Analytic Anisotropic Weighted Regularity in Polyhedra & Exponential Convergence of *h*-*p* Methods

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Outline

Outline

Main framework

Variational problem: Ingredients

- Ω polyhedron in $\mathbb{R}^3$ (or, sometimes, polygon in $\mathbb{R}^2$)
  - Polyhedron $=$ 3D open set with piecewise flat boundary. Includes cracks.
  - Most of results valid for smooth domains do not hold for polyhedra.
- $\mathbb{V}$ variational space for scalar functions or N-component vectors

$$H^1_0(\Omega)^N \subset \mathbb{V} \subset H^1(\Omega)^N$$

- a 1st order $\mathbb{V}$-coercive form, homogeneous with constant coefficients

$$a(\boldsymbol{u},\boldsymbol{v}) = \sum_{|\alpha|=1}\sum_{|\beta|=1}\sum_{i=1}^{N}\sum_{j=1}^{N}\int_\Omega a^{\alpha,\beta}_{i,j}\,\partial^\alpha u_i\,\partial^\beta v_j\,d\boldsymbol{x}$$

  - $\mathbb{V}$-coercive means: $\exists C_a > 0,\ \gamma \in \mathbb{R}$,

$$\forall \boldsymbol{u} \in \mathbb{V},\ a(\boldsymbol{u},\boldsymbol{v}) + \gamma\|\boldsymbol{u}\|^2_{L^2(\Omega)} \geq C_a\|\boldsymbol{u}\|^2_{H^1(\Omega)}$$

  - Examples: Δ, div $\mathbf{A}\nabla$, Lamé, general linear elasticity,...
  - Part of theory can be extended to variable coefficients.

Continuous Problem $\mathscr{P}$

Variational problem

- $\mathbb{P}$ underlying second order operator: $\mathbb{P}\boldsymbol{u} = \boldsymbol{f}$ with $\boldsymbol{f} = (f_1, \ldots, f_N)$ and

$$f_j = -\sum_{|\alpha|=1} \sum_{|\beta|=1} \sum_{i=1}^{N} \partial^\beta a_{i,j}^{\alpha,\beta} \, \partial^\alpha u_i \, .$$

- For $\boldsymbol{f} \in L^2(\Omega)$, the variational problem is

$\mathscr{P}$ $$\boxed{\text{Find } \boldsymbol{u} \in \mathbb{V}, \quad \forall \boldsymbol{v} \in \mathbb{V}, \quad a(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle}$$

- We will write $\boxed{\mathbb{P}\boldsymbol{u} = \boldsymbol{f}}$ (so that $\mathbb{P}$ contains zero natural conditions).

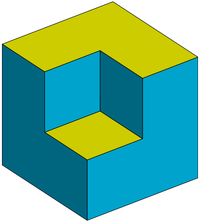
Question

How to design optimal Galerkin projection methods (e.g. FEM)

$\mathscr{C}$ $$\boxed{\text{Find } \boldsymbol{u}_n \in \mathbb{V}_n, \quad \forall \boldsymbol{v} \in \mathbb{V}_n, \quad a(\boldsymbol{u}_n, \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle}$$

Prototype

- Take Ω as the interior of Fichera corner (cube minus smaller cube)



- Solve the Laplace equation, i.e. with form $a(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v}$:

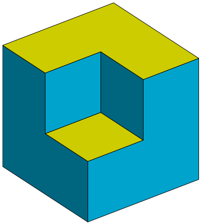
$$-\Delta \boldsymbol{u} = \boldsymbol{f} \quad \text{in} \quad \Omega$$

- Complete with "covering" boundary conditions, e.g. Dirichlet on yellow faces and Neumann on blue faces

Three typical zones: Regular, Edges, Corners

Let $\mathscr{E}$ be the set of the (open) edges $\boldsymbol{e}$ of Ω (21 elements for Fichera)

$$\mathfrak{E} = \bigcup_{\boldsymbol{e}\in\mathscr{E}} \boldsymbol{e}$$



Let $\mathscr{C}$ be the set of the corners $\boldsymbol{c}$ of Ω (14 elements for Fichera)

$$\mathfrak{C} = \bigcup_{\boldsymbol{c}\in\mathscr{C}} \boldsymbol{c}$$

- Regular zone: Let Ω_0 s.t. $\overline{\Omega}_0 \cap (\mathfrak{E} \cup \mathfrak{C}) = \emptyset$.
- Edge zone: Let $\Omega_{\mathfrak{E}}$ s.t. $\overline{\Omega}_{\mathfrak{E}} \cap \mathfrak{E} \neq \emptyset$ and $\overline{\Omega}_{\mathfrak{E}} \cap \mathfrak{C} = \emptyset$.
  For any $\boldsymbol{e} \in \mathscr{E}$, denote $\Omega_{\boldsymbol{e}}$ such a domain if $\overline{\Omega}_{\boldsymbol{e}} \cap \mathfrak{E} \subset \boldsymbol{e}$.
- Corner zone: Let $\Omega_{\mathfrak{C}}$ s.t. $\overline{\Omega}_{\mathfrak{C}} \cap \mathfrak{C} \neq \emptyset$.
  For any $\boldsymbol{c} \in \mathscr{C}$, denote $\Omega_{\boldsymbol{c}}$ such a domain if $\overline{\Omega}_{\boldsymbol{c}} \cap \mathfrak{C} = \boldsymbol{c}$.

Singularities and asymptotics

Assume $\boldsymbol{f}$ real analytic (convergence of Taylor Series around any $\boldsymbol{x}_0 \in \overline{\Omega}$).
What are the regularity properties of solution $\boldsymbol{u}$?

- Regular zone Ω_0.
  Then $\boldsymbol{u}$ is real analytic in $\overline{\Omega}_0$ (cv of TS around any $\boldsymbol{x}_0 \in \overline{\Omega}_0$).
- Edge zone $\Omega_{\boldsymbol{e}}$ for $\boldsymbol{e} \in \mathscr{E}$. Cylindrical coord. (r, θ, z) associated with $\boldsymbol{e}$.
  ω is the opening of $\boldsymbol{e}$. Then $\boldsymbol{u}$ has a singular expansion in $\Omega_{\boldsymbol{e}}$ starting as

$$\begin{cases} \gamma_{\boldsymbol{e}}(z)\, r^{\frac{\pi}{2\omega}} \sin(\frac{\pi}{2\omega}\theta) & \text{if Dirichlet-Neumann edge} \\ \gamma_{\boldsymbol{e}}(z)\, r^{\frac{\pi}{\omega}} \cos(\frac{\pi}{\omega}\theta) & \text{if Neumann-Neumann edge} \end{cases}$$

  Here the coefficient $z \mapsto \gamma_{\boldsymbol{e}}(z)$ is an analytic function in $\boldsymbol{e} \cap \Omega_{\boldsymbol{e}}$.
  Exponents of higher order are $\ell + (k + \frac{1}{2})\frac{\pi}{\omega}$ $\left[\text{or } \ell + k\frac{\pi}{\omega}\right]$ with $k, \ell \in \mathbb{N}$.
- Corner zone $\Omega_{\boldsymbol{c}}$ for $\boldsymbol{c} \in \mathscr{C}$. Polar coord. $(\rho, \varphi) \in \mathbb{R}_+ \times \mathbb{S}^2$ associated with $\boldsymbol{c}$. Then $\boldsymbol{u}$ has a singular expansion in $\Omega_{\boldsymbol{c}}$ with terms of type

$$\gamma_{\boldsymbol{c}}\, \rho^{\lambda_{\boldsymbol{c}}}\, \Phi_{\boldsymbol{c}}(\varphi), \quad \lambda_{\boldsymbol{c}} > 0, \ \ \Phi_{\boldsymbol{c}} \in H^1(G_{\boldsymbol{c}}), \ \ \gamma_{\boldsymbol{c}} \in \mathbb{R}.$$

  Here $G_{\boldsymbol{c}} = \mathbb{S}^2 \cap \Omega_{\boldsymbol{c}}$. Functions $\Phi_{\boldsymbol{c}}$ have singularities... at edges.

Problematics

Singular expansions are difficult to handle in 3D:

1. Corner singularities contribute to edge singularities.
2. No canonical splitting between edges and corners.
3. Singularity spaces are infinite dimensional.
4. Corner singularities are not directly explicit.

Aim

Compute solution $\boldsymbol{u}$ with optimal efficiency.

Solution proposed by Babuška and Guo :

1. Find families of “countably normed spaces” to which sol. u belongs.
2. Use h-p finite element approximation
3. Obtain exponential convergence.

2D versus 3D

The program was performed by B&G in 2D ('90), but was pending in 3D.

