Approximation of the LBB constant on corner domains

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Collaboration with Monique Dauge, Michel Crouzeix, Christine Bernardi, Vivette Girault, Yvon Lafranche

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Publications

- M. Dauge, C. Bernardi, M. Costabel, V. Girault
  On Friedrichs constant and Horgan-Payne angle for LBB condition
  Monogr. Mat. Garcia Galdeano, 39 (2014), 87–100.
- M. Costabel, M. Dauge
  On the inequalities of Babuška–Aziz, Friedrichs and Horgan–Payne
  Arch. Rational Mech. Anal. 217(3) (2015), 873–898.
- M. Costabel, M. Crouzeix, M. Dauge, Y. Lafranche
  The inf-sup constant for the divergence on corner domains
  Numer. Methods Partial Differential Equations 31(2) (2015), 439–458.
- M. Costabel
  Inequalities of Babuška-Aziz and Friedrichs-Velte for differential forms.
  arXiv:1507.08464, to appear in Operator Theory, Advances and Applications
- C. Bernardi, M. Costabel, M. Dauge, V. Girault
  Continuity properties of the inf-sup constant for the divergence
  SIAM J. Math. Anal., 48 (2016), pp. 1250–1271.

The LBB constant or inf-sup constant: Definition

- Ω bounded domain in $\mathbb{R}^d$ ($d \geq 1$). No regularity assumptions.

The inf-sup constant of Ω

$$\beta(\Omega) = \inf_{q \in L^2_\circ(\Omega)} \sup_{\boldsymbol{v} \in H^1_0(\Omega)^d} \frac{\displaystyle\int_\Omega \operatorname{div} \boldsymbol{v}\, q}{|\boldsymbol{v}|_1 \, \|q\|_0}$$

- $L^2(\Omega)$ space of square integrable functions q on Ω. Norm $\|q\|_0$
- $H^1(\Omega)$ Sobolev space of $v \in L^2(\Omega)$ with gradient $\nabla v \in L^2(\Omega)^d$
- $L^2_\circ(\Omega)$ subspace of $q \in L^2(\Omega)$ with $\int_\Omega q = 0$.
- $H^1_0(\Omega)$ closure in $H^1(\Omega)$ of $C^\infty_0(\Omega)$ (zero trace on $\partial\Omega$)
  (Semi-)Norm $|v|_1 = \|\nabla v\|_0$ equivalent to norm $\|v\|_{H^1(\Omega)}$
- $0 < \beta(\Omega) \leq 1$.
- $\beta(\Omega)$ is invariant with respect to translations, rotations, dilations.
- We will often talk about $\boxed{\sigma(\Omega) = \beta(\Omega)^2}$ instead of $\beta(\Omega)$.

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Main motivation: LBB condition and the Stokes system

The inf-sup condition or LBB condition

$$\beta(\Omega) > 0$$

The complete Stokes system

Find $u \in H^1_0(\Omega)^d$, $p \in L^2_\circ(\Omega)$:

$$\begin{aligned} -\Delta u + \nabla p &= f \qquad \text{in } \Omega \\ \operatorname{div} u &= g \qquad \text{in } \Omega \end{aligned}$$

Theorem

The mapping $(u, p) \mapsto (f, g) : H^1_0(\Omega)^d \times L^2_\circ(\Omega) \to H^{-1}(\Omega)^d \times L^2_\circ(\Omega)$ is an isomorphism if and only if $\beta(\Omega) > 0$.

Proved (more or less) by L. Cattabriga (1961) for smooth domains
Standard reference:
V. Girault, A. Raviart: Finite Element Methods for Navier-Stokes Equations, Springer 1986

The inf-sup condition or LBB condition

$$\beta(\Omega) > 0$$

Classical:
This is true for bounded Lipschitz domains.
Not true for domains with outward cusps.

Main motivation: LBB condition and the Stokes system

The inf-sup condition or LBB condition

$$\beta(\Omega) > 0$$

Now known [Acosta et al, 2006–2016]: For bounded domains, this is basically equivalent to Ω being a John domain.
(More general than Lipschitz ▸ Digression: John domains).

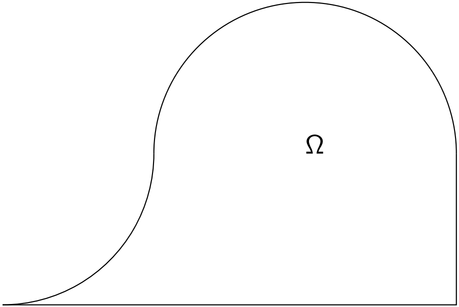

Figure: Not a John domain: Outward cusp, $\beta(\Omega) = 0$ [Friedrichs 1937]

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LBB

From Wikipedia, the free encyclopedia

LBB may stand for:

- *Lactobacillus delbrueckii subsp. bulgaricus*, a bacterium used in the production of yogurt.
- Lubbock Preston Smith International Airport, the IATA code
- Little Brown Bird - birdwatchers acronym for indistinct or unknown small dark bird
- Liberty Bible dataBase (.lbb file extension)
- Ladyzhenskaya-Babuska-Brezzi conditions for stability in mixed finite element analysis

The LBB condition

Since ∼1980, the inf-sup condition for the divergence is often called LBB condition, after

- **L**adyzhenskaya Added by J. T. Oden ca 1980, on suggestion by J.-L. Lions
- **B**abuška [Babuška 1971-73]
- **B**rezzi [Brezzi 1974]

Discrete LBB condition

Let $X_N \subset X = H_0^1(\Omega)^d$ and $M_N \subset M = L^2_\circ(\Omega)$ be sequences of closed subspaces. Define

$$\beta_N = \inf_{q \in M_N} \sup_{v \in X_N} \frac{\int_\Omega \operatorname{div} v \, q}{|v|_1 \, \|q\|_0}$$

The uniform discrete inf-sup condition

$$\beta_N(\Omega) \ge \beta_* > 0 \qquad \forall N$$

is also simply called Babuška-Brezzi condition or LBB condition.

Application

Stability and convergence of finite element methods for the Stokes system.

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Why is it important to know the value of $\beta(\Omega)$?

The Stokes system of incompressible fluid dynamics for $\boldsymbol{u} \in H_0^1(\Omega)^d$, $p \in L^2_\circ(\Omega)$

$$\boxed{\begin{aligned} -\Delta \boldsymbol{u} + \nabla p &= \mathbf{f} \qquad \text{in } \Omega \\ \operatorname{div} \boldsymbol{u} &= 0 \qquad \text{in } \Omega \end{aligned}}$$

has the variational form

$$\begin{aligned} \langle \nabla \boldsymbol{u}, \nabla \boldsymbol{v} \rangle - \langle \operatorname{div} \boldsymbol{v}, p \rangle &= \langle \boldsymbol{v}, \mathbf{f} \rangle \quad && \forall\, \boldsymbol{v} \in H_0^1(\Omega)^d \\ \langle \operatorname{div} \boldsymbol{u}, q \rangle &= 0 && \forall\, q \in L^2_\circ(\Omega) \end{aligned}$$

Pressure Stability for the Stokes problem

$$|\boldsymbol{u}|_1 \le |\mathbf{f}|_{-1}$$

$$\|p\|_0 \le \frac{1}{\beta(\Omega)} |\mathbf{f}|_{-1}$$

Also: Error reduction factor for iterative algorithms such as Uzawa.

Outline

1. History of this circle of ideas
2. Review of basic properties
3. Approximation problems
4. Corner domains

Time frame: Cosserat EVP

1898-1901	E.&F. Cosserat: 9 papers in CR Acad Sci Paris
1924	L. Lichtenstein: a boundary integral equation method
1967	V. Maz'ya – S. Mikhlin: "On the Cosserat spectrum. . . "
1973	S. Mikhlin: "The spectrum of an operator pencil. . . "
1993-1999	A. Kozhevnikov: Cosserat EV distribution, 100 years of history

Time frame: Cosserat EVP, Inf-Sup

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1994-2000	E. Chizhonkov – M. Olshanskiĭ: "On the optimal constant in the inf-
1999-2009	G. Stoyan: discrete inequalities
2000-2004	S. Zsuppán: conformal mappings
2006-	C. Simader – W. v. Wahl – S. Weyers: L^q, unbounded domains
2006-	G. Acosta – R.G. Durán – M.A. Muschietti: John domains
2000-2016	C. Bernardi, M. Co., M. Dauge, V. Girault . . .

