Implementation of functional approach to a posteriori error control for curvilinear Timoshenko beams

Olga Chistiakova, Maxim Frolov

Peter the Great St.Petersburg Polytechnic University
Department of Applied Mathematics

2016

A posteriori error estimates

A posteriori error estimates provide an opportunity to evaluate a norm of difference between exact and given approximate solutions without using the exact solution itself.

Functional approach is developed by Prof. S. Repin and his colleagues. It is reliable and can be used for a wide range of approximations.

In applied mechanics there are several beam models of various complexity. Known results for functional type error estimates:

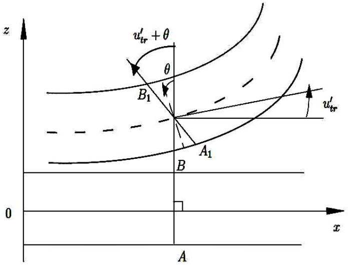
- Euler-Bernoulli beam: O. Mali, M. Frolov, 2009-2011
- Straight Timoshenko beam: M. Frolov, 2010
- Curvilinear Timoshenko beam: O. Chistiakova, M. Frolov, 2015-2016

Beam models and underlying assumptions

- Plane cross sections remain plane after bending
- All longitudinal sections resist bending independently
- **Euler-Bernoulli beam:** plane cross sections remain orthogonal to the deformed axis (works fine for thin beams and small deflections)
- **Timoshenko beam:** shear effects considered, plane cross sections may not remain orthogonal to the deformed axis (can be applied to beams of moderate thickness and even composite beams)

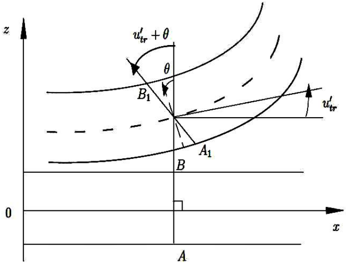
Curvilinear Timoshenko beam: Notation

u_{tr} – traverse displacement,
F_{tr} – traverse force,
u_{ax} – axial displacement,
F_{ax} – axial force,
θ – rotation of beam cross-section,
m – bending moment,
k – shear correction factor,
A – area of the cross-section,
E – Young's modulus,
G – shear modulus,
I_S – sectional moment of inertia,
R – curvature radius;



$'$ denotes the derivative $\frac{d}{dx}$ where $x \in \mathcal{I} = (0, L)$, L – the beam's length.

Curvilinear Timoshenko beam: Problem statement



Equilibrium equations:

$$\begin{cases} \left(EA\left(u'_{ax}+\frac{u_{tr}}{R}\right)\right)' + \frac{kGA}{R}\left(u'_{tr}+\theta-\frac{u_{ax}}{R}\right)+F_{ax}=0, \\ -\frac{EA}{R}\left(u'_{ax}+\frac{u_{tr}}{R}\right)+\left(kGA\left(u'_{tr}+\theta-\frac{u_{ax}}{R}\right)\right)'+F_{tr}=0, \\ \left(EI_S\theta'\right)'-kGA\left(u'_{tr}+\theta-\frac{u_{ax}}{R}\right)+m=0, \end{cases} \quad (1)$$

Curvilinear Timoshenko beam: Problem statement

A clamped beam is assumed:

$$\begin{cases} u_{tr}(0) = u_{tr}(L) = 0, \\ u_{ax}(0) = u_{ax}(L) = 0, \\ \theta(0) = \theta(L) = 0, \end{cases}$$

where L is the beam's length.
For any conforming approximate solution $(\tilde{u}_{ax}, \tilde{u}_{tr}, \tilde{\theta})$ the following error measure is estimated:

$$\epsilon^2 = 2\left(J(\tilde{u}_{ax}, \tilde{u}_{tr}, \tilde{\theta}) - J(u_{ax}, u_{tr}, \theta)\right) =$$
$$\left\|\sqrt{EA}\left(e'_{ax} + \frac{e_{tr}}{R}\right)\right\|^2 + \left\|\sqrt{EI_S}e'_{\theta}\right\|^2 + \left\|\sqrt{kGA}\left(e'_{tr} + e_{\theta} - \frac{e_{ax}}{R}\right)\right\|^2$$

where $e_{ax} = u_{ax} - \tilde{u}_{ax}$, $e_{tr} = u_{tr} - \tilde{u}_{tr}$, $e_{\theta} = \theta - \tilde{\theta}$.