Outline

Countably normed spaces

Defined by a sequence of semi-norms for functions $\boldsymbol{u}$ set on Ω

$$\boldsymbol{u} \longmapsto |\boldsymbol{u}|_{\mathbb{X}^\ell}\,, \quad \ell \in \mathbb{N} = \{0, 1, \ldots, \}$$

Associate normed spaces:

1. $\mathbb{X}^k = \{\boldsymbol{u} :\ |\boldsymbol{u}|_{\mathbb{X}^\ell} < \infty,\ 0 \le \ell \le k\}$ and $\|\boldsymbol{u}\|_{\mathbb{X}^k} = \sup_{\ell=0}^{k} |\boldsymbol{u}|_{\mathbb{X}^\ell}$
2. $\mathbb{X}^\infty = \{\boldsymbol{u} :\ |\boldsymbol{u}|_{\mathbb{X}^\ell} < \infty,\ \forall \ell \in \mathbb{N}\}$
3. $\mathbb{X}^\varpi = \left\{\boldsymbol{u} \in \mathbb{X}^\infty :\ \sup_{\ell \ge 1} \left(\frac{1}{\ell!} |\boldsymbol{u}|_{\mathbb{X}^\ell}\right)^{1/\ell} < \infty\right\}$ — analytic class

Example of Sobolev norms. If $\mathbb{X}^\ell$ s-norm is Sobolev s-norm $H^\ell(\Omega)$

1. $\mathbb{X}^k = H^k(\Omega)$,
2. $\mathbb{X}^\infty = \mathscr{C}^\infty(\overline{\Omega})$
3. $\mathbb{X}^\varpi = H^\varpi(\overline{\Omega})$: if $\boldsymbol{u} \in H^\varpi(\overline{\Omega})$, $\exists C_{\boldsymbol{u}} > 0$, $\forall \ell \ge 1$, $|\boldsymbol{u}|_{\mathbb{X}^\ell} \le C_{\boldsymbol{u}}^\ell \, \ell!$

Similar, but distinct, definitions for right hand sides: $\boldsymbol{f} \longmapsto |\boldsymbol{f}|_{\mathbb{Y}^\ell}\,, \quad \ell \in \mathbb{N}$

Rationale of the analytic regularity

Remind that $\mathbb{V}$ is the variational space:

$$\mathrm{H}^1_0(\Omega) \subset \mathbb{V} \subset H^1(\Omega)$$

Find suitable families of semi-norms $|\cdot|_{\mathbb{X}^m}$ and $|\cdot|_{\mathbb{Y}^m}$ such that

1. $\mathbb{X}^1$ is a subspace of H^1
2. The embedding of $\mathbb{X}^2$ in H^1 is compact
3. Real analytic functions $\boldsymbol{f}$ on $\overline{\Omega}$ belong to $\mathbb{Y}^\varpi$
4. The following analytic regularity holds

$$\boxed{\boldsymbol{u} \in \mathbb{V} \quad \text{and} \quad \mathbb{P}\boldsymbol{u} = \boldsymbol{f} \in \mathbb{Y}^\varpi \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{X}^\varpi}$$

We will see that, when Ω is a polygon or a polyhedron, it is possible to find such families that are, moreover, suitable to prove the exponential convergence of h-p FEM.

2D first. Polygons, choosing weights

Remind

- Ω polygon with corner set $\mathfrak{C} = \cup_{\boldsymbol{c}\in\mathscr{C}}\{\boldsymbol{c}\}$.
- 2[nd] order operator $\mathbb{P}$ (e.g. Δ)

Semi-norms $|\cdot|_{\mathbb{X}^\ell}$ and $|\cdot|_{\mathbb{Y}^\ell}$ are taken as weighted norms:

$$|\boldsymbol{u}|_{\mathbb{X}^\ell} = \sum_{|\alpha|=\ell} \left\| w_\ell\, \partial_{\boldsymbol{x}}^\alpha \boldsymbol{u} \right\|_{L^2(\Omega)} \quad \text{and} \quad |\boldsymbol{f}|_{\mathbb{Y}^\ell} = \sum_{|\alpha|=\ell-2} \left\| w_\ell\, \partial_{\boldsymbol{x}}^\alpha \boldsymbol{f} \right\|_{L^2(\Omega)}$$

where $w_0(\boldsymbol{x})$, $w_1(\boldsymbol{x})$,..., $w_\ell(\boldsymbol{x})$, ... family of weights of general type

$$w_\ell(\boldsymbol{x}) = \mathrm{r}(\boldsymbol{x})^{\gamma(\ell)}, \quad \mathrm{r}(\boldsymbol{x}) = dist(\boldsymbol{x}, \mathfrak{C}),$$

with a sequence $\ell \mapsto \gamma(\ell)$ to be chosen.

2D first. Homogeneous norms

Homogeneous norms: The Kondrat'ev spaces

Pick $\beta \in \mathbb{R}$ & set $\boxed{\gamma(\ell) = \beta + \ell}$

Semi-norms $|\cdot|_{\mathbb{X}^\ell}$ and $|\cdot|_{\mathbb{Y}^\ell}$ are the weighted norms:

$$|\boldsymbol{u}|_{\mathbb{X}^\ell} = \sum_{|\alpha|=\ell} \|\mathrm{r}(\boldsymbol{x})^{\beta+|\alpha|}\, \partial_{\boldsymbol{x}}^\alpha \boldsymbol{u}\|_{L^2(\Omega)}$$

and

$$|\boldsymbol{f}|_{\mathbb{Y}^\ell} = \sum_{|\alpha|=\ell-2} \|\mathrm{r}(\boldsymbol{x})^{\beta+\ell}\, \partial_{\boldsymbol{x}}^\alpha \boldsymbol{f}\|_{L^2(\Omega)} = \sum_{|\alpha|=\ell-2} \|\mathrm{r}(\boldsymbol{x})^{\beta+2+|\alpha|}\, \partial_{\boldsymbol{x}}^\alpha \boldsymbol{f}\|_{L^2(\Omega)}$$

1. $\mathbb{X}^1$ is a subspace of $H^1 \iff \beta \leq -1$
   Because $\|\boldsymbol{u}\|_{\mathbb{X}^1} \simeq \|\mathrm{r}^\beta\, \boldsymbol{u}\|_{L^2(\Omega)} + \|\mathrm{r}^{\beta+1}\, \nabla \boldsymbol{u}\|_{L^2(\Omega)}$.
2. The embedding of $\mathbb{X}^2$ in H^1 is compact $\implies \beta < -1$
3. Real analytic functions $\boldsymbol{f}$ belong to $\mathbb{Y}^\varpi \implies \beta > -3$
   Because constant functions c satisfy $\mathrm{r}^{\beta+2} c \in L^2$ iff $\beta + 2 > -1$.

Good for Dirichlet. Bad for Neumann

The Kondrat'ev spaces are **good for Dirichlet** but **bad for Neumann**

For Dirichlet problem $\mathbb{V} = H^1_0(\Omega)$. By virtue of angular Poincaré inequality

$$\boldsymbol{u} \in \mathbb{V} \Longrightarrow \mathrm{r}^{-1}\boldsymbol{u} \in L^2(\Omega).$$

Exists $b = b(\Omega, \mathbb{P}) > 0$ (the smallest singularity exponent — for Δ, $b = \min_{\boldsymbol{c}\in\mathscr{C}} \frac{\pi}{\omega_{\boldsymbol{c}}}$) so that [Kondrat'ev, 1967]

$$(\star)\quad -1-b < \beta < -1 \quad \Longrightarrow \quad \left[\boldsymbol{u} \in \mathbb{V} \text{ and } \mathbb{P}\boldsymbol{u} \in \mathbb{Y}^2 \;\Rightarrow\; \boldsymbol{u} \in \mathbb{X}^2\right]$$

- With explicit notation: $\boldsymbol{u} \in \mathbb{V}$ and $\mathrm{r}^{\beta+2}\mathbb{P}\boldsymbol{u} \in L^2 \Rightarrow \mathrm{r}^{\beta+|\alpha|}\partial^\alpha \boldsymbol{u} \in L^2$, $|\alpha| \le 2$.
- Note that $(\star)$ implies that singularities belong to $\mathbb{X}^\ell$ for any $\ell \in \mathbb{N}$.

For Neumann problem, independent pointwise values arise at each corner. The constant function $1 \notin \mathbb{X}^2$ if $\beta < -1$ because $w_0 = \mathrm{r}^\beta \notin L^2(\Omega)$.

2D. A choice for all seasons: inhomogeneous norms

Take

$$\star \; -1-b<\beta<-1 \quad \text{and} \quad 0 \; \gamma(\ell)=\max\{0,\ell+\beta\}$$

so that

$$w_\ell(\boldsymbol{x})=1 \text{ if } \ell<-\beta \quad \text{and} \quad w_\ell(\boldsymbol{x})=\mathrm{r}(\boldsymbol{x})^{\ell+\beta} \text{ if } \ell\geq-\beta.$$

With 0, condition 3 of embedding for analytical rhs is always satisfied.

Theorem [Mazya-Plamenevskii, 1984]

For Dirichlet, Neumann or mixed conditions, exists $b=b(\Omega,\mathbb{P})>0$ so that: With ⋆ $\beta\in(-1-b,-1)$ and 0 $w_\ell=\mathrm{r}^{\max\{0,\ell+\beta\}}$, then $(\forall m\geq 2)$

$$\boxed{\boldsymbol{u}\in\mathbb{V} \quad \text{and} \quad \mathbb{P}\boldsymbol{u}\in\mathbb{Y}^m \quad \Longrightarrow \quad \boldsymbol{u}\in\mathbb{X}^m}$$

Theorem [Babuška-Guo 1988, 1989, 1993]

There exists $\beta\in(-2,-1)$ such that with the weights $w_\ell=\mathrm{r}^{\max\{0,\ell+\beta\}}$:

$$\boxed{\boldsymbol{u}\in\mathbb{V} \quad \text{and} \quad \mathbb{P}\boldsymbol{u}\in\mathbb{Y}^\varpi \quad \Longrightarrow \quad \boldsymbol{u}\in\mathbb{X}^\varpi}$$

3D: Corners, edges, distance functions and weights

Ω polyhedron in $\mathbb{R}^3$. Distance to singular points: $\boldsymbol{x} \mapsto \mathrm{r}(\boldsymbol{x})$

- Corner set $\mathfrak{C} = \cup_{\boldsymbol{c}\in\mathscr{C}}\{\boldsymbol{c}\}$, distance functions: $\mathrm{r}_{\boldsymbol{c}}$ to $\boldsymbol{c}$, $\mathrm{r}_{\mathfrak{C}}$ to $\mathfrak{C}$,
- Edges $\boldsymbol{e}$, set of edges $\mathscr{E}$, distance functions: $\mathrm{r}_{\boldsymbol{e}}$ to $\boldsymbol{e}$.