Ball in $\mathbb{R}^d$: $\sigma(\Omega) = \frac{1}{d}$ [Ellipsoids in 3D: E.&F. Cosserat 1898]

In 2D:

Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} < 1$, $a < b$: $\sigma(\Omega) = \frac{a^2}{a^2 + b^2}$

Some other simple 2D domains, for example:

Annulus $a < r < 1$: $\sigma(\Omega) = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{1-a^2}{1+a^2}\frac{1}{\log 1/a}}$

[Chizhonkov-Olshanskii 2000]

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[Chizhonkov-Olshanskii 2000]

The inf-sup Constant: Known Values, Example

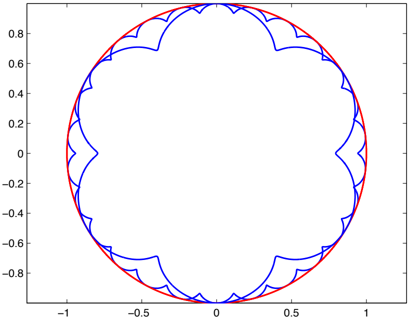
An example from [Zsuppán 2004] "Epitrochoid"

Conformal mapping $g_{m,c} : \Omega = \{z \in \mathbb{C} \mid |z| < 1\} \to \Omega_{m,c}$

$$g_{m,c}(z) = \frac{z - \frac{c}{m} z^m}{1 + \frac{c}{m}}$$

[Zsuppán 2004]

For $0 < c < 1$ and $m \in \mathbb{N}$ odd: $\beta(\Omega_{m,c})^2 = \frac{1}{2}(1 - \frac{m+1}{2m} c)$



Showing:

$\Omega_{m,c}$ for $c = 0.8$
$m = 7$ and $m = 27$

Observation: Non-convergence

As $m \to \infty$: $g_{m,c}(z) \to z$

$\Omega_{m,c} \to \Omega$, but

$\beta(\Omega_{m,c})^2 \to \frac{1}{2} - \frac{c}{4} \neq \frac{1}{2} = \beta(\Omega)^2$

Optimality

Known: $\beta(\Omega) \le \frac{1}{\sqrt{2}}$ for any bounded domain.

Hence: For $d = 2$, the ball is optimal: β is maximal.

Unknown: For $d \ge 3$, is the ball optimal? $\beta(\Omega) \le \frac{1}{\sqrt{d}}$?

The square $\Omega = (0,1) \times (0,1) =: \square \subset \mathbb{R}^2$

$\beta(\square)$ is still unknown !

Current Conjecture

$$\sigma(\square) = \frac{1}{2} - \frac{1}{\pi} \approx 0.18169\ldots \qquad (\to\ \beta(\square) = \sqrt{\frac{1}{2} - \frac{1}{\pi}} \approx 0.42625)$$

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Related inequalities

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Def: $$J(q) = \sup_{\boldsymbol{v}\in H^1_0(\Omega)^d} \frac{\big\langle \operatorname{div} \boldsymbol{v}, q \big\rangle}{\big|\boldsymbol{v}\big|_1} \quad \Big(= |\nabla q|_{-1}\ ,\ \text{dual norm}\ \Big)$$

Lemma: sup = max

$$J(q) = \frac{\big\langle \operatorname{div} \boldsymbol{w}(q), q \big\rangle}{\big|\boldsymbol{w}(q)\big|_1} = \big|\boldsymbol{w}(q)\big|_1$$

where $\boldsymbol{w}(q) \in H^1_0(\Omega)^d$ is the solution $\boldsymbol{w}$ of the vector Dirichlet problem $\Delta \boldsymbol{w} = \nabla q$, or in variational form

$$\langle \nabla \boldsymbol{w}, \nabla \boldsymbol{v} \rangle = \langle \operatorname{div} \boldsymbol{v}, q \rangle \quad \forall\, \boldsymbol{v} \in H^1_0(\Omega)^d$$

We write

$$\boldsymbol{w}(q) = \Delta^{-1} \nabla q$$

Recall the Stokes system of incompressible fluid dynamics for $\boldsymbol{u} \in H^1_0(\Omega)^d$, $p \in L^2_\circ(\Omega)$

$$\boxed{\begin{aligned} -\Delta \boldsymbol{u} + \nabla p = \mathbf{f} \quad & \text{in } \Omega \\ \operatorname{div} \boldsymbol{u} = 0 \quad & \text{in } \Omega \end{aligned}}$$

Definition

The Schur complement operator $\mathscr{S}$ for the Stokes system is

$$\mathscr{S} = \operatorname{div} \Delta^{-1} \nabla : \quad L^2_\circ \xrightarrow{\nabla} \boldsymbol{H}^{-1} \xrightarrow{\Delta^{-1}} \boldsymbol{H}^1_0 \xrightarrow{\operatorname{div}} L^2_\circ$$

$\mathscr{S}$ is a bounded positive selfadjoint operator in $L^2_\circ(\Omega)$.

Observation

Define

$$\sigma(\Omega) = \min \mathrm{Sp}(\mathscr{S})$$

Then

$$\sigma(\Omega) = \beta(\Omega)^2$$

Proof: $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ is the Riesz isometry. For $q \in L^2_\circ(\Omega)$:

$$\begin{aligned}\langle \mathscr{S} q, q \rangle &= \langle \operatorname{div} \Delta^{-1} \nabla q, q \rangle \\ &= \langle -\Delta^{-1} \nabla q, \nabla q \rangle \\ &= |\nabla q|^2_{-1} \\ &= J(q)^2\end{aligned}$$

$$\sigma(\Omega) = \inf_{q \in L^2_\circ(\Omega)} \frac{\langle \mathscr{S} q, q \rangle}{\langle q, q \rangle} = \beta(\Omega)^2$$

Schur complement and Cosserat eigenvalue problem

A well known lemma

Let $A : X \to Y$ and $B : Y \to X$ be linear operators. Then

$$Sp(AB) \setminus \{0\} \equiv Sp(BA) \setminus \{0\}\,.$$

Recall $\mathscr{S} = \operatorname{div} \Delta^{-1} \nabla$.

Corollary

The eigenvalue problem for the Schur complement of the Stokes system

$$\mathscr{S} p = \sigma p \quad \text{in } L^2_\circ(\Omega)$$

is, for $\sigma \neq 0$, equivalent to the eigenvalue problem

$$\Delta^{-1} \nabla \operatorname{div} u = \sigma u \quad \text{in } H^1_0(\Omega)^d$$

which is the same as

$$\sigma \Delta u = \nabla \operatorname{div} u \quad \text{in } H^{-1}_0(\Omega)^d\,.$$

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A general approximation result: Upper Semicontinuity

Theorem [BCDG 2016]

Let $X_N \subset X = H_0^1(\Omega)^d$ and $M_N \subset M = L^2_\circ(\Omega)$ be sequences of closed subspaces.
If $(M_N)_N$ is asymptotically dense in M, then

$$\limsup_{N\to\infty} \beta_N \le \beta(\Omega)$$

Proof: Recall the definition of $J(q)$ and define similarly

$$J_N(q) = \sup_{\boldsymbol{v}\in X_N} \frac{\langle \operatorname{div} \boldsymbol{v}, q\rangle}{|\boldsymbol{v}|_1}, \quad \text{so that } \beta(\Omega) = \inf_{q\in M} \frac{J(q)}{\|q\|_0} \text{ and } \beta_N = \inf_{q_N\in M_N} \frac{J_N(q_N)}{\|q_N\|_0}$$

Now for $q \in M$ given, choose $q_N \in M_N$ so that $q_N \to q$ in $L^2_\circ(\Omega)$. Then one has

$$\beta_N \le \frac{J_N(q_N)}{\|q_N\|_0} \le \frac{J(q_N)}{\|q_N\|_0} \to \frac{J(q)}{\|q\|_0}$$

Now assume that $\beta_N \to \beta_\infty$. Then $\beta_\infty \le \frac{J(q)}{\|q\|_0}$ for any $q \in M$ and, taking the inf,

finally $\beta_\infty \le \beta(\Omega)$.

