2_{L^2(\mathbb{R}^3\setminus\Gamma)^3}\right).
\end{aligned}
$$

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# Positivity Results for the Calderón Operator

**Lemma**

*With the constant $\tilde{\beta}$ from Lemma 1 we have*

$$\int_0^T e^{-\frac{2t}{T}} \left\langle \begin{pmatrix} \phi(\cdot,t) \\ \psi(\cdot,t) \end{pmatrix}, B(\partial_t) \begin{pmatrix} \phi \\ \psi \end{pmatrix} (\cdot,t) \right\rangle_\Gamma dt$$
$$\geq \tilde{\beta} \int_0^T e^{-\frac{2t}{T}} \left( \|\partial_t^{-1}\phi(\cdot,t)\|^2_{H^{-\frac{1}{2}}(\Gamma)} + \|\partial_t^{-1}\psi(\cdot,t)\|^2_{H^{\frac{1}{2}}(\Gamma)} \right),$$

*for any* $T > 0$ *and for all* $\phi \in C^4([0,T], H^{-\frac{1}{2}})$ *and all* $\psi \in C^3([0,T], H^{\frac{1}{2}})$ *with* $\phi(\cdot,0) = \partial_t\phi(\cdot,0) = \ldots = \partial_t^3\phi(\cdot,0) = 0$, $\psi(\cdot,0) = \partial_t\phi(\cdot,0) = \partial_t^2\phi(\cdot,0) = 0$.
*Here,* $c_T = \min(T^{-1}, T^{-3})$.

# First Order System

We take a look at the first order system and apply a similar approach as [Banjai,Lubich and Sayas (2015)] for the acoustic wave equation, but we have to extend our system in the following way:

$$\begin{aligned}\rho\dot{u} &= \mu\nabla\cdot V + \lambda\nabla\omega + \rho f,\\ \dot{V} &= 2\varepsilon(u) = \nabla u + (\nabla u)^T\\ \dot{\omega} &= \nabla\cdot u,\end{aligned}$$

$$B(\partial_t)\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\gamma u\\ -\gamma(\mu V + \lambda\omega I)n\end{pmatrix}.$$

In addition, the time-dependent Calderón operator $B(\partial_t)$ reads as:

$$\begin{aligned}B(\partial_t)\begin{pmatrix}\phi\\ \psi\end{pmatrix} &= \frac{1}{2}\begin{pmatrix}\gamma u\\ -\partial_t^{-1}\sigma(u)n\end{pmatrix}\\ &= \frac{1}{2}\begin{pmatrix}\gamma u\\ -\partial_t^{-1}\left(\mu 2\varepsilon(u) + \lambda((\nabla\cdot u)I)\right)n\end{pmatrix}\\ &= \frac{1}{2}\begin{pmatrix}\gamma u\\ -\partial_t^{-1}(\mu\dot{V} + \lambda(\dot{\omega}I))n\end{pmatrix} \quad \text{on } \Gamma.\end{aligned}$$

# Weak Formulation

We go over to the weak formulation of our first-order system and start with the following background by using integration by parts and Green's formula: With $\varepsilon^*(u) := \frac{1}{2}\nabla \cdot (u + u^T)$, we obtain

$$(\nabla \cdot V, Z) = \frac{1}{2}(\nabla \cdot V, Z) - \frac{1}{2}(V, \nabla u) + \frac{1}{2}\langle \gamma V, \gamma Z \rangle_\Gamma,$$

$$(\nabla \omega, x) = -\frac{1}{2}(\omega, \nabla \cdot x) + \frac{1}{2}(\nabla \omega, x) + \frac{1}{2}\langle \gamma \omega, \gamma x \rangle_\Gamma,$$

$$2(\varepsilon(u), Z) = (\nabla u + (\nabla u)^T, Z) = \frac{1}{2}(\nabla u, Z + Z^T) - \frac{1}{2}(u, \varepsilon^*(Z)) + \frac{1}{2}\langle \gamma u, \gamma(Z + Z^T) \rangle_\Gamma.$$

Due to the symmetric properties, that $V = V^T$ and $\omega = \omega^T$, it can be easily calculated that the following relation holds:

$$\mu \frac{1}{2}\langle \gamma u, \gamma V \rangle_\Gamma = \frac{1}{2}\langle \psi, \gamma \mu V n \rangle_\Gamma$$

$$\lambda \frac{1}{2}\langle \gamma u, \gamma \omega \rangle_\Gamma = \frac{1}{2}\langle \psi, \gamma \lambda \omega n \rangle_\Gamma.$$