Two ways of generating weights (using **inhomogeneous norm** choice)

1. A simple way: choose $\beta \in \mathbb{R}$ and use powers of r $\quad w_\ell = \mathrm{r}^{\max\{0,\ell+\beta\}}$
2. A finer tool: choose a multi-β, i.e. $\underline{\beta} = (\beta_{\boldsymbol{c}}, \beta_{\boldsymbol{e}})$

$$w_\ell = \prod_{\boldsymbol{c}\in\mathscr{C}} \mathrm{r}_{\boldsymbol{c}}^{\max\{0,\ell+\beta_{\boldsymbol{c}}\}} \times \prod_{\boldsymbol{e}\in\mathscr{E}} \Big(\frac{\mathrm{r}_{\boldsymbol{e}}}{\mathrm{r}_{\mathfrak{C}}}\Big)^{\max\{0,\ell+\beta_{\boldsymbol{e}}\}}$$

Remarks :

- $\boldsymbol{e} \in \mathscr{E}$ ends with two corners $\boldsymbol{c}$, $\boldsymbol{c}' \in \mathscr{C}$.
  Function $\mathrm{r}_{\boldsymbol{e}}/\mathrm{r}_{\mathfrak{C}}$ is $\simeq$ to $\mathrm{r}_{\boldsymbol{e}}/\mathrm{r}_{\boldsymbol{c}}$ near $\boldsymbol{c}$, to $\mathrm{r}_{\boldsymbol{e}}$ in the middle, to $\mathrm{r}_{\boldsymbol{e}}/\mathrm{r}_{\boldsymbol{c}'}$ near $\boldsymbol{c}'$
- If $\beta_{\boldsymbol{c}} \equiv \beta_{\boldsymbol{e}} \equiv \beta$, then $\prod_{\boldsymbol{c}\in\mathscr{C}} \mathrm{r}_{\boldsymbol{c}}^{\ell+\beta_{\boldsymbol{c}}} \times \prod_{\boldsymbol{e}\in\mathscr{E}} \big(\frac{\mathrm{r}_{\boldsymbol{e}}}{\mathrm{r}_{\mathfrak{C}}}\big)^{\ell+\beta_{\boldsymbol{e}}} \simeq \mathrm{r}^{\ell+\beta}$.
- *Simple option does not allow to take advantage of anisotropy*

Finite regularity in polyhedral domains

Coercive variational formulation of operator $\mathbb{P}$ in $\mathbb{V} \subset \boldsymbol{H}^1(\Omega)$

Theorem B [Mazya-Rossmann 2003] Revisited [CoDaNi, 2012]

Exists optimal numbers $b_{\boldsymbol{c}}(\Omega, \mathbb{P}) > -\frac{1}{2}$ and $b_{\boldsymbol{e}}(\Omega, \mathbb{P}) > 0$ so that:

- If $\underline{\beta}$ satisfies $\beta_{\boldsymbol{c}} \in (-b_{\boldsymbol{c}} - \frac{3}{2}, -1)$ and $\beta_{\boldsymbol{e}} \in (-1 - b_{\boldsymbol{e}}, -1)$
- If the weights are $w_\ell = \prod_{\boldsymbol{c} \in \mathscr{C}} r_{\boldsymbol{c}}^{\max\{0, \ell+\beta_{\boldsymbol{c}}\}} \times \prod_{\boldsymbol{e} \in \mathscr{E}} \left(\frac{r_{\boldsymbol{e}}}{r_{\mathfrak{e}}}\right)^{\max\{0, \ell+\beta_{\boldsymbol{e}}\}}$

Then $(\forall m \geq 2)$ $\boxed{\boldsymbol{u} \in \mathbb{V} \quad \text{and} \quad \mathbb{P}\boldsymbol{u} \in \mathbb{Y}^m \quad \Longrightarrow \quad \boldsymbol{u} \in \mathbb{X}^m}$

BUT

The 3D h-p FEM takes ***anisotropy*** *into account.*
It results in exponential convergence only if the additional regularity of solutions along edges is used for designing meshes.

Anisotropic weights

Weights w_ℓ providing isotropic semi-norms

$$|u|_{\mathbb{X}^\ell} = \sum_{|\alpha|=\ell} \|w_\ell\, \partial_x^\alpha u\|_{L^2(\Omega)}$$

have to be replaced by weights $w_{e,\alpha}$ depending on directions of derivation in each edge e. Let $e \in \mathscr{E}$ and let $\mathcal{V}_e$ be a neighborhood of e. Near the ends of e (that are corners) $\mathcal{V}_e$ is a conical neighborhood.

The new space $\mathbb{X}$ is defined on each $\mathcal{V}_e$ with semi-norms

$$\big|u|_{\mathcal{V}_e}\big|_{\mathbb{X}^\ell} = \sum_{|\alpha|=\ell} \|w_{e,\alpha} \partial_x^\alpha u\|_{L^2(\mathcal{V}_e)}$$

where multi-indices $\alpha = (\alpha_e^\perp, \alpha_e^\parallel)$ correspond to tubular coordinates $x = (x_e^\perp, x_e^\parallel)$, — perpendicular and parallel to e. Typically we take

$$\boxed{w_{e,\alpha} = r_e^{\max\{0, \beta_e + |\alpha_e^\perp|\}}}$$

that is independent of derivatives $\partial_x^{\alpha_e^\parallel}$ along e.

Anisotropic weights (edges & corners)

To simplify exposition, assume that edges are parallel to coordinate axes. The [inhomogeneous] anisotropic weights are

(W) $$w_\alpha = \prod_{c\in\mathscr{C}} r_c^{\max\{0,\beta_c+|\alpha|\}} \times \prod_{e\in\mathscr{E}} \left(\frac{r_e}{r_{\mathfrak{c}}}\right)^{\max\{0,\beta_e+|\alpha_e^\perp|\}}$$

Corresponding $\mathbb{X}^\ell$ and $\mathbb{Y}^\ell$ semi-norms are

$$|u|_{\mathbb{X}^\ell} = \sum_{|\alpha|=\ell} \|w_\alpha\,\partial_x^\alpha u\|_{L^2(\Omega)} \quad \text{and} \quad |f|_{\mathbb{Y}^\ell} = \sum_{|\alpha|=\ell-2} \|w_\alpha\,\partial_x^\alpha f\|_{L^2(\Omega)}$$

Theorem A [CoDaNi, 2012]

$\Omega \subset \mathbb{R}^3$ polyhedron and problem as in p.2.

With the same numbers $b_c(\Omega,\mathbb{P})$ and $b_e(\Omega,\mathbb{P})$ as in Theorem B:

- If β satisfies $\beta_c \in (-b_c - \frac{3}{2}, -1)$ and $\beta_e \in (-1 - b_e, -1)$
- Choose the weights according to (W)

Then $\boxed{u \in \mathbb{V} \quad \text{and} \quad \mathbb{P}u \in \mathbb{Y}^\varpi \implies u \in \mathbb{X}^\varpi}$

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20 years later...

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Outline

Meshes: Layers

Notations

- Ω polyhedron in $\mathbb{R}^3$.
- $\mathfrak{M} = (\mathfrak{M}_p)_{p\in\mathbb{N}}$ family of meshes with the following nested structure

$$\mathfrak{M}_0 = \mathfrak{T}_0$$
$$\mathfrak{M}_1 = \mathfrak{L}_0 \cup \mathfrak{T}_1$$
$$\mathfrak{M}_p = \mathfrak{L}_0 \cup \mathfrak{L}_1 \cup \ldots \cup \mathfrak{L}_{p-1} \cup \mathfrak{T}_p, \quad p \geq 1$$

  with the regular layers $\mathfrak{L}_\ell$ and the terminal layers $\mathfrak{T}_p$.
- Any of these submeshes are formed of (mapped) hexahedral elements K.
- Any $K \in \mathfrak{L}_\ell$ satisfies $\overline{K} \cap (\mathfrak{E} \cup \mathfrak{C}) = \emptyset$
- Any $K \in \mathfrak{T}_p$ satisfies $\overline{K} \cap \mathfrak{C} = \{\boldsymbol{c}\}$ or $\overline{K} \cap \mathfrak{E} \neq \emptyset$ & $\overline{K} \cap \mathfrak{E} \subset \boldsymbol{e}$. Both conditions can be satisfied for a same element.
- Size conditions are imposed, subject to the position of each element.