Curvilinear Timoshenko beam: Simplifying notation

The error measure can be rewritten as

$$\epsilon^2 = \left\| \frac{1}{\sqrt{EA}} e_\gamma \right\|^2 + \left\| \frac{1}{\sqrt{EI_S}} e_\psi \right\|^2 + \left\| \frac{1}{\sqrt{kGA}} e_\xi \right\|^2,$$

using the following notation:

$$\begin{cases} \gamma = EA\left(u'_{ax} + \frac{u_{tr}}{R}\right),\ \tilde{\gamma} = EA\left(\tilde{u}'_{ax} + \frac{\tilde{u}_{tr}}{R}\right),\ e_\gamma = \gamma - \tilde{\gamma}; \\ \xi = kGA\left(u'_{tr} + \theta - \frac{u_{ax}}{R}\right),\ \tilde{\xi} = kGA\left(\tilde{u}'_{tr} + \tilde{\theta} - \frac{\tilde{u}_{ax}}{R}\right),\ e_\xi = \xi - \tilde{\xi}; \\ \psi = EI_S\theta',\ \tilde{\psi} = EI_S\tilde{\theta}',\ e_\psi = \psi - \tilde{\psi}. \end{cases}$$

Curvilinear Timoshenko beam: Free variables

Introducing a triple of free elements $\hat{\gamma} \in \mathbf{H}^1(\mathcal{I})$, $\hat{\xi} \in \mathbf{H}^1(\mathcal{I})$, $\hat{\psi} \in \mathbf{H}^1(\mathcal{I})$, we get:

$$\epsilon^2 = \boxed{\int_0^L \frac{1}{EA}\left(\hat{\gamma} - \tilde{\gamma}\right) e_\gamma \, dx} + \int_0^L \left(\hat{\gamma}' + \frac{\hat{\xi}}{R} + F_{ax}\right) e_{ax} \, dx +$$

$$+ \boxed{\int_0^L \frac{1}{kGA}\left(\hat{\xi} - \tilde{\xi}\right) e_\xi \, dx} + \int_0^L \left(-\frac{\hat{\gamma}}{R} + \hat{\xi}' + F_{tr}\right) e_{tr} \, dx +$$

$$+ \boxed{\int_0^L \frac{1}{EI_S}(\hat{\psi} - \tilde{\psi}) e_\psi \, dx} + \int_0^L \left(\hat{\psi}' - \hat{\xi} + m\right) e_\theta \, dx.$$

Curvilinear Timoshenko beam: Estimating ϵ^2 terms

Estimating the encircled terms is straightforward.
For estimating terms that contain elements of equilibrium equations, an extra inequality needs to be utilized:

$$\frac{1}{C^2} \leq \inf_{(v_0, w_0, \phi_0)} \frac{\chi^2(v_0, w_0, \phi_0)}{\triangle^2(v_0, w_0, \phi_0)},$$

where $(v_0, w_0, \phi_0) \in \mathbf{H}_0^1(\mathcal{I}) \times \mathbf{H}_0^1(\mathcal{I}) \times \mathbf{H}_0^1(\mathcal{I})$,

$$\chi^2 = \left\| \sqrt{EA}\left(v_0' + \frac{w_0}{R}\right) \right\|^2 + \left\| \sqrt{EI_S}\phi_0{}' \right\|^2 + \left\| \sqrt{kGA}\left(w_0' + \phi_0 - \frac{v_0}{R}\right) \right\|^2,$$

$$\triangle^2 = \|C_{ax}^{1/2} v_0\|^2 + \|C_{tr}^{1/2} w_0\|^2 + \|C_\theta^{1/2} \phi_0\|^2.$$

Constants C_{ax}, C_{tr}, C_θ are chosen as normalized combinations of material and shape parameters: $C_{tr} = kGA/L^2$, $C_{ax} = EA/L^2$, $C_\theta = EI_S/L^2$.