A simple case where convergence holds

In general, one can have $\boxed{\beta_N \le \beta(\Omega)}$ or $\boxed{\beta_N \ge \beta(\Omega)}$.
No general criterion known.

Exception

$$\text{If } X_N = \Delta^{-1}\nabla M_N, \quad \text{then } \beta_N \ge \beta(\Omega).$$

Thus, if one knows a basis $(q_n)_{n\in\mathbb{N}}$ of $L^2_\circ(\Omega)$ for which the Dirichlet problem for $\boldsymbol{w}_n \in H^1_0(\Omega)^d$

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In other words, this is a Galerkin eigenvalue approximation of the exact Schur complement operator $\mathscr{S}$. In general cases, Δ^{-1} will have to be approximated, too.

A simple case where convergence holds

In general, one can have $\boxed{\beta_N \leq \beta(\Omega)}$ or $\boxed{\beta_N \geq \beta(\Omega)}$.
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Example: The rectangle

[M. Gaultier, M. Lezaun 1996] Let $\Omega = (0,a) \times (0,b)$. Then

$$q_{km}(x,y) = \cos(\kappa x)\cos(\mu y), \quad \kappa = \frac{k\pi}{a}, \mu = \frac{m\pi}{b}, \; k,m \geq 0, k+m > 0$$

defines an orthogonal basis of $L^2_\circ(\Omega)$. The Schur complement operator $\mathscr{S} = \operatorname{div}\Delta^{-1}\nabla = \partial_x\Delta^{-1}\partial_x + \partial_y\Delta^{-1}\partial_y$ can be computed analytically by solving 1D Dirichlet problems on $(0,a)$ and $(0,b)$

$$\mathscr{S} q_{km} = -\kappa^2 \cos\kappa x\,(\partial_y^2 - \kappa^2)^{-1}[\cos\mu y] - \mu^2\cos\mu y\,(\partial_x^2 - \mu^2)^{-1}[\cos\kappa x]$$

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Numerical results. — We have performed a few numerical tests. Let K be a positive integer. We have computed an approximate value of the smallest eigenvalue α_K of the matrix A_K by means of the power of Mises [2, pp. 226-227]. We stopped this calculatioin when the relative error was less than 10^{-9}. We have ascertained that sequence $\{\alpha_K\}_{K>0}$ converges quickly.

The above mentioned values of the constant $P(\Omega)^{-1}$ have been rounded up to the 3-th decimal place.

$$L=1\,,\quad \ell=1: P(\Omega)^{-1} = 0.226$$
$$L=2\,,\quad \ell=1: P(\Omega)^{-1} = 0.151$$
$$L=4\,,\quad \ell=1: P(\Omega)^{-1} = 0.047\,.$$

$(L,\ell) = (a,b)$
$K = N$
$P(\Omega)^{-1} = \sigma(\Omega) = \beta(\Omega)^2$

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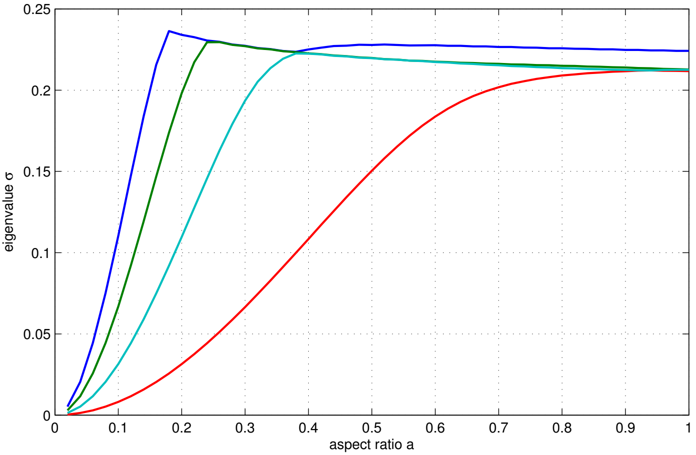
The rectangle: First 4 Cosserat eigenvalues, Gaultier-Lezaun method



Algorithm for computing the discrete LBB constant

Cosserat eigenvalue problem

Find $\boldsymbol{u} \in H_0^1(\Omega)^d \setminus \{0\}$, $\sigma \in \mathbb{C}$ such that

$$\sigma \Delta \boldsymbol{u} - \nabla \operatorname{div} \boldsymbol{u} = 0\,.$$

The Cosserat eigenvalue problem is a Stokes eigenvalue problem

Find $\boldsymbol{u} \in H_0^1(\Omega)^d$, $p \in L^2_\circ(\Omega) \setminus \{0\}$, $\sigma \in \mathbb{C}$:

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Variational form: Find $\boldsymbol{u} \in X$, $p \in M$, $\sigma \in \mathbb{C}$:

$$\begin{aligned} \langle \nabla \boldsymbol{u}, \nabla \boldsymbol{v} \rangle - \langle \operatorname{div} \boldsymbol{v}, p \rangle &= 0 && \forall \boldsymbol{v} \in X \\ \langle \operatorname{div} \boldsymbol{u}, q \rangle &= \sigma \langle p, q \rangle && \forall q \in M \end{aligned}$$

Galerkin discretization: $X \curvearrowright X_N$, $M \curvearrowright M_N \implies \min \sigma = \beta_N^2$

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Remarks on Two Stokes eigenvalue problems

Stokes eigenvalue problem, first kind

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1st kind:
- Appears in dynamic problems (time stepping, Laplace transform)
- Elliptic eigenvalue problem, compact resolvent,
- Known conditions for convergence of numerical algorithms (discrete LBB condition...)

2nd kind:
- Provides the (continuous and discrete) inf-sup constant: $\beta_N^2 = \min \sigma_{X_N, M_N}$
- Not an elliptic eigenvalue problem
- Not covered by any general theory of numerical approximation of eigenvalue problems

- Both eigenvalue problems are discretized with the same code!
- Standard code available: Stokes + matrix eigenvalue problem

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A general approximation result: Convergence

We now assume two conditions for the function spaces, with some s satisfying $0 < s < \frac{1}{2}$

1. An inverse inequality for M_N

$$\forall q \in M_N : \quad \|q\|_s \leq \eta_{N,s} \|q\|_0$$

2. An approximation property for X_N

$$\forall \boldsymbol{u} \in H^{1+s}(\Omega) \cap H_0^1(\Omega) : \quad \inf_{\boldsymbol{v} \in X_N} |\boldsymbol{u} - \boldsymbol{v}|_1 \leq \varepsilon_{N,s} \|\boldsymbol{u}\|_{1+s}$$

Theorem [BCDG 2016]

Let Ω have H^{1+s} regularity for the Dirichlet problem for some $0 < s < \frac{1}{2}$, with an estimate

$$\|\Delta^{-1}\|_{H^{-1+s} \to H^{1+s}} \leq C_s$$

and let conditions 1 and 2 be satisfied. Then

$$\beta_N \geq \beta(\Omega) - C_s \eta_{N,s} \varepsilon_{N,s} .$$

In particular, if $\eta_{N,s}\varepsilon_{N,s} \to 0$ and M_N is asymptotically dense, then

$$\boxed{\lim_{N \to \infty} \beta_N = \beta(\Omega) .}$$

Proof. For $q \in M_N$, let $\boldsymbol{w} = \Delta^{-1}\nabla q$ and $\boldsymbol{w}_N = \Delta_N^{-1}\nabla q$. Then

$$\left|\boldsymbol{w} - \boldsymbol{w}_N\right|_1 = \inf_{\boldsymbol{v}\in X_N} \left|\boldsymbol{w} - \boldsymbol{v}\right|_1$$

hence

$$\begin{aligned} J(q) - J_N(q) &= \left|\boldsymbol{w}\right|_1 - \left|\boldsymbol{w}_N\right|_1 \leq \left|\boldsymbol{w} - \boldsymbol{w}_N\right|_1 \\ &\leq \varepsilon_{N,s}\left\|\boldsymbol{w}\right\|_{1+s} \\ &\leq C_s\varepsilon_{N,s}\left\|\nabla q\right\|_{-1+s} \leq C_s\varepsilon_{N,s}\left\|q\right\|_s \\ &\leq \eta_{N,s}C_s\varepsilon_{N,s}\left\|q\right\|_0 \end{aligned}$$