σ-meshes

$\sigma \in (0,1)$ is a parameter of the family $\mathfrak{M}$. One often takes $\boxed{\sigma = \frac{1}{2}}$

- Regular Zone: Ω_0 s.t. $\overline{\Omega}_0 \cap (\mathfrak{E} \cup \mathfrak{C}) = \emptyset$
  - Ω_0 intersects a finite number of layers $\mathfrak{L}_\ell$, $\ell \leq L$
  - Ω_0 is disjoint from terminal layers $\mathfrak{T}_p$ for $p > L$
- Pure Edge Zone: $\Omega_{\boldsymbol{e}}$ s.t. $\overline{\Omega}_{\boldsymbol{e}} \cap \mathfrak{C} = \emptyset$, $\overline{\Omega}_{\boldsymbol{e}} \cap \mathfrak{E} \neq \emptyset$ and $\subset \boldsymbol{e}$.
  - In $\Omega_{\boldsymbol{e}}$, all elements K are aligned with tubular coordinates $(x_{\boldsymbol{e}}^{\perp}, x_{\boldsymbol{e}}^{\parallel})$, i.e.
  $$K = K_{\boldsymbol{e}}^{\perp} \times K_{\boldsymbol{e}}^{\parallel}$$
  - For $K \in \mathfrak{T}_p$, component $K_{\boldsymbol{e}}^{\perp}$ has size $\mathcal{O}(\sigma^p)$ and $K_{\boldsymbol{e}}^{\parallel}$ size $\mathcal{O}(1)$
  - For $K \in \mathfrak{L}_\ell$, component $K_{\boldsymbol{e}}^{\perp}$ has size $\mathcal{O}(\sigma^\ell)$ and $K_{\boldsymbol{e}}^{\parallel}$ size $\mathcal{O}(1)$
  Moreover the distance $r_{\boldsymbol{e}}$ to the edge $\boldsymbol{e}$ is equivalent to σ^ℓ in $K_{\boldsymbol{e}}^{\perp}$
- Corner zone: $\Omega_{\boldsymbol{c}}$ s.t. $\overline{\Omega}_{\boldsymbol{c}} \cap \mathfrak{C} = \boldsymbol{c}$. Splits into
  - Pure Corner Zone: $\Omega_{\boldsymbol{c},0}$ s.t. $\overline{\Omega}_{\boldsymbol{c},0} \cap \mathfrak{E} = \emptyset$
  $K \in \mathfrak{T}_p$ has size $\mathcal{O}(\sigma^p)$, and $K \in \mathfrak{L}_\ell$ has size $\mathcal{O}(\sigma^\ell)$ & $r_{\boldsymbol{c}}\big|_K \sim \sigma^\ell$
  - Edge-Corner Zone: $\Omega_{\boldsymbol{c},\boldsymbol{e}}$ s.t. $\overline{\Omega}_{\boldsymbol{c},\boldsymbol{e}} \cap \mathfrak{E} \subset \boldsymbol{e}$

Pure Edge Zone [with anisotropy]

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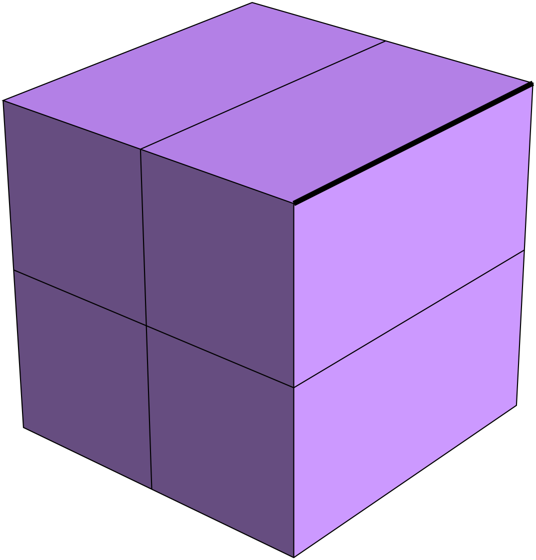

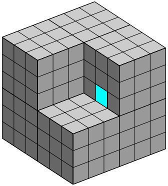

$\mathfrak{T}_0$ 4 elem.

Pure Edge Zone [with anisotropy]

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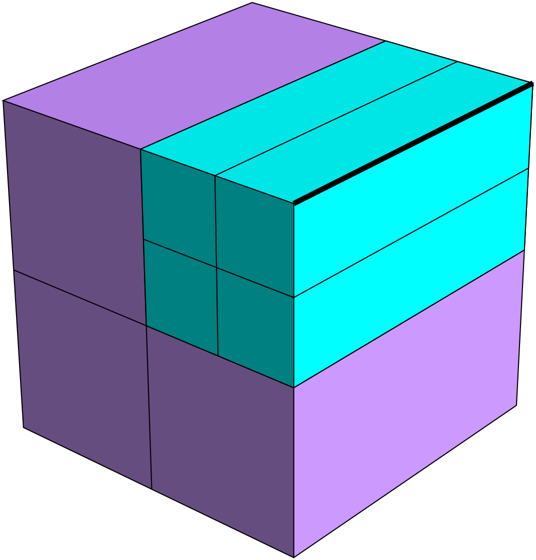
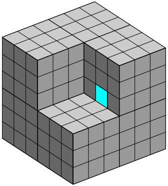
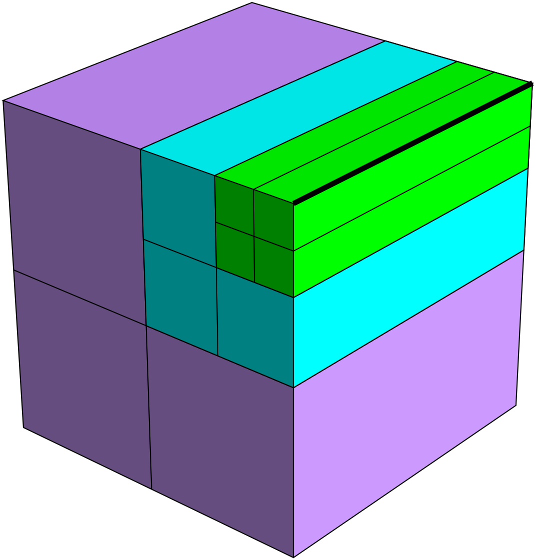


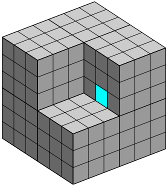


$\mathfrak{L}_0$ 3 elem.
$\mathfrak{T}_1$ 4 elem.

Pure Edge Zone [with anisotropy]

DRAWN WITH FIG4TEX





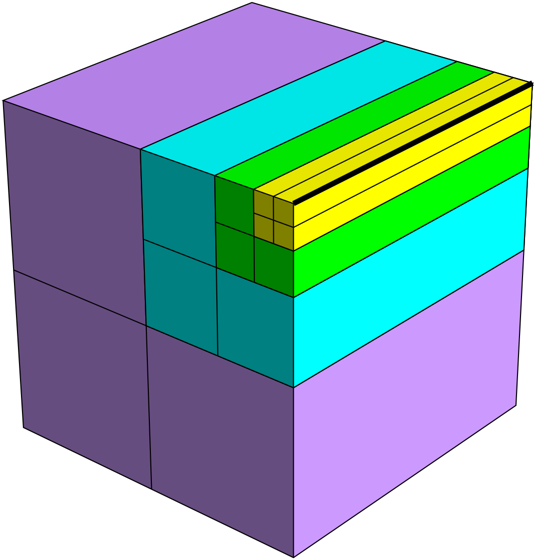
$\mathfrak{L}_0$ 3 elem.

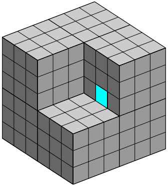
$\mathfrak{L}_1$ 3 elem.

$\mathfrak{T}_2$ 4 elem.

Pure Edge Zone [with anisotropy]

DRAWN WITH FIG4TEX





$\mathfrak{L}_0$ 3 elem.

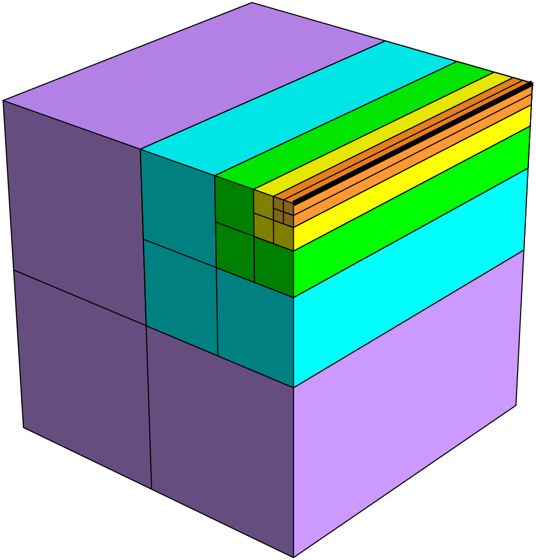
$\mathfrak{L}_1$ 3 elem.

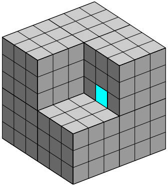
$\mathfrak{L}_2$ 3 elem.

$\mathfrak{T}_3$ 4 elem.

Pure Edge Zone [with anisotropy]

DRAWN WITH FIG4TEX





$\mathfrak{L}_0$ 3 elem.

$\mathfrak{L}_1$ 3 elem.

$\mathfrak{L}_2$ 3 elem.

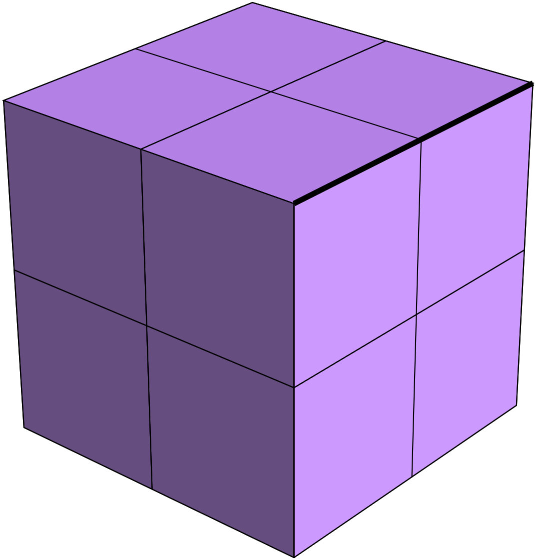
$\mathfrak{L}_3$ 3 elem.

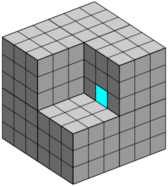
$\mathfrak{T}_4$ 4 elem.