Curvilinear Timoshenko beam: A posteriori error estimate

Finally,

$$\boxed{\epsilon \leq M = M_1 + M_2} \quad or \quad \boxed{\epsilon^2 \leq (1+\beta)\, M_1^2 + \left(1+\beta^{-1}\right) M_2^2}$$

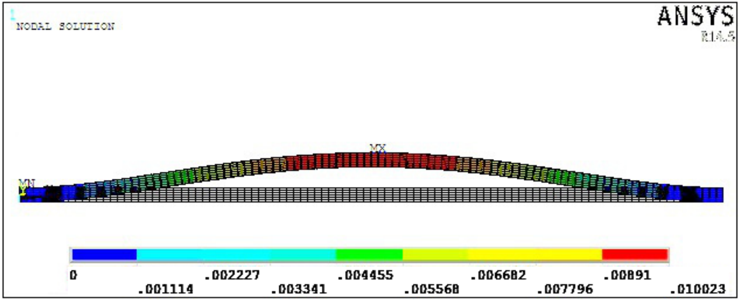
where

$$\begin{cases} M_1^2 = \left\| \frac{1}{\sqrt{EA}} \left(\hat{\gamma} - \tilde{\gamma}\right) \right\|^2 + \left\| \frac{1}{\sqrt{kGA}} \left(\hat{\xi} - \tilde{\xi}\right) \right\|^2 + \left\| \frac{1}{\sqrt{EI_S}} \left(\hat{\psi} - \tilde{\psi}\right) \right\|^2, \\ M_2^2 = C^2 \mathcal{R}^2, \\ \mathcal{R}^2 = \left\| C_{ax}^{-1/2} \left(\hat{\gamma}' + \frac{\hat{\xi}}{R} + F_{ax} \right) \right\|^2 + \left\| C_{tr}^{-1/2} \left(-\frac{\hat{\gamma}}{R} + \hat{\xi}' + F_{tr} \right) \right\|^2 + \\ \quad + \left\| C_{\theta}^{-1/2} \left(\hat{\psi}' - \hat{\xi} + m \right) \right\|^2 \end{cases}$$

and β is an arbitrary positive parameter.

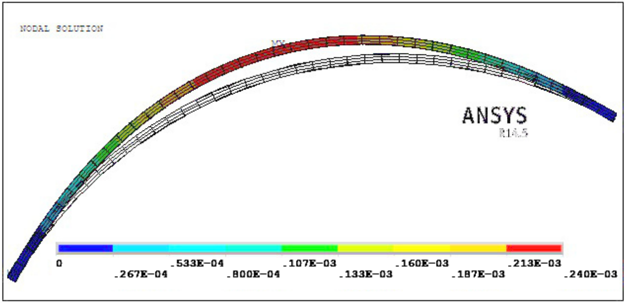
Numerical example 1: Thin straight beam

M. Frolov, Russ. J. NAMM, 2010



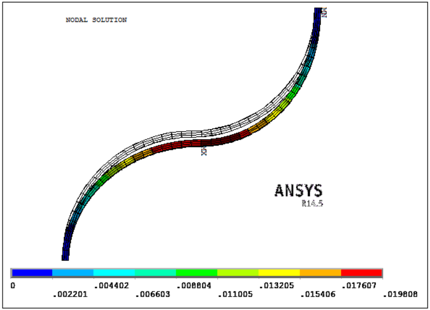
Number of elements	ϵ	M	*ratio*
100	7.91×10^{-2}	7.99×10^{-2}	1.0
200	4.01×10^{-2}	4.05×10^{-2}	1.0
400	2.01×10^{-2}	2.04×10^{-2}	1.0
800	1.01×10^{-2}	1.04×10^{-2}	1.0

Numerical example 2: Thin curvilinear beam



Number of elements	ϵ	M	*ratio*
100	7.87×10^{-6}	8.58×10^{-6}	1.1
200	3.96×10^{-6}	4.36×10^{-6}	1.1
400	2.02×10^{-6}	2.22×10^{-6}	1.1
800	1.07×10^{-6}	1.20×10^{-6}	1.1

Numerical example 3:
Curvilinear beam of moderate thickness



Number of elements	ϵ	M	*ratio*
100	11.6×10^{-2}	14.3×10^{-2}	1.2
200	5.93×10^{-2}	7.31×10^{-2}	1.2
400	2.98×10^{-2}	3.73×10^{-2}	1.3
800	1.74×10^{-2}	2.25×10^{-2}	1.3

Conclusions

- A new functional a posteriori error estimate is derived for curvilinear Timoshenko beams.
- The estimate is implemented and numerically tested on a number of examples with ANSYS.
- The estimate is reliable, can be applied to any conforming approximate solution and appears to be a generalization of a functional a posteriori error estimate for straight Timoshenko beams known from the literature.
- In all considered cases overestimation of the true error is sufficiently small to conclude that the estimate is efficient and may become a practically valuable tool for engineering computations with black-box commercial software for CAE.

Thank you for your attention!