For $\left\|q\right\|_0 = 1$:

$$\begin{aligned} \beta(\Omega) \leq J(q) &= J_N(q) + (J(q) - J_N(q)) \\ &\leq J_N(q) + \eta_{N,s}C_s\varepsilon_{N,s} \end{aligned}$$

Minimizing over $q \in M_N$ gives the result

$$\beta(\Omega) \leq \beta_N + \eta_{N,s}C_s\varepsilon_{N,s}\,.$$

Consequences for finite element approximations

A. *h* version of the FEM

Let X_N and M_N be conforming finite element spaces defined on quasi-regular meshes with meshwidths h_{X_N} and h_{M_N}. Direct and inverse estimates are well known:

$$\eta_{N,s} = C\, h_{M_N}^{-s}\,; \qquad \varepsilon_{N,s} = C\, h_{X_N}^{s} \quad (\text{any } s \in (0, \tfrac{1}{2}))$$

Corollary, *h* version

If $\lim\limits_{N\to\infty} \dfrac{h_{X_N}}{h_{M_N}} = 0$, then $\lim\limits_{N\to\infty} \beta_N = \beta(\Omega)$.

B. *p* version of the FEM

Let X_N and M_N be finite element spaces of degrees p_{X_N} and p_{M_N} on fixed meshes. The known direct and inverse estimates are

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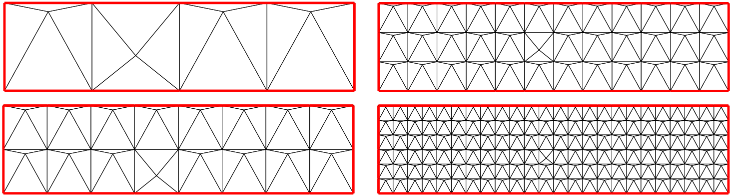
A. *h* version: Yes, sort of

Theorem [BCDG2016]

(iii) Given a polygon Ω, there exists $\beta_0 > 0$ such that for arbitrary $\beta_\infty \in (0, \beta_0)$ one can construct a finite element method with $h_{X_N} = h_{M_N}$ for which

$$\lim_{N\to\infty} \beta_N = \beta_\infty$$

Exemple: Scott-Vogelius P_4-P_3^{dc} elements on "near-singular meshes"



Are these conditions necessary?

B. *p* version: Probably not

Numerical observations:

1. $p_{X_N} \sim p_{M_N} + k$: No convergence
   (Known [Bernardi-Maday 1999]: $\beta_N \sim p^{-1/2} \to 0$)
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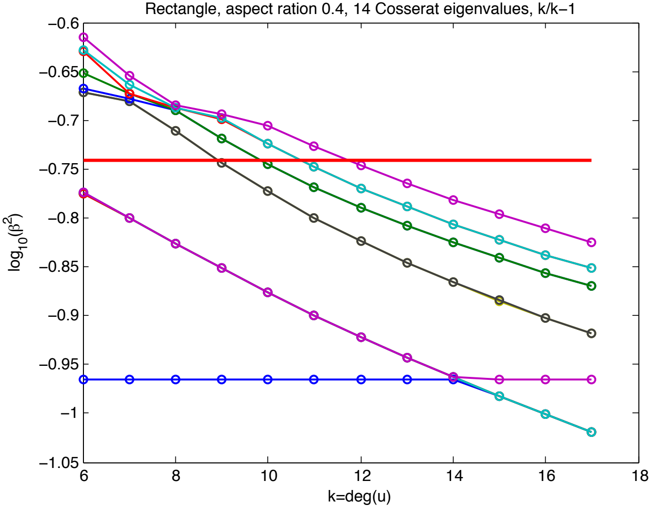
Conjecture for the *p* version

As soon as the method is inf-sup stable, $\lim_{N\to\infty} \beta_N = \beta(\Omega)$

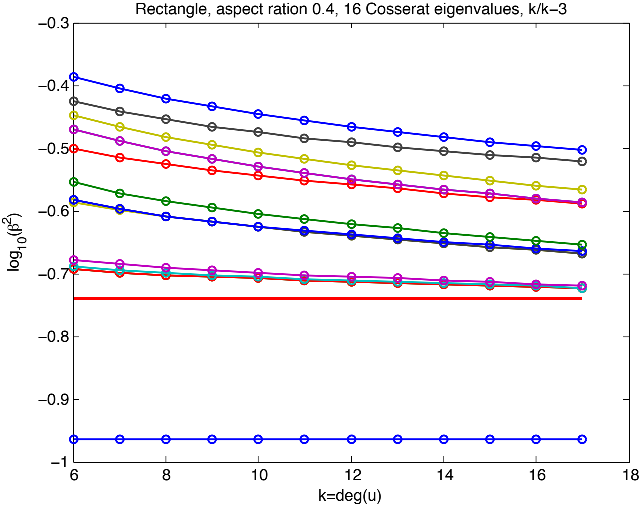
Rectangle, aspect ratio 0.4

First 13 Cosserat eigenvalues, (Q_k, Q_{k-1}) "Taylor-Hood"



Rectangle, aspect ratio 0.4

First 13 Cosserat eigenvalues, (Q_k, Q_{k-3})



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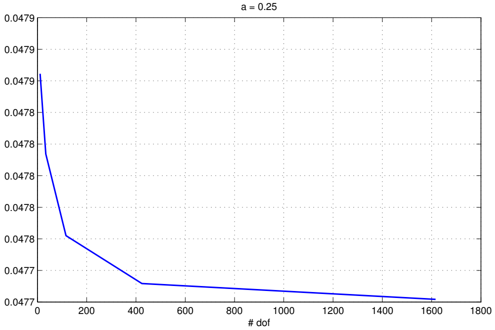
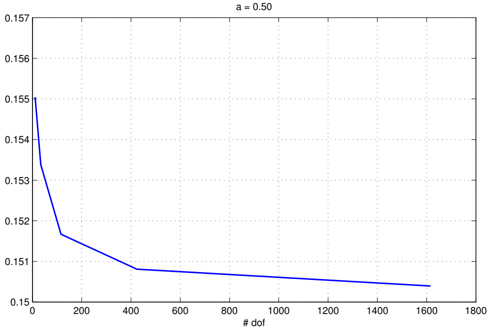
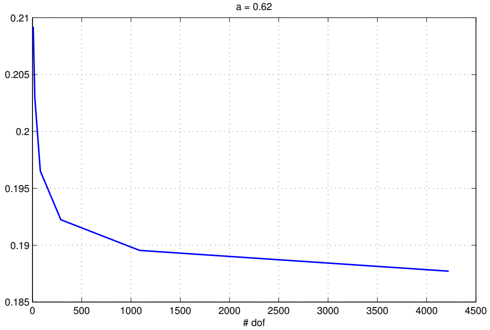


calc	extr	err	ord
0.047864495			
0.047813740			
0.047762043	0.050601250	0.002839206	1.233
0.047731711	0.047688648	0.000043063	1.661
0.047721678	0.047716719	0.000004959	1.653

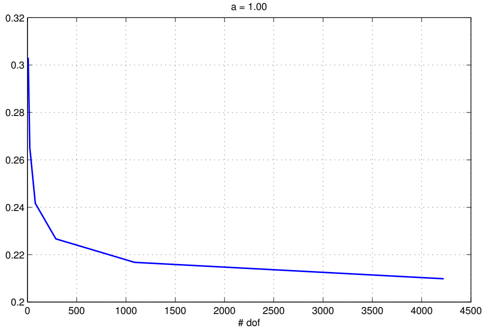


calc	extr	err	ord
0.155045590			
0.153379211			
0.151668328	0.217441888	0.065773559	0.572
0.150808779	0.149940934	0.000867845	1.165
0.150394460	0.150008906	0.000385553	1.064