$\# \, 3p + 4$

NB: Edge Zone without anisotropy

DRAWN WITH FIG4TEX

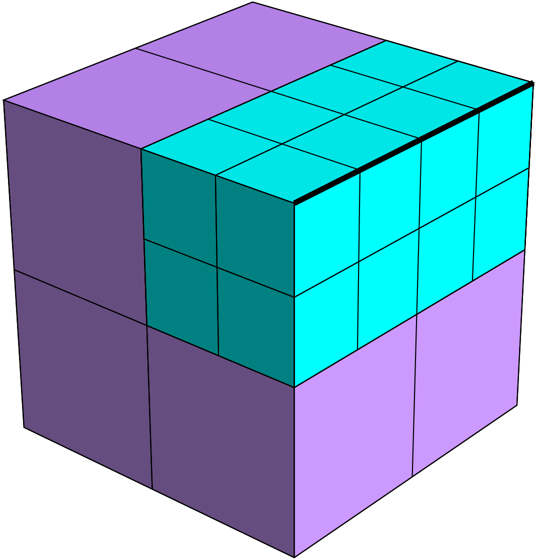


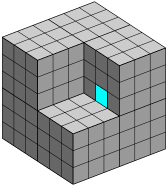


$\mathfrak{T}_0$ 8 elem.

NB: Edge Zone without anisotropy

DRAWN WITH FIG4TEX

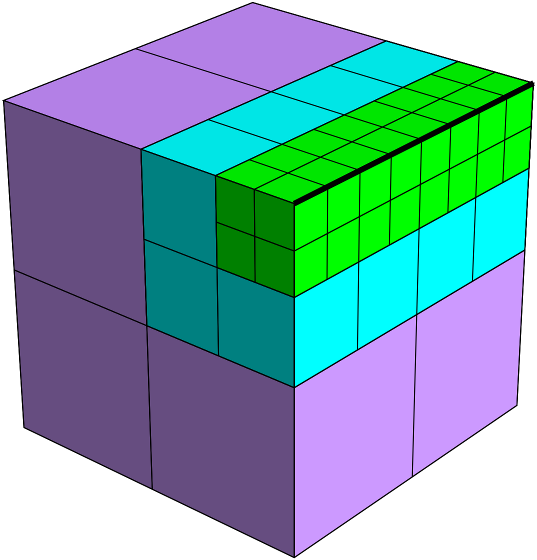


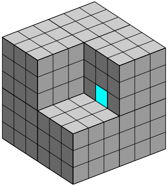


$\mathfrak{L}_0$ 6 elem.
$\mathfrak{T}_1$ 16 elem.

NB: Edge Zone without anisotropy

DRAWN WITH FIG4TEX





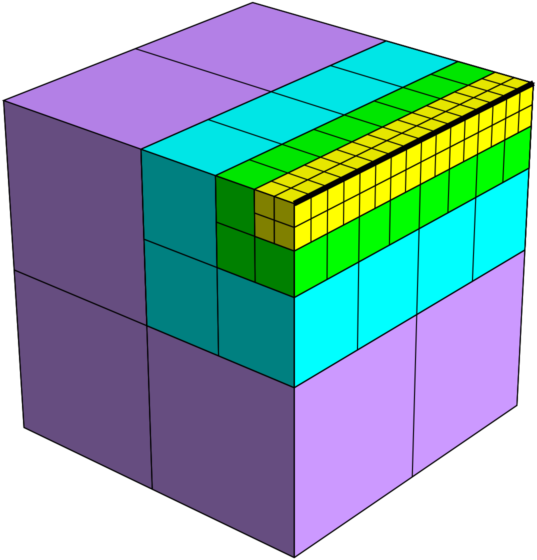
$\mathfrak{L}_0$ 6 elem.

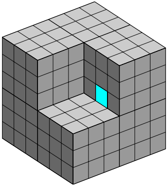
$\mathfrak{L}_1$ 12 elem.

$\mathfrak{T}_2$ 32 elem.

NB: Edge Zone without anisotropy

[DRAWN WITH FIG4TEX]

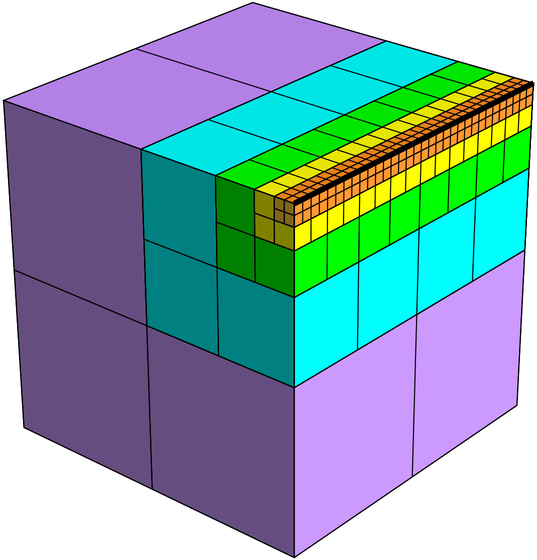


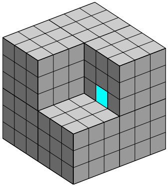


$\mathfrak{L}_0$ 6 elem.
$\mathfrak{L}_1$ 12 elem.
$\mathfrak{L}_2$ 24 elem.
$\mathfrak{T}_3$ 64 elem.

NB: Edge Zone without anisotropy

DRAWN WITH FIG4TEX





$\mathfrak{L}_0$ 6 elem.

$\mathfrak{L}_1$ 12 elem.

$\mathfrak{L}_2$ 24 elem.

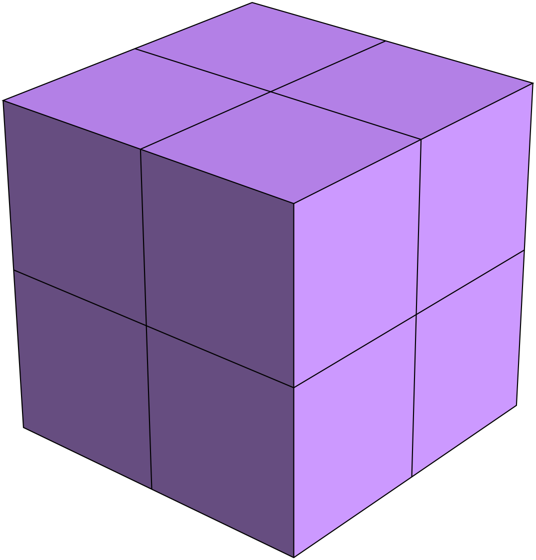
$\mathfrak{L}_3$ 48 elem.

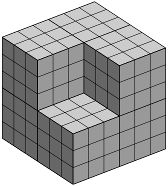
$\mathfrak{T}_4$ 128 elem.

$\# > 6 \cdot 2^p$

Pure Corner Zone

DRAWN WITH FIG4TEX

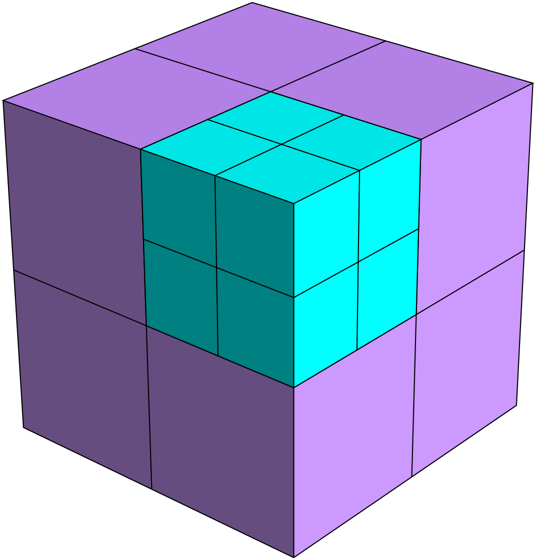


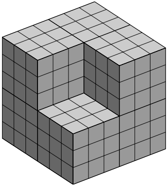


$\mathfrak{T}_0$ 8 elem.

Pure Corner Zone

DRAWN WITH FIG4TEX

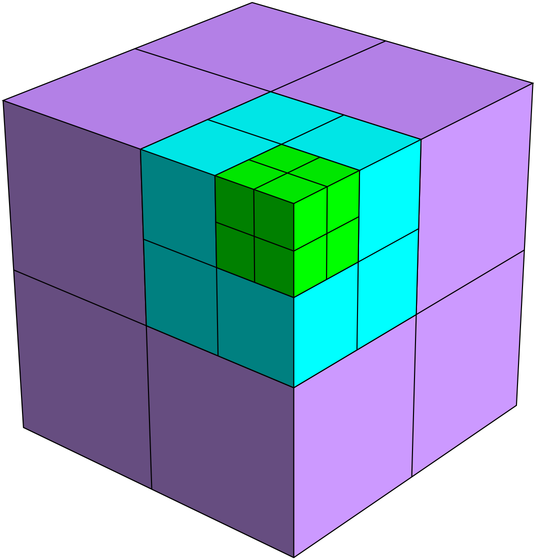


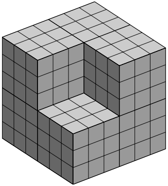


$\mathfrak{L}_0$ 7 elem.
$\mathfrak{T}_1$ 8 elem.

Pure Corner Zone

DRAWN WITH FIG4TEX





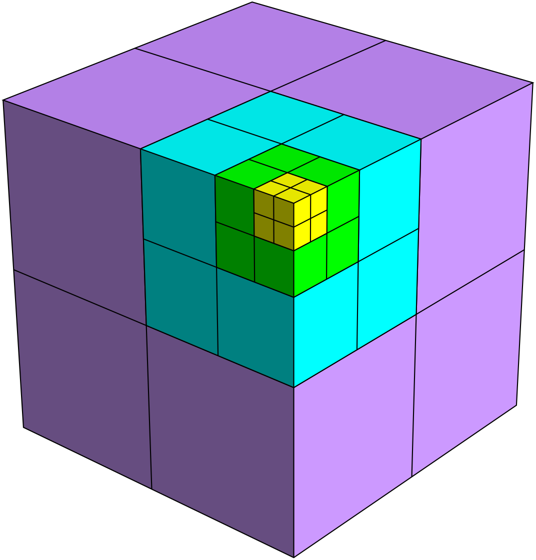
$\mathfrak{L}_0$ 7 elem.