Further on, we should keep in mind that:

$$\psi = \gamma u,$$

$$\phi = -\gamma(\mu V + \lambda \omega) n.$$

# Weak Formulation

Thus, we get

$$\begin{aligned}
\rho(\dot{u}, u) =& -\mu\frac{1}{2}(V, \nabla u) + \mu\frac{1}{2}(\nabla \cdot V, u) - \lambda\frac{1}{2}(\omega, \nabla \cdot u) + \lambda\frac{1}{2}(\nabla\omega, u) \\
&+ \underbrace{\frac{1}{2}\langle \gamma(\mu V + \lambda\omega)n, \gamma u\rangle_\Gamma}_{=-\frac{1}{2}\langle \phi, \gamma u\rangle} + \rho(f, u), \\
\mu(\dot{V}, \frac{1}{2}V) =& \mu\frac{1}{2}(\nabla u, V) - \mu\frac{1}{2}(u, \nabla \cdot V) + \mu\frac{1}{2}\langle \gamma u, \gamma V\rangle_\Gamma \\
\lambda(\dot{\omega}, \omega) =& -\lambda\frac{1}{2}(u, \nabla\omega) + \lambda\frac{1}{2}(\nabla \cdot u, \omega) + \lambda\frac{1}{2}\langle \gamma u, \gamma\omega\rangle_\Gamma, \\
\left\langle \begin{pmatrix} \phi \\ \psi \end{pmatrix}, B(\partial_t)\begin{pmatrix} \phi \\ \psi \end{pmatrix} \right\rangle_\Gamma =& \frac{1}{2}\langle \phi, \gamma u\rangle_\Gamma - \frac{1}{2}\langle \psi, \gamma\mu V n\rangle_\Gamma - \frac{1}{2}\langle \psi, \gamma\lambda\omega n\rangle_\Gamma.
\end{aligned}$$

# Weak Formulation

Thus, we get

$$\rho(\dot{u},u) = -\mu\frac{1}{2}(V,\nabla u) + \mu\frac{1}{2}(\nabla\cdot V,u) - \lambda\frac{1}{2}(\omega,\nabla\cdot u) + \lambda\frac{1}{2}(\nabla\omega,u) + \underbrace{\frac{1}{2}\langle\gamma(\mu V+\lambda\omega)n,\gamma u\rangle_\Gamma}_{=-\frac{1}{2}\langle\phi,\gamma\boldsymbol{u}\rangle} + \rho(f,u),$$

$$\mu(\dot{V},\frac{1}{2}V) = \mu\frac{1}{2}(\nabla u,V) - \mu\frac{1}{2}(u,\nabla\cdot V) + \mu\frac{1}{2}\langle\gamma u,\gamma V\rangle_\Gamma$$

$$\lambda(\dot{\omega},\omega) = -\lambda\frac{1}{2}(u,\nabla\omega) + \lambda\frac{1}{2}(\nabla\cdot u,\omega) + \lambda\frac{1}{2}\langle\gamma u,\gamma\omega\rangle_\Gamma,$$

$$\left\langle\begin{pmatrix}\phi\\ \psi\end{pmatrix}, B(\partial_t)\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right\rangle_\Gamma = \frac{1}{2}\langle\phi,\gamma u\rangle_\Gamma - \frac{1}{2}\langle\psi,\gamma\mu Vn\rangle_\Gamma - \frac{1}{2}\langle\psi,\gamma\lambda\omega n\rangle_\Gamma.$$

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## Weak Formulation

Thus, we get

$$\rho(\dot{u},u) = -\mu\frac{1}{2}(V,\nabla u) + \mu\frac{1}{2}(\nabla\cdot V,u) - \lambda\frac{1}{2}(\omega,\nabla\cdot u) + \lambda\frac{1}{2}(\nabla\omega,u) + \underbrace{\frac{1}{2}\langle\gamma(\mu V+\lambda\omega)n,\gamma u\rangle_\Gamma}_{=-\frac{1}{2}\langle\phi,\gamma u\rangle} + \rho(f,u),$$

$$\mu(\dot{V},\frac{1}{2}V) = \mu\frac{1}{2}(\nabla u,V) - \mu\frac{1}{2}(u,\nabla\cdot V) + \mu\frac{1}{2}\langle\gamma u,\gamma V\rangle_\Gamma$$

$$\lambda(\dot{\omega},\omega) = -\lambda\frac{1}{2}(u,\nabla\omega) + \lambda\frac{1}{2}(\nabla\cdot u,\omega) + \lambda\frac{1}{2}\langle\gamma u,\gamma\omega\rangle_\Gamma,$$

$$\left\langle \begin{pmatrix}\phi\\ \psi\end{pmatrix}, B(\partial_t)\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right\rangle_\Gamma = \frac{1}{2}\langle\phi,\gamma u\rangle_\Gamma - \frac{1}{2}\langle\psi,\gamma\mu V n\rangle_\Gamma - \frac{1}{2}\langle\psi,\gamma\lambda\omega n\rangle_\Gamma.$$