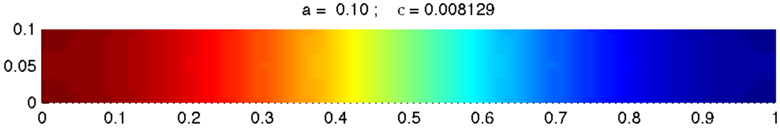


calc	extr	err	ord
0.209208446			
0.203020937			
0.196526729	0.334039166	0.137512437	0.452
0.192239043	0.183907261	0.008331782	0.845
0.189536554	0.184929274	0.004607279	0.769
0.187710936	0.183910061	0.003800875	0.667

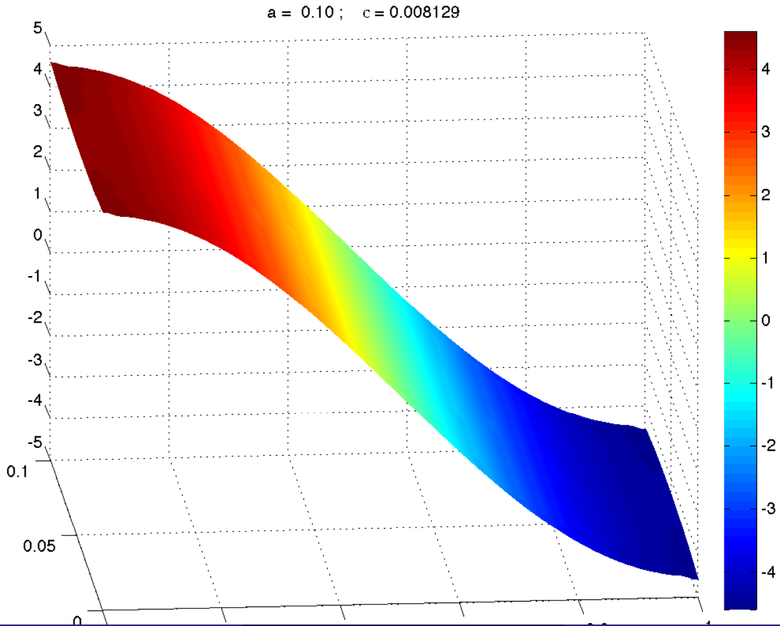
The rectangle: Convergence of 1st Cosserat eigenvalue, $a = 1$



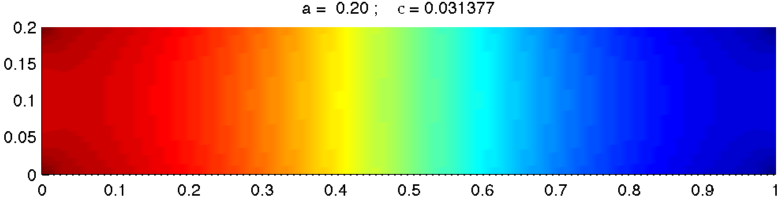
calc	extr	err	ord
0.303075403			
0.265273420			
0.241665485	0.202400120	0.039265365	0.738
0.226676132	0.200606788	0.026069343	0.644
0.216753160	0.197318117	0.019435043	0.563
0.209836989	0.193928572	0.015908412	0.496


a = 0.10 ; c = 0.008129
0.1
0.05
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

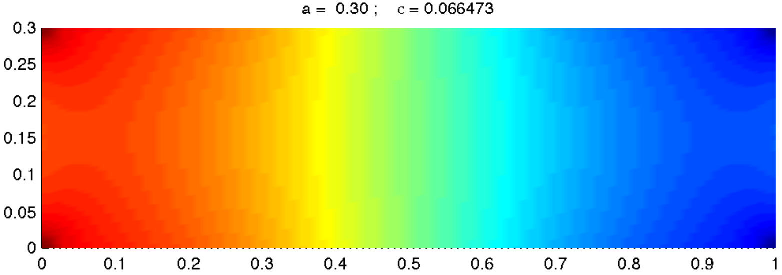
First Cosserat eigenfunction on rectangles : a=0.10



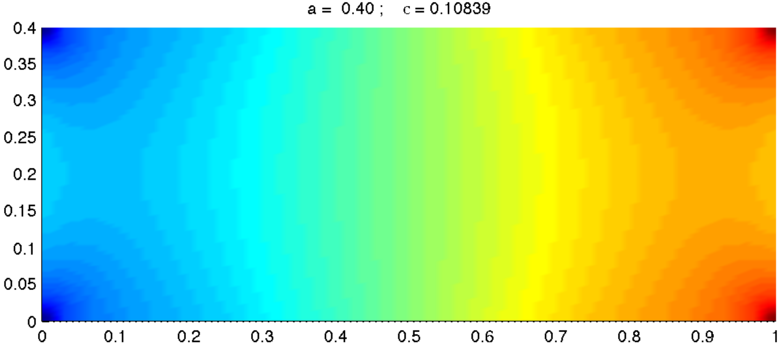
First Cosserat eigenfunction on rectangles : a=0.20



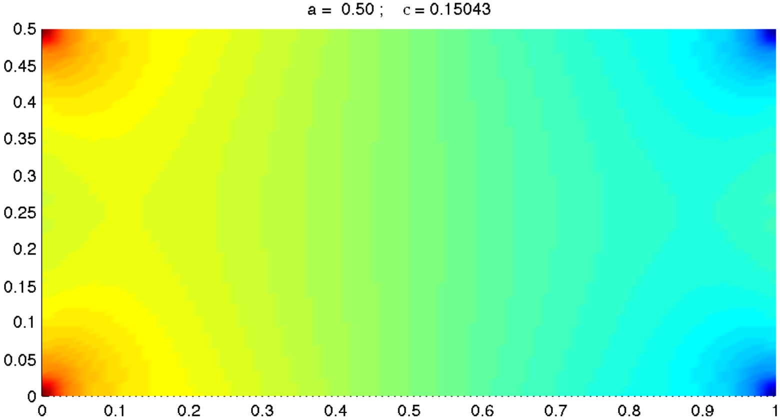
First Cosserat eigenfunction on rectangles : a=0.30



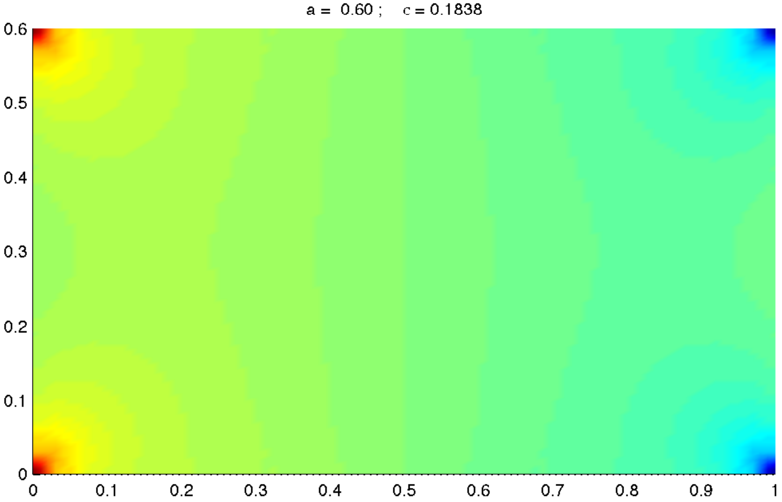
First Cosserat eigenfunction on rectangles : a=0.40