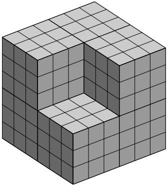
$\mathfrak{L}_1$ 7 elem.

$\mathfrak{T}_2$ 8 elem.

Pure Corner Zone

DRAWN WITH FIG4TEX

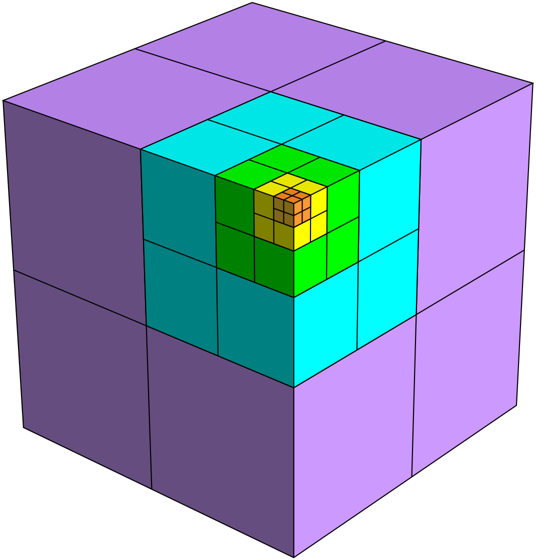


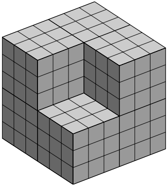


$\mathfrak{L}_0$ 7 elem.
$\mathfrak{L}_1$ 7 elem.
$\mathfrak{L}_2$ 7 elem.
$\mathfrak{T}_3$ 8 elem.

Pure Corner Zone

DRAWN WITH FIG4TEX



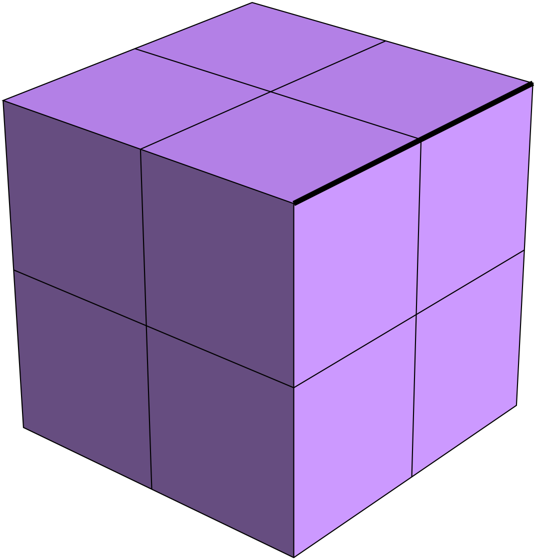


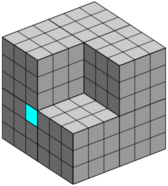
$\mathfrak{L}_0$	7 elem.
$\mathfrak{L}_1$	7 elem.
$\mathfrak{L}_2$	7 elem.
$\mathfrak{L}_3$	7 elem.
$\mathfrak{T}_4$	8 elem.

$\# \, 7p + 8$

Corner Edge Zone (one edge)

DRAWN WITH FIG4TEX

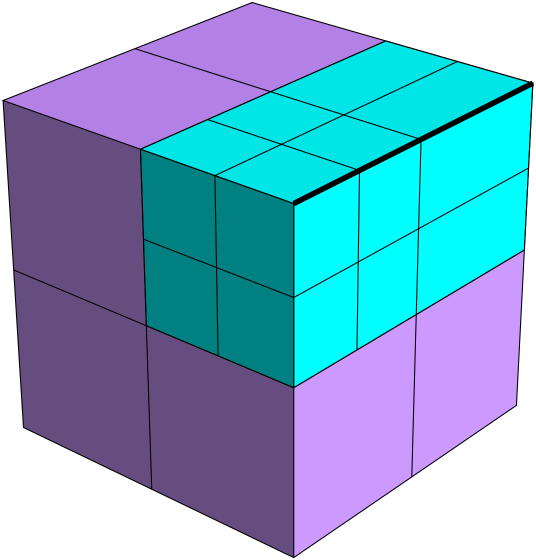


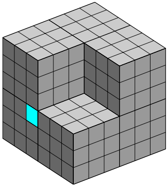


$\mathfrak{T}_0$ 8 elem.

Corner Edge Zone (one edge)

DRAWN WITH FIG4TEX



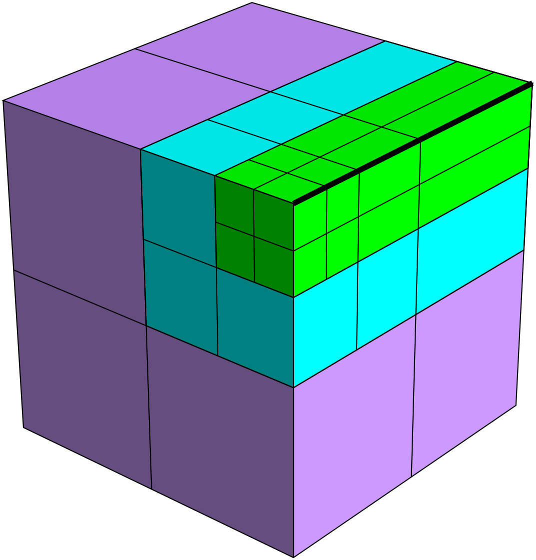


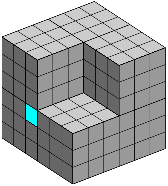
$\mathfrak{L}_0$ 6 elem.

$\mathfrak{T}_1$ 12 elem.

Corner Edge Zone (one edge)

DRAWN WITH FIG4TEX





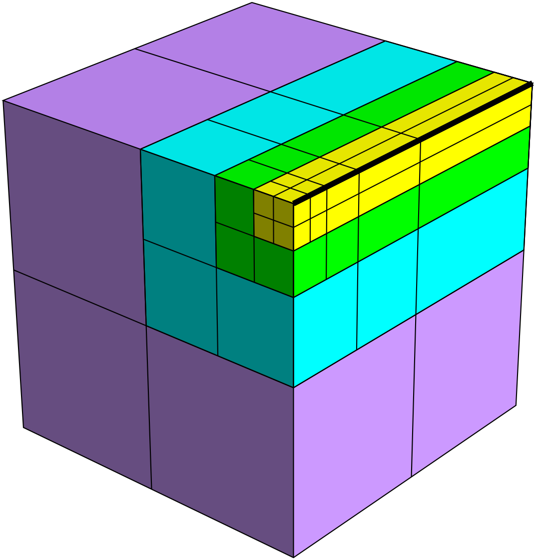
$\mathfrak{L}_0$ 6 elem.

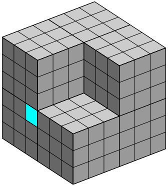
$\mathfrak{L}_1$ 9 elem.

$\mathfrak{T}_2$ 16 elem.

Corner Edge Zone (one edge)

DRAWN WITH FIG4TEX





$\mathfrak{L}_0$ 6 elem.

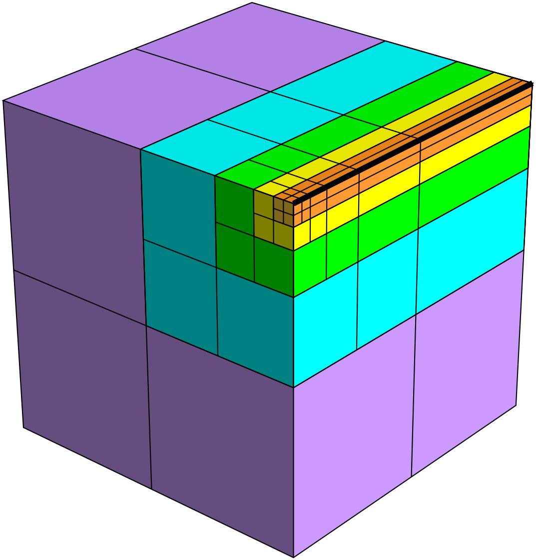
$\mathfrak{L}_1$ 9 elem.

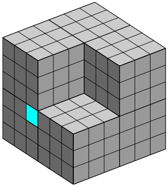
$\mathfrak{L}_2$ 12 elem.

$\mathfrak{T}_3$ 20 elem.

Corner Edge Zone (one edge)

DRAWN WITH FIG4TEX





$\mathfrak{L}_0$ 6 elem.

$\mathfrak{L}_1$ 9 elem.

$\mathfrak{L}_2$ 12 elem.

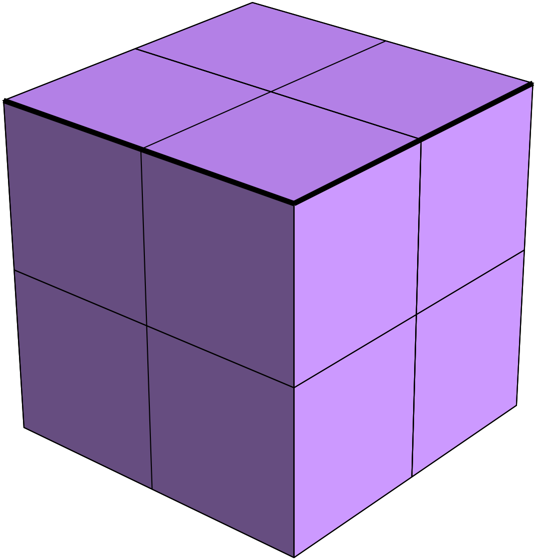
$\mathfrak{L}_3$ 15 elem.

