An Enriched Discontinuous Galerkin Method for Resolving Eddy Current Singularities by P-Refinement

Raffael Casagrande

ETH Zürich

raffael.casagrande@sam.math.ethz.ch

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2D, Time-harmonic, Scalar Eddy Current Problem

$$-\Delta u + i\sigma u = 0 \quad \text{in } \Omega \qquad (1a)$$

$$u = 1 \quad \text{on } \Gamma_D \qquad (1b)$$

$$\mathbf{grad}\, u \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_N \qquad (1c)$$

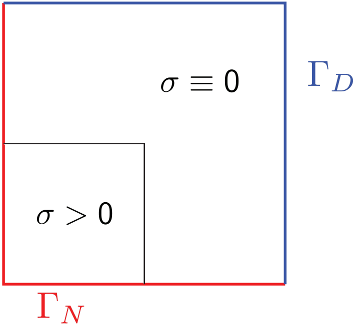

Figure: Domain Ω

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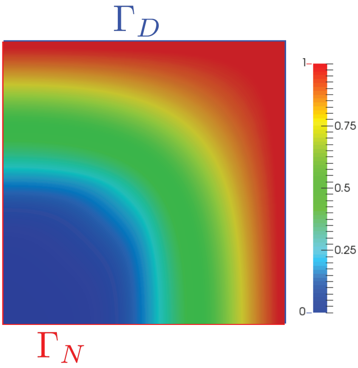

Figure: Reference Solution u

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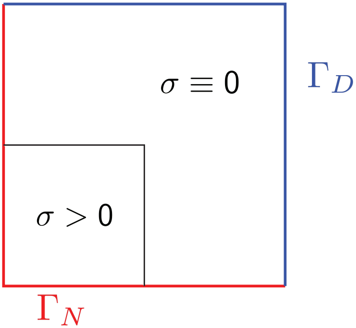
[Dauge et al, 2014]: leading singularity is

$$r^2 \log r$$



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Approximate PDE's with singular behavior

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- Enrich approximation space:
  - Multiply with smooth Cutoff [Strang and Fix, 1973]
  - Partition of Unity (PUM) [Babuška and Melenk, 1997]
  - Auxiliary mapping [Hae-Soo and Babuska, 1992]
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Goal:

No Mesh refinement, easy implementation, exponential convergence, efficient.

Multiply with smooth Cutoff [Strang and Fix, 1973]

Standard FEM: Find $u_h \subset V_h \subset H^1(\Omega)$ s.t.

$$a(u_h, v_h) = \ell(v_h) \qquad \text{for all } v_h \in V_h.$$

where

$$a(w, v) := \int_\Omega \mathbf{grad} w \cdot \overline{\mathbf{grad} v} + i\sigma w \overline{v} \mathrm{d}\mathbf{x}$$

Céa's Lemma

$$\|u - u_h\|_{H^1(\Omega)} \leq C \min_{v \in V_h} \|u - v\|_{H^1(\Omega)}$$

Multiply with smooth Cutoff [Strang and Fix, 1973]

Approximation space V_h

$$V_h := \mathcal{P}_p(\mathcal{T}_h) \oplus \left\{ \chi(r)\, \mathfrak{s}_1^{k,0/1}(r,\theta) \,\middle|\, k = 1 \ldots p-1 \right\}$$

Increase $p \Rightarrow$ exponential convergence.

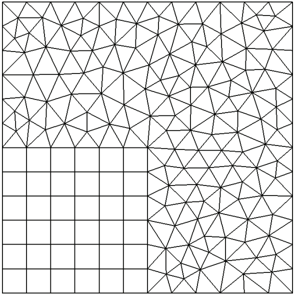

Figure: Triangulation $\mathcal{T}_h$

Multiply with smooth Cutoff [Strang and Fix, 1973]

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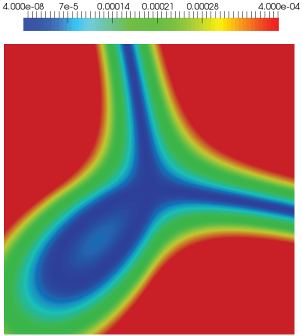