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$$\mu(\dot{V},\frac{1}{2}V) = \mu\frac{1}{2}(\nabla u,V) - \mu\frac{1}{2}(u,\nabla\cdot V) + \mu\frac{1}{2}\langle\gamma u,\gamma V\rangle_\Gamma$$

$$\lambda(\dot{\omega},\omega) = -\lambda\frac{1}{2}(u,\nabla\omega) + \lambda\frac{1}{2}(\nabla\cdot u,\omega) + \lambda\frac{1}{2}\langle\gamma u,\gamma\omega\rangle_\Gamma,$$

$$\left\langle \begin{pmatrix}\phi\\ \psi\end{pmatrix}, B(\partial_t)\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right\rangle_\Gamma = \frac{1}{2}\langle\phi,\gamma u\rangle_\Gamma - \frac{1}{2}\langle\psi,\gamma\mu Vn\rangle_\Gamma - \frac{1}{2}\langle\psi,\gamma\lambda\omega n\rangle_\Gamma.$$

# Weak Formulation

Thus, we get

$$\rho(\dot{u}, u) = -\mu\frac{1}{2}(V, \nabla u) + \mu\frac{1}{2}(\nabla\cdot V, u) - \lambda\frac{1}{2}(\omega, \nabla\cdot u) + \lambda\frac{1}{2}(\nabla\omega, u)$$

$$+ \underbrace{\frac{1}{2}\langle\gamma(\mu V + \lambda\omega)n, \gamma u\rangle_\Gamma}_{=-\frac{1}{2}\langle\phi,\gamma u\rangle} + \rho(f, u),$$

$$\mu(\dot{V}, \frac{1}{2}V) = \mu\frac{1}{2}(\nabla u, V) - \mu\frac{1}{2}(u, \nabla\cdot V) + \mu\frac{1}{2}\langle\gamma u, \gamma V\rangle_\Gamma$$

$$\lambda(\dot{\omega}, \omega) = -\lambda\frac{1}{2}(u, \nabla\omega) + \lambda\frac{1}{2}(\nabla\cdot u, \omega) + \lambda\frac{1}{2}\langle\gamma u, \gamma\omega\rangle_\Gamma,$$

$$\left\langle \begin{pmatrix}\phi\\ \psi\end{pmatrix}, B(\partial_t)\begin{pmatrix}\phi\\ \psi\end{pmatrix} \right\rangle_\Gamma = \frac{1}{2}\langle\phi, \gamma u\rangle_\Gamma - \frac{1}{2}\langle\psi, \gamma\mu V n\rangle_\Gamma - \frac{1}{2}\langle\psi, \gamma\lambda\omega n\rangle_\Gamma.$$

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Thus, we get

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$$+ \underbrace{\frac{1}{2}\langle\gamma(\mu V+\lambda\omega)n,\gamma u\rangle_\Gamma}_{=-\frac{1}{2}\langle\phi,\gamma u\rangle} + \rho(f,u),$$

$$\mu(\dot{V},\frac{1}{2}V) = \mu\frac{1}{2}(\nabla u,V) - \mu\frac{1}{2}(u,\nabla\cdot V) + \mu\frac{1}{2}\langle\gamma u,\gamma V\rangle_\Gamma$$

$$\lambda(\dot{\omega},\omega) = -\lambda\frac{1}{2}(u,\nabla\omega) + \lambda\frac{1}{2}(\nabla\cdot u,\omega) + \lambda\frac{1}{2}\langle\gamma u,\gamma\omega\rangle_\Gamma,$$

$$\left\langle \begin{pmatrix}\phi\\ \psi\end{pmatrix}, B(\partial_t)\begin{pmatrix}\phi\\ \psi\end{pmatrix} \right\rangle_\Gamma = \frac{1}{2}\langle\phi,\gamma u\rangle_\Gamma - \frac{1}{2}\langle\psi,\gamma\mu V n\rangle_\Gamma - \frac{1}{2}\langle\psi,\gamma\lambda\omega n\rangle_\Gamma.$$