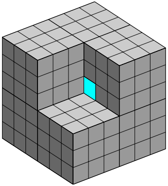
$\mathfrak{T}_4$ 24 elem.

$$\frac{3}{2}p(p+3)+4p+8$$

Corner Edge Zone (two edges)

DRAWN WITH FIG4TEX

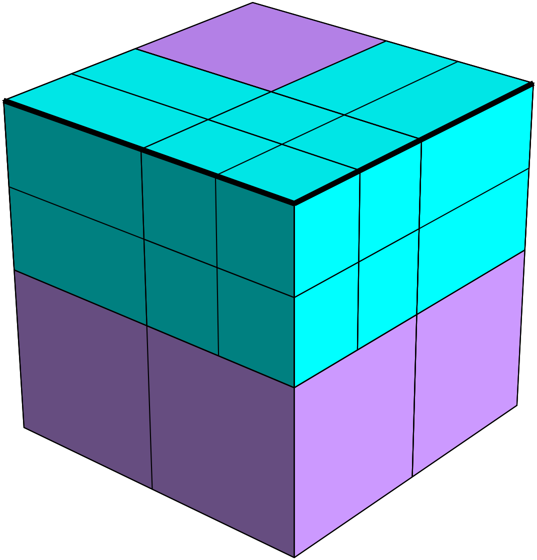


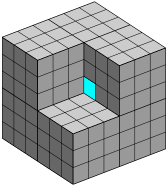


$\mathfrak{T}_0$ 8 elem.

Corner Edge Zone (two edges)

DRAWN WITH FIG4TEX



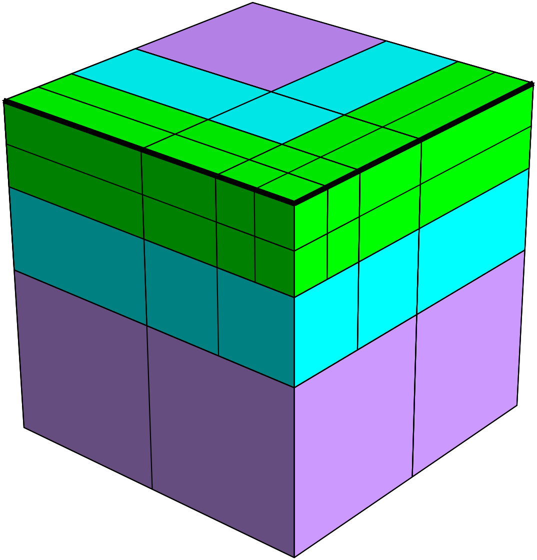


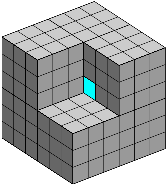
$\mathfrak{L}_0$ 5 elem.

$\mathfrak{T}_1$ 16 elem.

Corner Edge Zone (two edges)

DRAWN WITH FIG4TEX





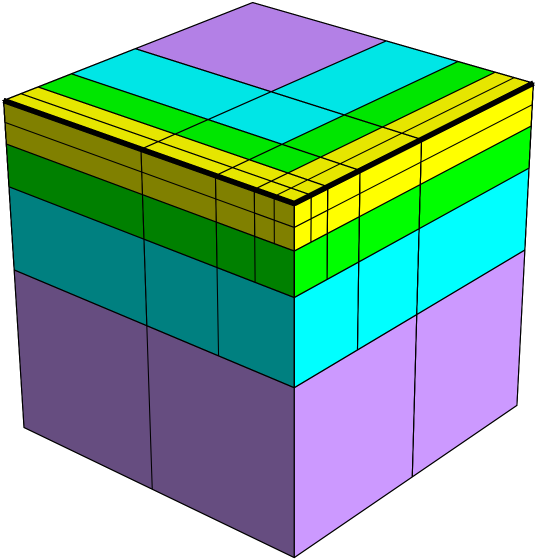
$\mathfrak{L}_0$ 5 elem.

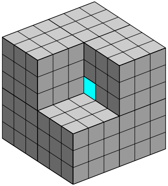
$\mathfrak{L}_1$ 11 elem.

$\mathfrak{T}_2$ 24 elem.

Corner Edge Zone (two edges)

DRAWN WITH FIG4TEX

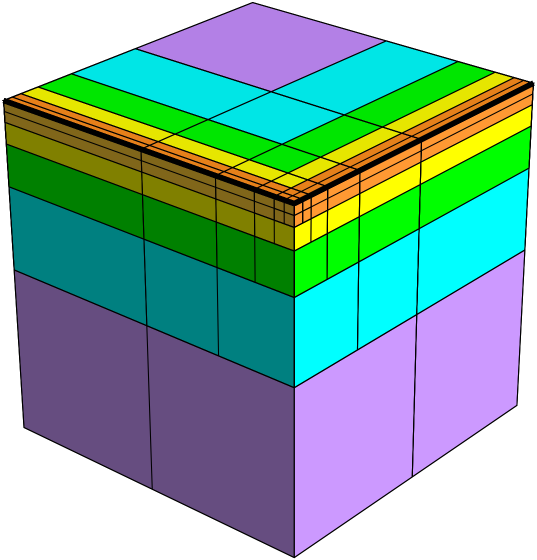


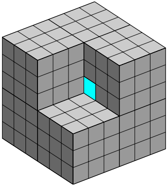


$\mathfrak{L}_0$ 5 elem.
$\mathfrak{L}_1$ 11 elem.
$\mathfrak{L}_2$ 17 elem.
$\mathfrak{T}_3$ 32 elem.

Corner Edge Zone (two edges)

DRAWN WITH FIG4TEX





$\mathfrak{L}_0$ 5 elem.
$\mathfrak{L}_1$ 11 elem.
$\mathfrak{L}_2$ 17 elem.
$\mathfrak{L}_3$ 23 elem.
$\mathfrak{T}_4$ 40 elem.

$3p^2 + 10p + 8$

Finite element spaces

With a mesh family $\mathfrak{M} = (\mathfrak{M}_p)_{p\geq 1}$ on polyhedron Ω at hands, we introduce for any $p \geq 1$ a discrete space $\mathbb{V}_p$.

- **Simple option**, based on (mapped) polynomial spaces of partial degree p: $\mathbb{Q}_p(K) = \mathbb{Q}_p(K_1) \otimes \mathbb{Q}_p(K_2) \otimes \mathbb{Q}_p(K_3)$

$$\begin{cases} \mathbb{V}_p^{\mathrm{DG}} = \{\boldsymbol{v} \in \mathrm{L}^2, \quad \forall K \in \mathfrak{M}_p,\ \boldsymbol{v}\big|_K \in \mathbb{Q}_p(K)\} & \text{DG version} \\ \mathbb{V}_p = \{\boldsymbol{v} \in \mathbb{V}, \quad \forall K \in \mathfrak{M}_p,\ \boldsymbol{v}\big|_K \in \mathbb{Q}_p(K)\} & \text{conforming version} \end{cases}$$

- **More elaborate option**, based on an anisotropic distribution of polynomial degrees $\mathbb{Q}_{\boldsymbol{p}(K)}(K) = \mathbb{Q}_{p^\perp}(K^\perp) \otimes \mathbb{Q}_{p^\parallel}(K^\parallel)$, with function $\boldsymbol{p} : \mathfrak{M}_p \ni K \mapsto (p^\perp, p^\parallel) \in \{0, \ldots, p\}^2$

$$\begin{cases} \mathbb{V}_{\boldsymbol{p}}^{\mathrm{DG}} = \{\boldsymbol{v} \in \mathrm{L}^2, \quad \forall K \in \mathfrak{M}_p,\ \boldsymbol{v}\big|_K \in \mathbb{Q}_{\boldsymbol{p}(K)}(K)\} & \text{DG version} \\ \mathbb{V}_{\boldsymbol{p}} = \{\boldsymbol{v} \in \mathbb{V}, \quad \forall K \in \mathfrak{M}_p,\ \boldsymbol{v}\big|_K \in \mathbb{Q}_{\boldsymbol{p}(K)}(K)\} & \text{conforming version} \end{cases}$$

  Principle:
  - $p^\perp$ increases from 0 to p when the layer index ℓ decreases from p to 0
  - $p^\parallel$ increases from 0 to p when the distance $r_{\boldsymbol{c}}\big|_K$ increases...

Finite element spaces: # of DOF

elements in $\mathfrak{L}_\ell$: $\mathcal{O}(\ell+1)$, $\ell = 0, \ldots, p-1$
elements in $\mathfrak{T}_p$: $\mathcal{O}(p+1)$

Isotropy of degrees

Dimension of $\mathbb{Q}_p$: $(p+1)^3$
Dimension of $\mathbb{V}_p$ (with prefactor):

$$\mathcal{O}\Big(\sum_{\ell=0}^{p}(\ell+1)(p+1)^3\Big) = \mathcal{O}(\frac{p^5}{2})$$

Anisotropy of degrees

Dimension of $\mathbb{Q}_{\boldsymbol{p}}$: $(p^\perp+1)^2(p^\parallel+1)$
Dimension of $\mathbb{V}_{\boldsymbol{p}}$: less than $\dim \mathbb{V}_p$, but greater than

$$\mathcal{O}\Big(\sum_{\ell=0}^{p}(\ell+1)(p+1-\ell)^3\Big) = \mathcal{O}(\frac{p^5}{20})$$

Conclusion: Degree anisotropy provides us with a smaller prefactor. But the power of p is unchanged.