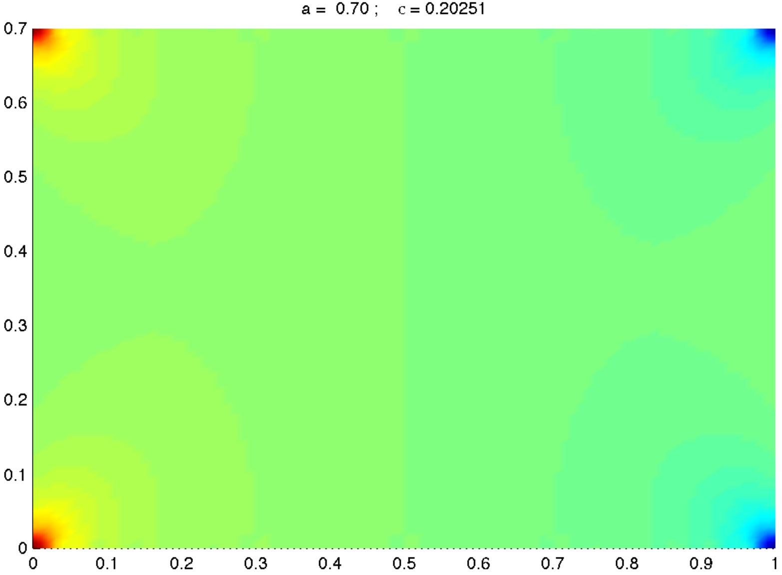
First Cosserat eigenfunction on rectangles : a=0.50



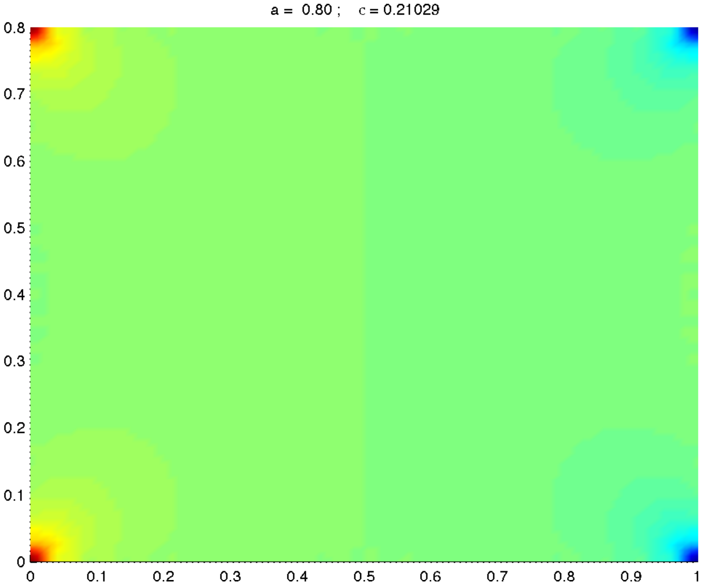
First Cosserat eigenfunction on rectangles : a=0.60



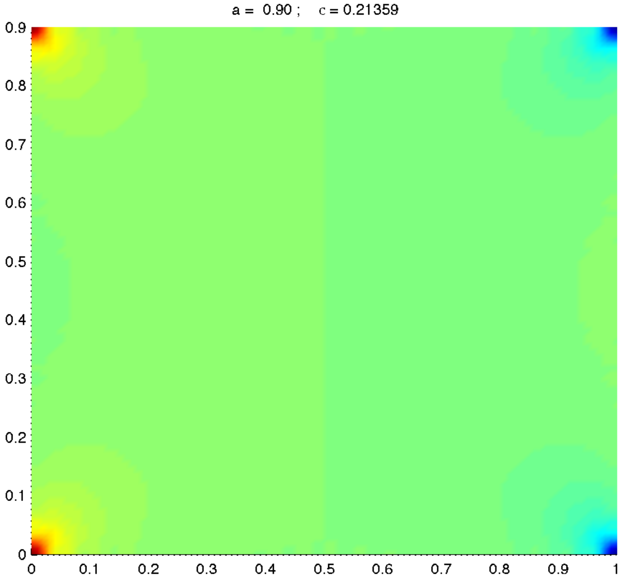
First Cosserat eigenfunction on rectangles : a=0.70



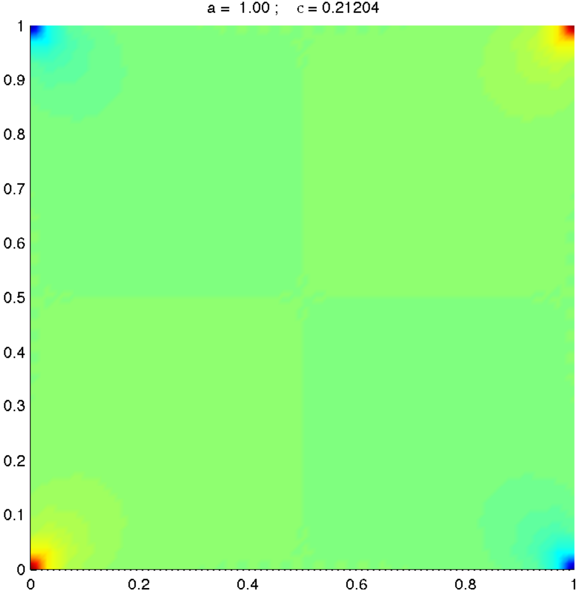
First Cosserat eigenfunction on rectangles : a=0.80



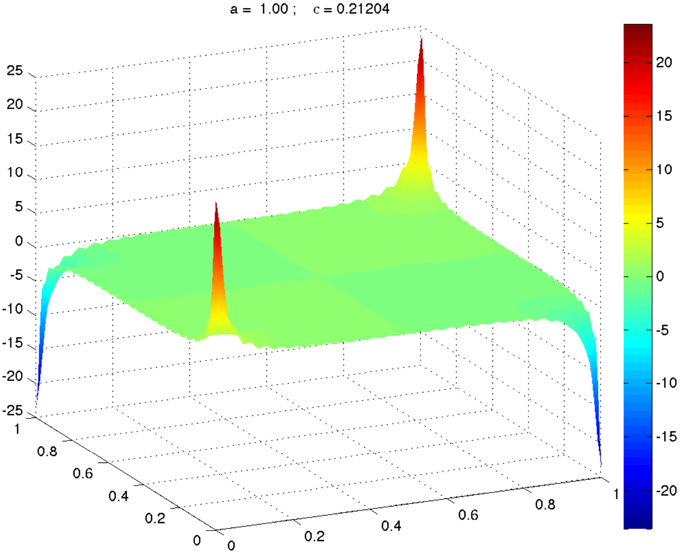
First Cosserat eigenfunction on rectangles : a=0.90



First Cosserat eigenfunction on rectangles : a=1.00



First Cosserat eigenfunction on rectangles : a=1.00



Corner singularities by Kondrat'ev's method

For $\sigma \notin \{0, \frac{1}{2}, 1\}$, the operator $A_\sigma = -\sigma\Delta + \nabla \operatorname{div}$ is elliptic.
If $\Omega \subset \mathbb{R}^2$ has a corner of opening ω, one can therefore determine the corner singularities via **Kondrat'ev**'s method of Mellin transformation:
Look for solutions $\boldsymbol{u}$ of the form $r^\lambda \phi(\theta)$ in a sector. $\to q \sim r^{\lambda-1}\phi(\theta)$
Characteristic equation (Lamé system, known!) for a corner of opening ω:

$$(*) \qquad (1-2\sigma)\omega \frac{\sin\lambda\omega}{\lambda\omega} = \pm \sin\omega.$$

Theorem [Kondrat'ev 1967]

For $\sigma \in [0,1] \setminus \{0, \frac{1}{2}, 1\}$, $A_\sigma : H_0^1(\Omega) \to \mathbf{H}^{-1}(\Omega)$ is Fredholm iff the equation $(*)$ has no solution on the line $\operatorname{Re}\lambda = 0$.