## Weak Formulation

Thus, we get

$$\rho(\dot{u}, u) = -\mu\frac{1}{2}(V, \nabla u) + \mu\frac{1}{2}(\nabla \cdot V, u) - \lambda\frac{1}{2}(\omega, \nabla \cdot u) + \lambda\frac{1}{2}(\nabla\omega, u) + \underbrace{\frac{1}{2}\langle \gamma(\mu V + \lambda\omega)n, \gamma u\rangle_\Gamma}_{= -\frac{1}{2}\langle \phi, \gamma u\rangle} + \rho(f, u),$$

$$\mu(\dot{V}, \frac{1}{2}V) = \mu\frac{1}{2}(\nabla u, V) - \mu\frac{1}{2}(u, \nabla \cdot V) + \mu\frac{1}{2}\langle \gamma u, \gamma V\rangle_\Gamma$$

$$\lambda(\dot{\omega}, \omega) = -\lambda\frac{1}{2}(u, \nabla\omega) + \lambda\frac{1}{2}(\nabla \cdot u, \omega) + \lambda\frac{1}{2}\langle \gamma u, \gamma\omega\rangle_\Gamma,$$

$$\left\langle \begin{pmatrix}\phi\\ \psi\end{pmatrix}, B(\partial_t)\begin{pmatrix}\phi\\ \psi\end{pmatrix} \right\rangle_\Gamma = \frac{1}{2}\langle \phi, \gamma u\rangle_\Gamma - \frac{1}{2}\langle \psi, \gamma\mu V n\rangle_\Gamma - \frac{1}{2}\langle \psi, \gamma\lambda\omega n\rangle_\Gamma.$$

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## Weak Formulation

Thus, we get

$$\rho(\dot{u}, u) = -\mu\frac{1}{2}(V, \nabla u) + \mu\frac{1}{2}(\nabla\cdot V, u) - \lambda\frac{1}{2}(\omega, \nabla\cdot u) + \lambda\frac{1}{2}(\nabla\omega, u) + \underbrace{\frac{1}{2}\langle\gamma(\mu V + \lambda\omega)n, \gamma u\rangle_\Gamma}_{=-\frac{1}{2}\langle\phi,\gamma u\rangle} + \rho(f, u),$$

$$\mu(\dot{V}, \frac{1}{2}V) = \mu\frac{1}{2}(\nabla u, V) - \mu\frac{1}{2}(u, \nabla\cdot V) + \mu\frac{1}{2}\langle\gamma u, \gamma V\rangle_\Gamma$$

$$\lambda(\dot{\omega}, \omega) = -\lambda\frac{1}{2}(u, \nabla\omega) + \lambda\frac{1}{2}(\nabla\cdot u, \omega) + \lambda\frac{1}{2}\langle\gamma u, \gamma\omega\rangle_\Gamma,$$

$$\left\langle \begin{pmatrix}\phi\\ \psi\end{pmatrix}, B(\partial_t)\begin{pmatrix}\phi\\ \psi\end{pmatrix} \right\rangle_\Gamma = \frac{1}{2}\langle\phi, \gamma u\rangle_\Gamma - \frac{1}{2}\langle\psi, \gamma\mu V n\rangle_\Gamma - \frac{1}{2}\langle\psi, \gamma\lambda\omega n\rangle_\Gamma.$$

# Weak Formulation

Thus, we get

$$\rho(\dot{u},u) = -\mu\frac{1}{2}(V,\nabla u) + \mu\frac{1}{2}(\nabla\cdot V,u) - \lambda\frac{1}{2}(\omega,\nabla\cdot u) + \lambda\frac{1}{2}(\nabla\omega,u) + \underbrace{\frac{1}{2}\langle\gamma(\mu V+\lambda\omega)n,\gamma u\rangle_\Gamma}_{=-\frac{1}{2}\langle\phi,\gamma u\rangle} + \rho(f,u),$$

$$\mu(\dot{V},\frac{1}{2}V) = \mu\frac{1}{2}(\nabla u,V) - \mu\frac{1}{2}(u,\nabla\cdot V) + \mu\frac{1}{2}\langle\gamma u,\gamma V\rangle_\Gamma$$