Figure: Singularity $\mathfrak{s}_1^{k,0}$

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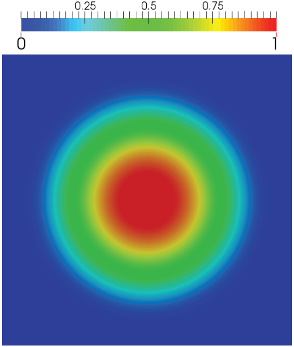

Figure: Smooth cutoff χ

Multiply with smooth Cutoff [Strang and Fix, 1973]

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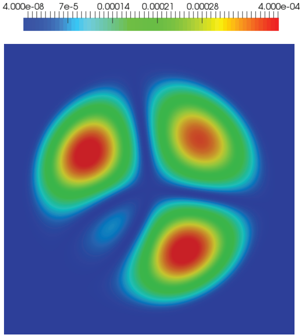
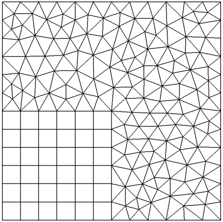


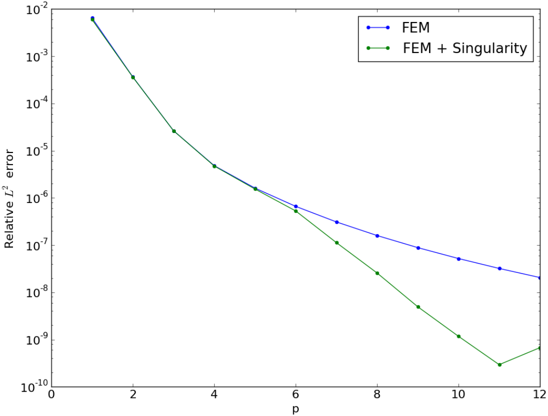
Figure: $\chi \cdot \mathfrak{s}_1^{k,0}$

Multiply with smooth Cutoff [Strang and Fix, 1973]

p-Convergence[1]



$\sigma = 32$



[1] w.r.t. reference solution on extremely fine mesh

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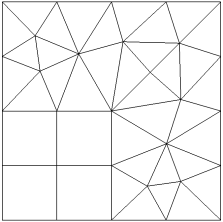

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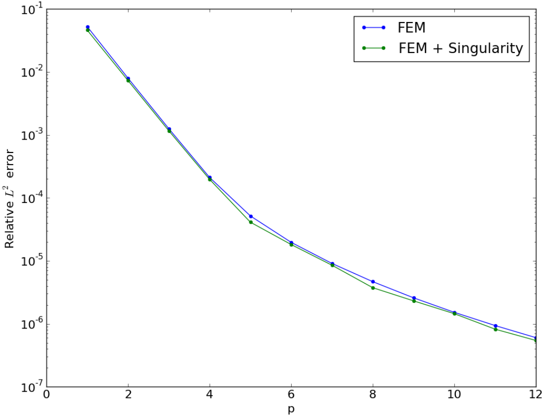

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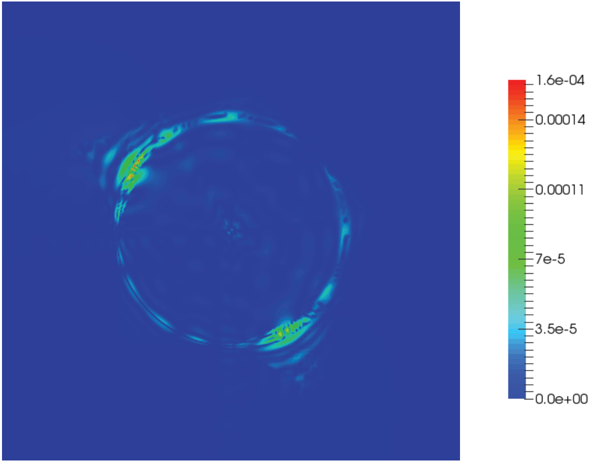
Pointwise error $|u - u^h|$, $p = 10$