Outline

Factorial estimates

1. $\widehat{\Lambda}$ reference interval $(-1,1)$ and π_0^p orthogonal projection on $\mathbb{Q}_p(\widehat{\Lambda})$.
   The fundamental p-version estimate is

$$\|u-\pi_0^p u\|^2_{\mathrm{L}^2(\widehat{\Lambda})} \le \frac{(p+1-k)!}{(p+1+k)!}\,\|u^{(k)}\|^2_{\mathrm{L}^2(\widehat{\Lambda})} \qquad 0\le k\le p+1$$

2. Let $\mathfrak{O}_p=\mathfrak{L}_0\cup\mathfrak{L}_1\cup\ldots\cup\mathfrak{L}_{p-1}$. Exists $C=C(\mathfrak{M})$:

$$\|\boldsymbol{u}-\Pi_1^p\boldsymbol{u}\|^2_{\underline{\mathbb{X}}^1(\mathfrak{O}_p)} \le C(\mathfrak{M})\,\frac{(p-k)!}{(p+k)!}\,|\boldsymbol{u}|^2_{\mathbb{X}^k(\mathfrak{O}_p)} \qquad 1\le k\le p+1$$

$\|\cdot\|^2_{\underline{\mathbb{X}}^1(\mathfrak{O}_p)}$ is the broken norm $\sum_{K\in\mathfrak{O}_p}\|\cdot\|^2_{\underline{\mathbb{X}}^1(K)}$

Key: The weights w_α are [equivalent to] constants on each $K\in\mathfrak{O}_p$.

3. If $\boldsymbol{u}\in\mathbb{X}^\varpi$, then by definition $|\boldsymbol{u}|_{\mathbb{X}^k(\mathfrak{O}_p)}\le C_{\boldsymbol{u}}^k(k!)$. Hence

$$\|\boldsymbol{u}-\Pi_1^p\boldsymbol{u}\|^2_{\underline{\mathbb{X}}^1(\mathfrak{O}_p)} \le C^{2k}\,(k!)^2\,\frac{(p-k)!}{(p+k)!} \qquad 1\le k\le p+1$$

Exponential estimate

By Stirling's formula $n! \simeq n^n e^{-n}\sqrt{2\pi n}$ there exists $\delta > 0$ such that

$$C^{2k}(k!)^2 \frac{(p-k)!}{(p+k)!} \leq \delta^{2k} \frac{(p-k)^{p-k}\, k^k\, k^k}{(p+k)^{p+k}} = \left(\frac{p-k}{p+k}\right)^{p-k} \left(\frac{\delta k}{p+k}\right)^{2k}.$$

Choosing $k = p/(\delta+1)$, we obtain

$$C^{2k}(k!)^2 \frac{(p-k)!}{(p+k)!} \leq \left(\frac{\delta k}{(\delta+2)k}\right)^{p-k} \left(\frac{\delta k}{(\delta+2)k}\right)^{2k} = \left(\frac{\delta}{\delta+2}\right)^{p(1+\frac{1}{\delta+1})}.$$

With $b := -\log\left(\frac{\delta}{\delta+2}\right)^{(1+\frac{1}{\delta+1})/2}$ we have proved, for $\ell = 0, 1$

Lemma

$$\|\boldsymbol{u} - \Pi_1^p \boldsymbol{u}\|_{\mathbb{X}^1(\mathfrak{O}_p)} \leq C\, e^{-bp} \quad \text{with} \quad b > 0 \quad \text{independent of } p.$$

Note that $p = \sqrt[5]{N}$ with $N = \#\mathrm{DOF}$.

On the way

We have [almost] obtained elementwise H^1 estimates for a best approximation of $\boldsymbol{u}$ in $\mathbb{V}_p^{\mathsf{DG}}$: It remains to estimate in terminal layers $\mathfrak{T}_p$.

1. If enough Dirichlet conditions are imposed, one can take homogeneous weights in $\mathbb{X}^{\varpi}$, and it suffices to use the zero interpolant in $\mathfrak{T}_p$.
2. If not, one has to define a special $\mathbb{Q}_1$ quasi-interpolant in $K \in \mathfrak{T}_p$.

Then, to end the task there are two options

1. Construct a suitable DG (Discontinuous Galerkin) method for the discretization of problem $\mathbb{P}\boldsymbol{u} = \boldsymbol{f}$ in $\mathbb{V}_p^{\mathsf{DG}}$
2. Convert elementwise H^1 estimates in full H^1-estimates by suitable patchwise lifting of traces, so that to keep the exponential best approximation. Then apply Céa Lemma and obtain exponential convergence of Galerkin projections.

Discontinuous Galerkin

DG with Interior Penalty for scalar $\mathbb{P} = \Delta$

- Mesh $\mathfrak{M}_p \in \mathfrak{M}$ and $\mathfrak{F}_p$ set of faces F of elements $K \in \mathfrak{M}_p$.
- $\boldsymbol{n}_K$ outward normal for $K \in \mathfrak{M}_p$
- $\{\boldsymbol{w}\}\big|_F = \frac{1}{2}(\boldsymbol{w}\big|_K + \boldsymbol{w}\big|_{K'})$ average of vector $\boldsymbol{w}$ on $F = K \cap K'$
- $[w]\big|_F = w\big|_K \boldsymbol{n}_K + w\big|_{K'} \boldsymbol{n}_{K'}$ jump of scalar w on $F = K \cap K'$

Set $\alpha\big|_F = \dfrac{(\max_{K',K''} p_K^{\perp})^2}{\min_{K',K''} h_K^{\perp}}$ on $F = K' \cap K''$, with $h_K^{\perp}$ the smallest size of K

For $u, v \in \mathbb{V}_p^{\mathrm{DG}}$

$$a^{\mathrm{DG}}(u,v) = \underline{a}(u,v) - \int_{\mathfrak{F}_p} \{\nabla u\} \cdot [v] + \theta \int_{\mathfrak{F}_p} \{\nabla v\} \cdot [u] + \gamma \int_{\mathfrak{F}_p} \alpha\, [u] \cdot [v]$$

$\mathscr{D}$ Find $\boldsymbol{u}_p^{\mathrm{DG}} \in \mathbb{V}_p^{\mathrm{DG}}, \quad \forall \boldsymbol{v} \in \mathbb{V}_p^{\mathrm{DG}}, \quad a^{\mathrm{DG}}(\boldsymbol{u}_p^{\mathrm{DG}}, \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle$

Discontinuous Galerkin: Exponential convergence

Theorem [Schötzau–Schwab–Wihler]

Let $\boldsymbol{f} \in \mathbb{Y}^{\varpi}$ (space associated with anisotropic weights with exponents $\beta_{\boldsymbol{e}}, \beta_{\boldsymbol{c}} < -1$, cf Theorem A). For $\theta = \pm 1$ and $\gamma > 0$ large enough, $\boldsymbol{u}_p^{\mathrm{DG}}$ converges exponentially to $\boldsymbol{u}$

$$\|\boldsymbol{u} - \boldsymbol{u}_p^{\mathrm{DG}}\|_{\underline{H}^1(\Omega)} \leq Ce^{-bp}, \quad b > 0, \text{ independent from } p$$

- D. Schötzau, C. Schwab, and T. P. Wihler, *hp-dGFEM for second-order elliptic problems in polyhedra I: Stability on geometric meshes*, SIAM J. Numer. Anal., **51** (2013) 1610–1633
- ———, *hp-DGFEM for second order elliptic problems in polyhedra II: Exponential convergence*, SIAM J. Numer. Anal., **51** (2013) 2005–2035
- ———, *hp-dGFEM for second-order mixed elliptic problems in polyhedra*, Math. Comp., **85** (2016) 1051–1083

Conforming Approximation: Exponential convergence

Discrete problems

$\mathscr{C}$

$$\text{Find } \boldsymbol{u}_p \in \mathbb{V}_p, \quad \forall \boldsymbol{v} \in \mathbb{V}_p, \quad a(\boldsymbol{u}_p, \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle$$

Theorem [Schötzau–Schwab]

Let $\boldsymbol{f} \in \mathbb{Y}^{\varpi}$ (space associated with anisotropic weights with exponents $\beta_{\boldsymbol{e}}, \beta_{\boldsymbol{c}} < -1$, cf Theorem A). Then $\boldsymbol{u}_p$ converges exponentially to $\boldsymbol{u}$

$$\|\boldsymbol{u} - \boldsymbol{u}_p\|_{H^1(\Omega)} \leq Ce^{-bp}, \quad b > 0, \text{ independent from } p$$

- D. SCHÖTZAU AND C. SCHWAB, *Exponential convergence for hp-version and spectral finite element methods for elliptic problems in polyhedra*, Math. Models Methods Appl. Sci., **25** (2015) 1617–1661
- ———, *Exponential convergence of hp-FEM for elliptic problems in polyhedra: Mixed boundary conditions and anisotropic polynomial degrees*, SAM Report, ETH Zürich, **2016-05** (2016)

Conclusion

30 years ago, the roots of the *h*–*p* method in 1D

- W. Gui and I. Babuška, *The h, p and h-p versions of the finite element method in 1 dimension. I. The error analysis of the p-version*, Numer. Math., **49** (1986) 577–612.
- ———, *The h, p and h-p versions of the finite element method in 1 dimension. II. The error analysis of the h- and h-p versions*, Numer. Math., **49** (1986) 613–657.
- ———, *The h, p and h-p versions of the finite element method in 1 dimension. III. The adaptive h-p version*, Numer. Math., **49** (1986) 659–683.

Today, the *h*–*p* analysis is essentially achieved in 3D

Essentially?

1. Generalization to elliptic systems with variable coefficients
2. Maxwell

1. needs coercivity and technical skills
2. needs new ideas?