$$\lambda(\dot{\omega},\omega) = -\lambda\frac{1}{2}(u,\nabla\omega) + \lambda\frac{1}{2}(\nabla\cdot u,\omega) + \lambda\frac{1}{2}\langle\gamma u,\gamma\omega\rangle_\Gamma,$$

$$\left\langle\begin{pmatrix}\phi\\\psi\end{pmatrix}, B(\partial_t)\begin{pmatrix}\phi\\\psi\end{pmatrix}\right\rangle_\Gamma = \frac{1}{2}\langle\phi,\gamma u\rangle_\Gamma - \frac{1}{2}\langle\psi,\gamma\mu Vn\rangle_\Gamma - \frac{1}{2}\langle\psi,\gamma\lambda\omega n\rangle_\Gamma.$$

## Field Energy

By adding the four Equations, we finally arrive at

$$\frac{d}{dt}\left(\rho\frac{1}{2}\|u\|^2_{L_2(\Omega)}+\mu\frac{1}{4}\|V\|^2_{L_2(\Omega)^3}=\lambda\frac{1}{2}\|\omega\|^2_{L_2(\Omega)}\right)+\left\langle\begin{pmatrix}\phi\\ \psi\end{pmatrix},B(\partial_t)\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right\rangle_\Gamma$$
$$=\rho(f,u)_{L_2(\Omega)}.$$

For $\rho\geq 0$, $\mu\geq 0$ and $\lambda\geq 0$, which is valid for common materials, and the positivity of the Calderón operator $B(\partial_t)$ from Lemma 2 this provides, that the field energy

$$E=\rho\frac{1}{2}\|u\|^2_{L_2(\Omega)}+\mu\frac{1}{4}\|V\|^2_{L_2(\Omega)^3}+\lambda\frac{1}{2}\|\omega\|^2_{L_2(\Omega)}$$

satisfies for $t>0$

$$E(t)+\tilde{\beta}c_T\int_0^T\left(\|\partial_t^{-1}\phi(\cdot,t)\|^2_{H^{-\frac{1}{2}}(\Gamma)}+\|\partial_t^{-1}\psi(\cdot,t)\|^2_{H^{\frac{1}{2}}(\Gamma)}\right)dt\leq e^2E(0).$$

## FEM-BEM Formulation

$$
\begin{aligned}
\rho \mathbf{M_0} \dot{\mathbf{u}} &= -\mu \mathbf{D}^T \mathbf{V} + \lambda \bar{\mathbf{D}} \omega - \mathbf{C_0} \phi + \rho \mathbf{M_0} \mathbf{f}, \\
\mathbf{M_1} \dot{\mathbf{V}} &= \mathbf{D} \mathbf{u} - \mathbf{C_1} \psi, \\
\mathbf{M_2} \dot{\boldsymbol{\omega}} &= -\bar{\mathbf{D}}^T \mathbf{u} - \bar{\mathbf{C}}_1 \psi, \\
\mathbf{B}(\partial_t) \begin{pmatrix} \phi \\ \psi \end{pmatrix} &= \begin{pmatrix} \mathbf{C_0}^T \mathbf{u} \\ \mu \mathbf{C_1}^T \mathbf{V} + \lambda \bar{\mathbf{C}}_1^T \omega \end{pmatrix}.
\end{aligned}
$$

Here, we have

$$
\begin{aligned}
\mathbf{M_0} &= (b_j^U, b_i^U), \mathbf{M_1} = \frac{1}{2}(b_j^V, b_i^V), \mathbf{M_2} = (b_j^W, b_i^W), \\
\mathbf{D}|_{ji} &= -\frac{1}{2}(b_j^V, \nabla b_i^U) + \frac{1}{2}(\nabla \cdot b_j^V, b_i^U), \\
\bar{\mathbf{D}}|_{ji} &= -\frac{1}{2}(b_j^W, \nabla b_i^U) + \frac{1}{2}(\nabla \cdot b_j^W, b_i^U), \\
\mathbf{C_0}|_{ki} &= -\frac{1}{2}\left\langle b_k^\Phi, \gamma b_i^U \right\rangle_\Gamma, \mathbf{C_1}|_{lj} = \frac{1}{2}\left\langle b_l^\Psi, \gamma b_j^V n \right\rangle_\Gamma, \bar{\mathbf{C}}_1|_{lj} = \frac{1}{2}\left\langle b_i^\Psi, \gamma b_j^W n \right\rangle_\Gamma.
\end{aligned}
$$

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# Time Discretization via Leapfrog-Convolution Quadrature