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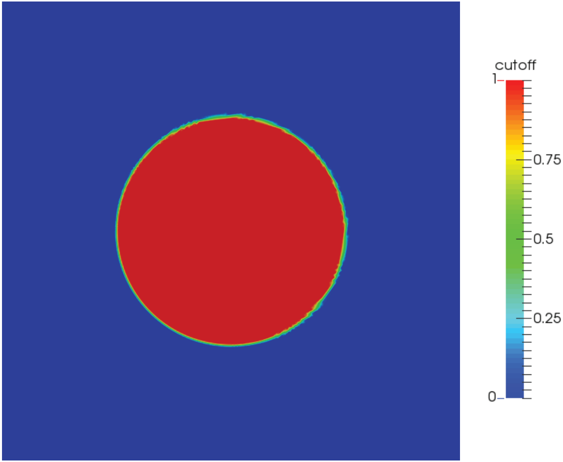

Enriched Discontinuous Galerkin

Nonsymmetric Interior Penalty (NIP):

Find $u_h \subset V_h \subset H^2(\mathcal{T}_h)$ s.t.

$$a^{\text{nip}}(u_h, v_h) = \ell^{\text{nip}}(v_h) \qquad \text{for all } v_h \in V_h. \tag{2}$$

where

$$a^{\text{nip}}(w, v) := a(w, v) - \sum_{F \in \mathcal{F}_h^i} \int_F \{\!\{\mathbf{grad}_h w\}\!\} \cdot \mathbf{n}_F \, [\![\overline{v}]\!] + \sum_{F \in \mathcal{F}_h^i} [\![w]\!] \, \{\!\{\overline{\mathbf{grad}_h v}\}\!\} \cdot \mathbf{n}_F + \sum_{F \in \mathcal{F}_h^i} \frac{\eta}{h_F} \int_F [\![w]\!] \, [\![\overline{v}]\!]$$

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$\eta > 0$ ensures coercivity

Enriched Discontinuous Galerkin

Theorem (Strang 2)

Assume $u \in H^2(\mathcal{T}_h)$ *and let* $u_h \in V_h \subset H^2(\mathcal{T}_h)$ *be the Nonsymmetric Interior Penalty solution. Then*

$$\|u - u_h\|_{ip} \leq (1 + CC_{tr^{-1}}) \min_{v_h \in V_h} \|u - v_h\|_{ip,*}$$

where C *is independent of* V_h *and*

$$\|w\|_{ip,*}^2 := \|w\|_{ip}^2 + \sum_{T \in \mathcal{T}_h} h_T \|\mathbf{grad} w\|_{L^2(\partial T)^2}^2$$

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Corollary (cf. [Rivière, Wheeler and Girault, 1999])

If $V_h = \mathcal{P}_p(\mathcal{T}_h)$, $\mathcal{T}_h$ is simplical, shape regular and u sufficiently smooth:

- $C_{tr^{-1}} \leq C\, p$
- $\min_{v_h \in V_h} \|u - v_h\|_{ip,*} \leq C \frac{h^p}{p^{p-1/2}} \|u\|_{H^{p+1}(\Omega)}$

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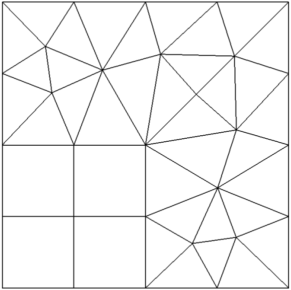

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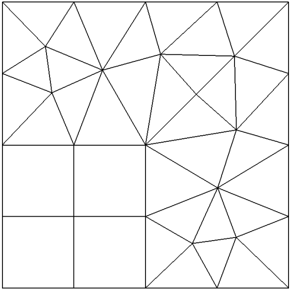

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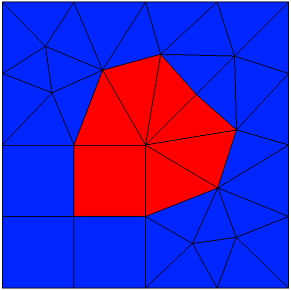

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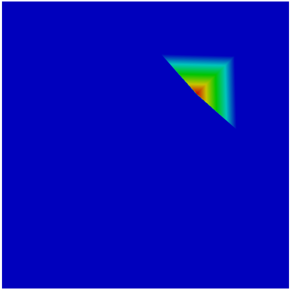