Result :

- $(*)$ has roots on the line $\operatorname{Re}\lambda = 0$ iff $|1-2\sigma|\omega \le |\sin\omega|$
- If $|1-2\sigma|\omega > |\sin\omega|$, there is a real root $\lambda \in (0,1)$

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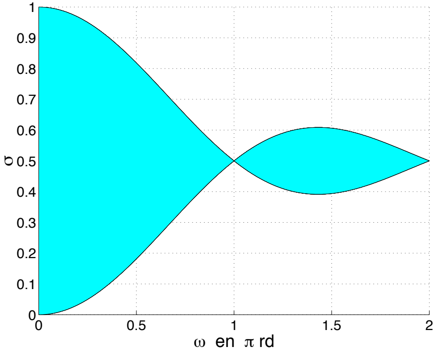
Result [Co-Crouzeix-Dauge-Lafranche 2015]

$\Omega \subset \mathbb{R}^2$ piecewise smooth with corners of opening ω_j.

$$\mathrm{Sp}_{\mathrm{ess}}(\mathscr{S}) = \bigcup_{\text{corners } j} \left[\frac{1}{2} - \frac{|\sin\omega_j|}{2\omega_j}, \frac{1}{2} + \frac{|\sin\omega_j|}{2\omega_j}\right] \cup \{1\}$$



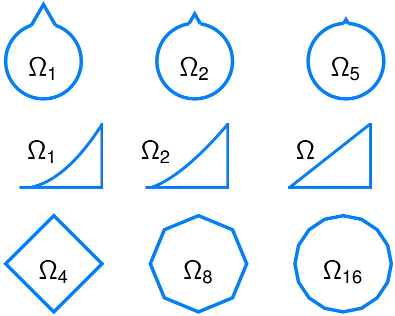
Essential spectrum: σ vs. opening ω

Example : Rectangle, $\omega = \frac{\pi}{2}$

$$\mathrm{Sp}_{\mathrm{ess}}(\mathscr{S}\big|_M) = [\tfrac{1}{2} - \tfrac{1}{\pi}, \tfrac{1}{2} + \tfrac{1}{\pi}]$$
$$\approx [0.181, 0.818]$$

Corollary: Square $\Omega = \square$

$$\beta(\square)^2 \le \frac{1}{2} - \frac{1}{\pi}$$



$\beta(\Omega_N) \leq \beta(\text{corner}) < \sqrt{\frac{1}{2}} = \beta(\Omega)$ (disc)
$\Longrightarrow$ No convergence

Cusps $0 < y < x^{1+1/N}$:
$\beta(\Omega_N) = 0$, $\to$ triangle $\beta(\Omega) > 0$
$\Longrightarrow$ No convergence

Regular polygons,
$0 \leq \beta(\Omega) - \beta(\Omega_N) \leq \frac{\pi}{2N}$: Convergence

Inner approximation: Upper semicontinuity

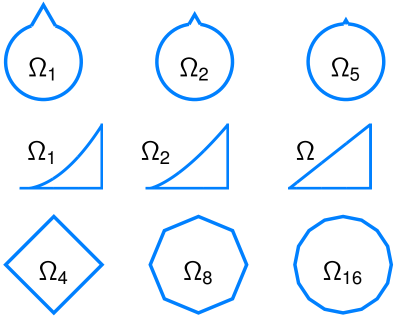
$\Omega_N \subset \Omega$ with $\text{meas}(\Omega \setminus \Omega_N) \to 0 \implies \limsup \beta(\Omega_N) \leq \beta(\Omega)$

Approximation in Lipschitz norm: Continuity

Let Ω_N converge to Ω in Lipschitz norm, that is: $\chi_N : \Omega_N \to \Omega$ is a bi-Lipschitz homeomorphism such that $\|\nabla(\chi_N - \text{Id})\|_{L^\infty} \to 0$.

Then $$\lim_{N\to\infty} \beta(\Omega_N) = \beta(\Omega)$$

Approximation of the domain Ω [BCDG 2016]



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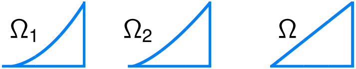

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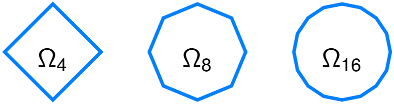

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Then
$$\lim_{N\to\infty} \beta(\Omega_N) = \beta(\Omega)$$

Thank you for your attention!

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John domains

Theorem [Acosta-Durán-Muschietti 2006], [Durán 2012]

Let $\Omega \subset \mathbb{R}^d$ be a bounded John domain. Then $\beta(\Omega) > 0$.

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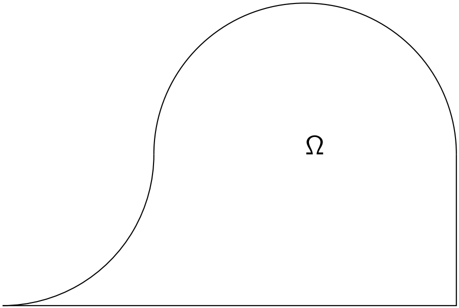

Figure: Not a John domain: Outward cusp, $\beta(\Omega) = 0$ [Friedrichs 1937]

Definition

A domain $\Omega \subset \mathbb{R}^d$ with a distinguished point $\boldsymbol{x}_0$ is called a John domain if it satisfies the following "twisted cone" condition:
There exists a constant $\delta > 0$ such that, for any $\boldsymbol{y}$ in Ω, there is a rectifiable curve γ: $[0,\ell] \to \Omega$ parametrized by arclength such that

$$\gamma(0) = \boldsymbol{y}, \quad \gamma(\ell) = \boldsymbol{x}_0, \quad \text{and} \quad \forall t \in [0,\ell] : \ \mathrm{dist}(\gamma(t), \partial\Omega) \geq \delta\, t\,.$$

Here $\mathrm{dist}(\gamma(t), \partial\Omega)$ denotes the distance of $\gamma(t)$ to the boundary $\partial\Omega$.

Example : Every weakly Lipschitz domain is a John domain.

Definition of John Domain

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Example : Every weakly Lipschitz domain is a John domain.

A John domain: Union of Lipschitz domains

San Juan de la Peña, Jaca 2013

Figure: A weakly Lipschitz domain: the self-similar zigzag

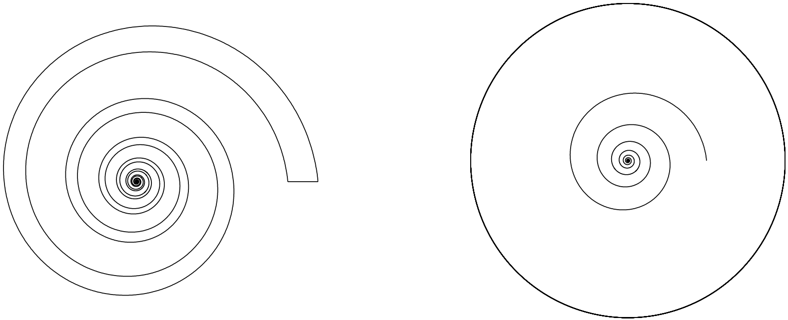
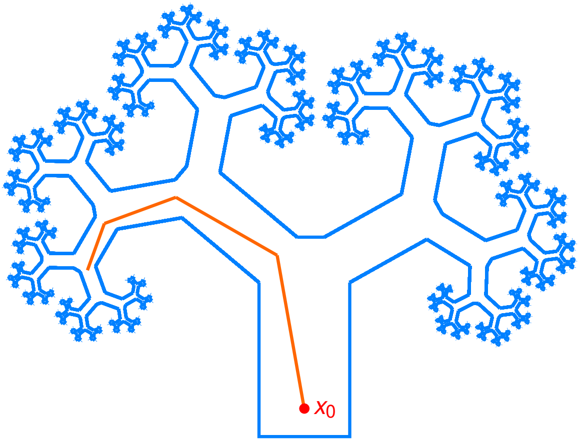


Figure: Weakly Lipschitz (left), John domain (right)



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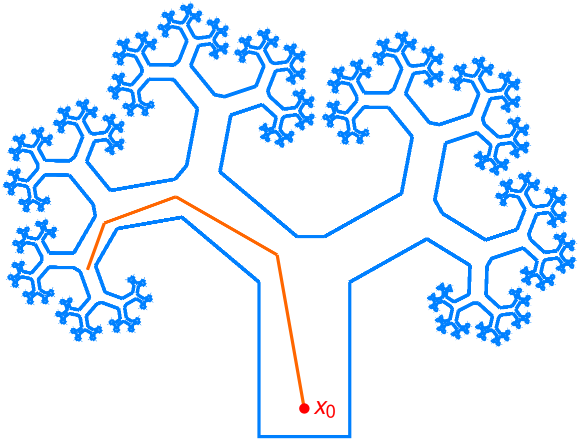
Fractal John domains: Tree or Lung