Our time-discretization is based on [Banjai,Lubich and Sayas (2015)]:

$$\mathbf{M}_1\mathbf{V}^{n+\frac{1}{2}} = \mathbf{M}_1\mathbf{V}^n + \frac{1}{2}\Delta t\mathbf{D}\mathbf{u}^n - \frac{1}{2}\Delta t\mathbf{C}_1\psi^n,$$

$$\mathbf{M}_2\omega^{n+\frac{1}{2}} = \mathbf{M}_2\omega^n - \frac{1}{2}\Delta t\bar{\mathbf{D}}^T\mathbf{u}^n - \frac{1}{2}\Delta t\bar{\mathbf{C}}_1\phi^n,$$

$$\rho\mathbf{M}_0\mathbf{u}^{n+1} = \mathbf{M}_0\mathbf{u}^n - \mu\Delta t\mathbf{D}^T\mathbf{V}^{n+\frac{1}{2}} + \lambda\Delta t\bar{\mathbf{D}}\omega^{n+\frac{1}{2}} + \Delta t\mathbf{M}_0\mathbf{f}^{n+\frac{1}{2}} + \Delta t\mathbf{C}_0\phi^{n+\frac{1}{2}},$$

$$\mathbf{M}_1\mathbf{V}^{n+1} = \mathbf{M}_1\mathbf{V}^{n+\frac{1}{2}} + \frac{1}{2}\Delta t\mathbf{D}\mathbf{u}^{n+1} - \frac{1}{2}\Delta t\mathbf{C}_1\psi^{n+1},$$

$$\mathbf{M}_2\omega^{n+1} = \mathbf{M}_2\omega^{n+\frac{1}{2}} - \frac{1}{2}\Delta t\bar{\mathbf{D}}^T\mathbf{u}^{n+1} + \frac{1}{2}\Delta t\bar{\mathbf{C}}_1\psi^{n+1},$$

$$\left[\mathbf{B}(\partial_t)\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right]^{n+\frac{1}{2}} =$$

$$\begin{pmatrix} \mathbf{C}_0{}^T\bar{\mathbf{u}}^{n+\frac{1}{2}} \\ \mu\mathbf{C}_1{}^T\left(\mathbf{V}^{n+\frac{1}{2}} - \alpha\Delta t^2\mathbf{M}_1{}^{-1}\bar{\mathbf{C}}_1\dot{\psi}^{n+\frac{1}{2}}\right) + \lambda\bar{\mathbf{C}}_1^T\left(\omega^{n+\frac{1}{2}} - \alpha\Delta t^2\mathbf{M}_2{}^{-1}\bar{\mathbf{C}}_1\dot{\psi}^{n+\frac{1}{2}}\right)\end{pmatrix}.$$

## Guideline for Stability Analysis

The background for our stability analysis for the full discretization is the stability of the spatial semidiscretization, where we showed estimates for the field energy, mechanical energy and boundary functions. We adopt these for the full discretization:

- Start with the perturbed problem.
- Take a look at the discrete field energy and its bounds.
- Apply similar approach for the discrete mechanical energy.
- Give the bounds for the boundary function.
- Finally, present the error bound for the full discretization.

# Setting of the Stability Analysis

We take a look at the perturbed discrete scheme

$$\mathbf{V}^{n+\frac{1}{2}} = \mathbf{V}^{n} + \frac{1}{2}\Delta t \mathbf{D}\mathbf{u}^{n} - \frac{1}{2}\Delta t \mathbf{C}_1 \psi^{n} + \frac{1}{2}\Delta t \mathbf{g}^{n},$$
$$\boldsymbol{\omega}^{n+\frac{1}{2}} = \boldsymbol{\omega}^{n} - \frac{1}{2}\Delta t \bar{\mathbf{D}}^{T}\mathbf{u}^{n} - \frac{1}{2}\Delta t \bar{\mathbf{C}}_1 \phi^{n} + \frac{1}{2}\Delta t \mathbf{h}^{n},$$

$$\rho\mathbf{u}^{n+1} = \rho\mathbf{u}^{n} - \mu\Delta t \mathbf{D}^{T}\mathbf{V}^{n+\frac{1}{2}} + \lambda\Delta t \bar{\mathbf{D}}\boldsymbol{\omega}^{n+\frac{1}{2}} + \Delta t \mathbf{M}_0 \mathbf{f}^{n+\frac{1}{2}} + \Delta t \mathbf{C}_0 \phi^{n+\frac{1}{2}},$$
$$\mathbf{V}^{n+1} = \mathbf{V}^{n+\frac{1}{2}} + \frac{1}{2}\Delta t \mathbf{D}\mathbf{u}^{n+1} - \frac{1}{2}\Delta t \mathbf{C}_1 \psi^{n+1} + \frac{1}{2}\Delta t \mathbf{g}^{n},$$
$$\boldsymbol{\omega}^{n+1} = \boldsymbol{\omega}^{n+\frac{1}{2}} - \frac{1}{2}\Delta t \bar{\mathbf{D}}^{T}\mathbf{u}^{n+1} + \frac{1}{2}\Delta t \bar{\mathbf{C}}_1 \psi^{n+1} + \frac{1}{2}\Delta t \mathbf{h}^{n},$$

$$\left[\mathbf{B}(\partial_t^{\Delta t})\begin{pmatrix}\phi\\ \bar{\psi}\end{pmatrix}\right]^{n+\frac{1}{2}} =$$
$$\begin{pmatrix} \mathbf{C}_0{}^{T}\bar{\mathbf{u}}^{n+\frac{1}{2}} + \boldsymbol{\rho}^{n+\frac{1}{2}} \\ \mu\mathbf{C}_1{}^{T}\left(\mathbf{V}^{n+\frac{1}{2}} - \alpha\Delta t^2 \mathbf{M}_1{}^{-1}\bar{\mathbf{C}}_1\dot{\psi}^{n+\frac{1}{2}}\right) + \lambda\bar{\mathbf{C}}_1^{T}\left(\boldsymbol{\omega}^{n+\frac{1}{2}} - \alpha\Delta t^2\bar{\mathbf{C}}_1\dot{\psi}^{n+\frac{1}{2}}\right) + \boldsymbol{\sigma}^{n+\frac{1}{2}} \end{pmatrix}$$

## Discrete Field Energy

The discrete energy is given by

$$E^n = \rho\frac{1}{2}|\mathbf{u}^n|^2 + \frac{1}{4}\left(\frac{1}{2}\mu(|\mathbf{V}^{n+\frac{1}{2}}|^2 + |\mathbf{V}^{n-\frac{1}{2}}|^2) + \lambda(|\boldsymbol{\omega}^{n+\frac{1}{2}}|^2 + |\boldsymbol{\omega}^{n-\frac{1}{2}}|^2)\right).$$

**Lemma**

*The discrete energy is bounded at $t = n\Delta t$ by*

$$\begin{aligned} E^n \leq C\Bigg( & E^0 + \frac{t}{2}\Delta t \sum_{j=0}^{n} |\mathbf{f}^{j+\frac{1}{2}}|^2 + |\mathbf{g}^j|^2 + |\mathbf{h}^j|^2 \\ & + \max(t^2, t^6)\Delta t \sum_{j=0}^{n} \left(|(\partial_t^{\Delta t})^2 \boldsymbol{\rho}^{j+\frac{1}{2}}|^2 + |(\partial_t^{\Delta t})^2 \boldsymbol{\sigma}^{j+\frac{1}{2}}|^2\right)\Bigg), \end{aligned}$$

*where $C$ is independent of $h$, $\Delta t$, and $n$.*

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# Second-Order Formulation

Differentiating the first and last equation of the perturbed first order system and eliminating $\mathbf{V}$ and $\boldsymbol{\omega}$ yields the second order formulation

$$\rho\ddot{\mathbf{u}} = -\mu\mathbf{D}^T(\mathbf{D}\mathbf{u} - \mathbf{C}_1\psi) + \lambda\bar{\mathbf{D}}(\bar{\mathbf{D}}^T\mathbf{u} - \bar{\mathbf{C}}_1\psi) - \mathbf{C}_0\dot{\phi} + \rho\dot{\mathbf{f}} - \mu\mathbf{D}^T\mathbf{g} - \lambda\bar{\mathbf{D}}^T\mathbf{h},$$

$$\mathbf{B}(\partial_t)\begin{pmatrix}\dot{\phi}\\ \dot{\psi}\end{pmatrix} = \begin{pmatrix}\mathbf{C}_0{}^T\dot{\mathbf{u}}\\ \mu\mathbf{C}_1{}^T(\mathbf{D}\mathbf{u} - \mathbf{C}_1\psi) + \lambda\bar{\mathbf{C}}_1^T(\bar{\mathbf{D}}^T\mathbf{u} - \bar{\mathbf{C}}_1\psi)\end{pmatrix} + \begin{pmatrix}\dot{\rho}\\ \dot{\sigma} + \mathbf{C}_1{}^T\mathbf{g} + \bar{\mathbf{C}}_1^T\mathbf{h}\end{pmatrix}.$$