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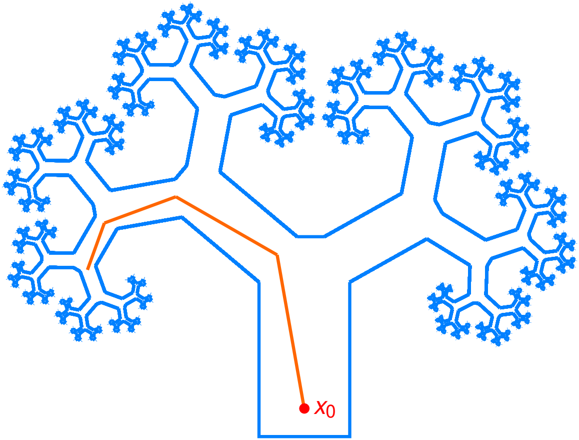

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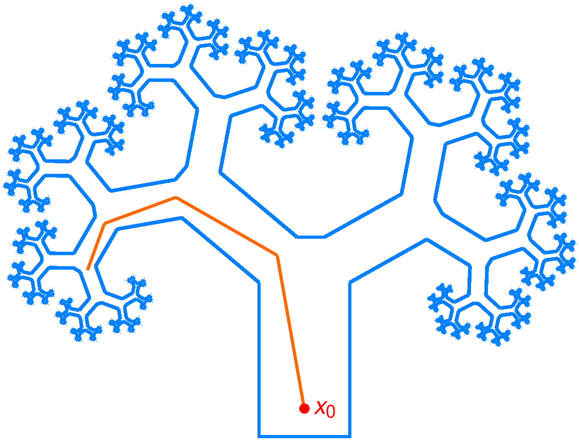

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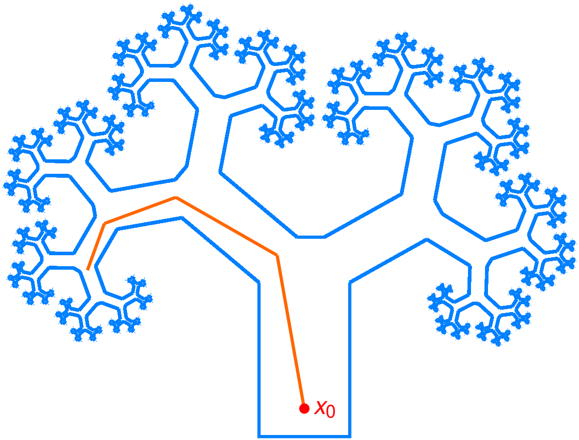

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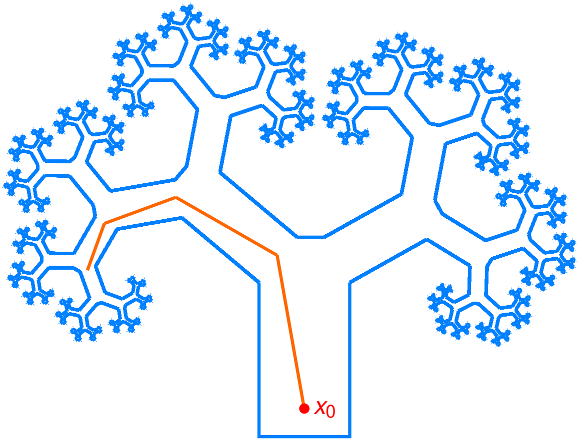

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Related inequalities: Equivalent reformulations

inf-sup condition

$$\inf_{q \in L^2_\circ(\Omega)} \sup_{\boldsymbol{v} \in H^1_0(\Omega)^d} \frac{\langle \operatorname{div} \boldsymbol{v}, q \rangle}{|\boldsymbol{v}|_1 \, \|q\|_0} \geq \beta > 0$$

Lions' lemma [Lions 1958, Nečas 1965]

$$\forall q \in L^2_\circ(\Omega) \; : \; \|q\|_0 \leq C \, |\nabla q|_{-1} \, ; \qquad C = \frac{1}{\beta} < \infty$$

Babuška-Aziz inequality [B-A 1971, Bogovskiĭ1979]

$$\forall q \in L^2_\circ(\Omega) \, \exists \boldsymbol{v} \in H^1_0(\Omega)^d \; : \; \operatorname{div} \boldsymbol{v} = q \text{ and } |\boldsymbol{v}|_1^2 \leq C \|q\|_0^2 \, ; \; C = \frac{1}{\beta^2} < \infty$$

Linearized strain tensor $e(u) = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^\top)$

Korn's second inequality

If $\nabla \boldsymbol{u} - (\nabla \boldsymbol{u})^\top \in L^2_\circ(\Omega)$, then

$$\|\nabla \boldsymbol{u}\|_0^2 \le K(\Omega)\, \|e(u)\|_0^2$$

If the LBB condition is satisfied for Ω, Korn's inequality follows:

$$\partial_i \partial_j u_k = \partial_i e_{jk} + \partial_j e_{jk} - \partial_k e_{ij}, \qquad e = e(u)$$

$$\Longrightarrow |\nabla\nabla \boldsymbol{u}|_{-1} \sim |\nabla e(u)|_{-1} \Longrightarrow \|\nabla \boldsymbol{u}\|_0 \sim \|e(u)\|_0$$

For $\Omega \subset \mathbb{R}^d$, LBB implies Korn: $K(\Omega) \le 1 + \frac{2(d-1)}{\beta(\Omega)^2}$.

1. Any bounded domain $\Omega \subset \mathbb{R}^d$: ($\quad C(\Omega) = \beta(\Omega)^{-2}$)
   LBB $\Longrightarrow$ Korn, $K(\Omega) \le 1 + 2(d-1)\,C(\Omega)$.
2. $d = 2$, Ω simply connected:
   LBB $\Longleftrightarrow$ Korn, $C(\Omega) \le K(\Omega) \le 1 + 2\,C(\Omega)$
3. $d = 2$, Ω simply connected, Lipschitz: $K(\Omega) = 2\,C(\Omega)$
   For smooth domains:
   C.O. Horgan, L.E. Payne, *On Inequalities of Korn, Friedrichs and Babuška-Aziz.* ARMA 82 (1983), 165–179.
   For Lipschitz domains: [Costabel-Dauge 201?]
4. $d = 2$, $\Omega = B_{r_2} \setminus \overline{B}_{r_1}$ (not simply connected): $K(\Omega) \ne 2\,C(\Omega)$
   [Dafermos 1968 (Korn), Chizhonkov–Olshanskii 2000 (LBB)]

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LBB $\longleftrightarrow$ Korn: Results

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Currently open problems:

1. Is Korn $\Longrightarrow$ LBB true for arbitrary domains?
2. What are the optimal bounds between Korn and LBB?
3. Is $K(\Omega) = 2\, C(\Omega)$ true for arbitrary simply connected domains in $\mathbb{R}^2$?

Friedrichs' inequality [named by Horgan-Payne 1983]

There exists a constant Γ such that for any holomorphic $f + ig \in L^2_\circ(\Omega)$

$$\|f\|_0^2 \leq \Gamma \|g\|_0^2$$

Theorem [Friedrichs 1937]

True for piecewise smooth domains $\Omega \subset \mathbb{R}^2$ with no outward cusps.

Definition: $\Gamma(\Omega) = \inf \Gamma$.

Theorem

[Horgan–Payne 1983] Let $\Omega \subset \mathbb{R}^2$ be bounded, simply connected, and C^2. Then

$$\frac{1}{\beta(\Omega)^2} = \Gamma(\Omega) + 1\,.$$

[Costabel-Dauge 2015] This is true for any bounded domain $\Omega \subset \mathbb{R}^2$.

Friedrichs-Velte inequality [Velte 1998]

Let $\Omega \subset \mathbb{R}^3$ be a bounded smooth domain. There exists a constant Γ_1 such that for any

$$f \in L^2_\circ(\Omega),\ \boldsymbol{g} \in L^2(\Omega)^3 \text{ such that } \nabla f = \mathbf{curl}\,\boldsymbol{g} : \quad \|f\|_0^2 \le \Gamma_1 \|\boldsymbol{g}\|_0^2 .$$

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Relations: The Friedrichs-Velte inequality

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