We go over to the discrete mechanical energy. Let $\dot{\mathbf{u}}^{n+\frac{1}{2}} = \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t}$,
$\dot{\mathbf{f}}^n = \frac{\mathbf{f}^{n+\frac{1}{2}} - \mathbf{f}^{n-\frac{1}{2}}}{\Delta t}$ etc. and $\bar{\mathbf{u}}^{n+\frac{1}{2}} = \frac{1}{2}(\mathbf{u}^{n+1} + \mathbf{u}^n)$, $\bar{\psi}^{n+\frac{1}{2}} = \frac{1}{2}(\psi^{n+1} + \psi^n)$.

# Discrete Mechanical Field Energy

## Lemma

*The discrete mechanical energy*

$$H^{n+\frac{1}{2}} = \rho\frac{1}{2}|\dot{\mathbf{u}}^{n+\frac{1}{2}}|^2 + \mu\frac{1}{2}|\mathbf{D}\bar{\mathbf{u}}^{n+\frac{1}{2}} - \mathbf{C}_1\bar{\psi}^{n+\frac{1}{2}}|^2 + \lambda\frac{1}{2}|\bar{\mathbf{D}}^T\bar{\mathbf{u}}^{n+\frac{1}{2}} - \bar{\mathbf{C}}_1\bar{\psi}^{n+\frac{1}{2}}|^2$$

*is bounded at* $t = (n+\frac{1}{2})\Delta t$ *by*

$$H^{n+\frac{1}{2}} \leq C\Bigg(H^{\frac{1}{2}} + \frac{t}{2}\sum_{j=0}^{n}|\rho\dot{\mathbf{f}}^j - \mu\mathbf{D}\mathbf{g}^j + \lambda\bar{\mathbf{D}}^T\mathbf{h}^j|^2$$

$$+ \max(t^2, t^6)\sum_{j=1}^{n}\Big(|(\partial_t^{\Delta t})^2\dot{\boldsymbol{\rho}}|^2 + |(\partial_t^{\Delta t})^2(\dot{\boldsymbol{\sigma}} + \mu\mathbf{C}_1{}^T\mathbf{g}^j + \lambda\bar{\mathbf{C}}_1{}^T\mathbf{h}^j)|^2\Big)\Bigg),$$

*where* $C$ *is independent on* $h$*, as well as* $\Delta t$ *and* $n$*.*

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## Error Bound for the Full Discretization

**Theorem**

*Assume that the initial values and the inhomogenity of the wave equation have their support in $\Omega$. Let the initial values for the semi-discretization be chosen as $u_h(0) = P_h u(0)$, $V_h(0) = P_h V(0)$, $\omega_h(0) = P_h \omega(0)$, where $P_h$ denotes the $L_2$-orthogonal projection onto the finite element space. If the solution of the wave equation is sufficiently smooth, then the error of the FEM and BEM with leapfrog and convolution quadrature full discretization under the CFL condition and the stability parameter $\alpha$ is bounded at $t = n\Delta t$ by*

$$\rho \| u_h^n - u(t) \|_{L_2(\Omega)} + \frac{1}{2}\mu \| V_h^n - V(t) \|_{L_2(\Omega)^3} + \lambda \| \omega^n - \omega(t) \|_{L_2(\Omega)^3}$$
$$+ \left( \Delta t \sum_{j=0}^{n-1} \| \phi_h^{j+\frac{1}{2}} - \phi(t_{j+\frac{1}{2}}) \|^2_{H^{-\frac{1}{2}}(\Gamma)} + \| \bar{\psi}_h^{j+\frac{1}{2}} - \psi(t_{j+\frac{1}{2}}) \|^2_{H^{\frac{1}{2}}(\Gamma)} \right)^{\frac{1}{2}}$$
$$\leq C(t)(h + \Delta t^2),$$

*where the constant $C(t)$ grows at most polynomially with $t$.*

# Conclusion and Outlook

- We have constructed a stable coupling of the interior and exterior problem for the elastic wave equation.
- The main result was the positivity of the Calderón operator, which was the basis to prove the stability and to find error bounds for the full discretization.
- Next step will be the implementation and numerical tests.

Thanks for your attention!

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