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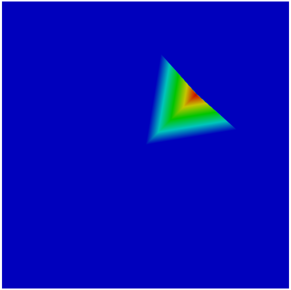

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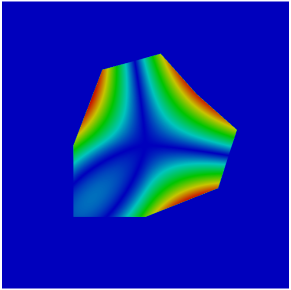

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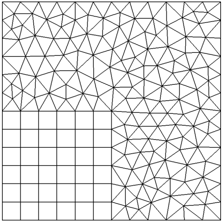
- $\mathcal{S}_p(\mathcal{T}_{h,c}) = \left\{ \mathfrak{s}_1^{k,0/1}\Big|_{\mathcal{T}_{h,c}} \,\middle|\, k = 1 \dots p-1 \right\}$
- Expect exponential convergence as $p \to \infty$

Enriched Discontinuous Galerkin

p-Convergence[2]



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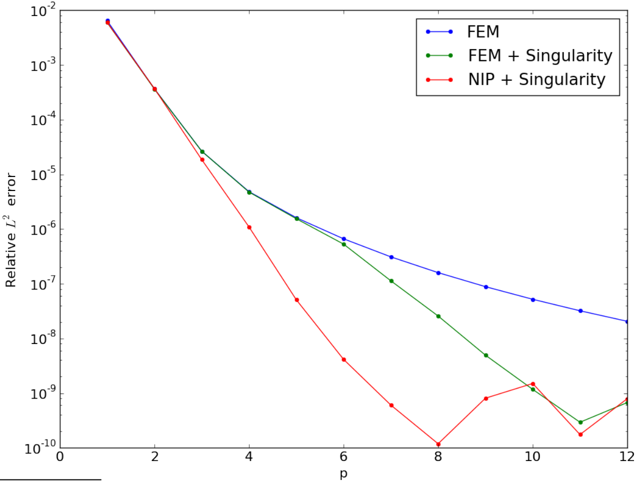

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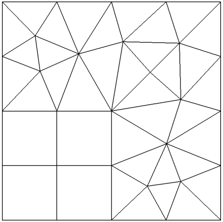

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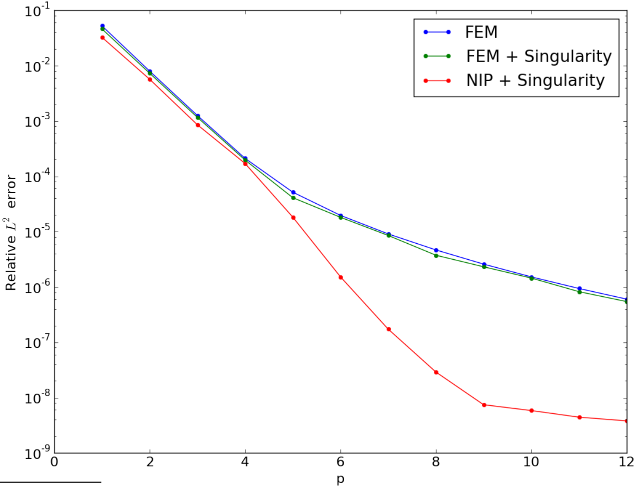

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Quadrature

Question: How to calculate $\int_K \mathfrak{s}_1^{p-1,0}(\mathbf{x}) \cdot P_p(\mathbf{x})\,\mathrm{d}\mathbf{x}$ where

$$\mathfrak{s}_1^{k,0}(\mathbf{x}(r,\theta)) \approx r^{k+2}\,(\log r \cos(k+2)\theta - \theta \sin(k+2)\theta)$$

- Analytic formula
  - Feasible for *h*-refinement [Strang and Fix, 1973]
  - Nightmare for arbitrary high-order *p*
- hp-Quadrature

hp-Quadrature

Input: # refinement levels N, integrand $f \in C^m(\overline{K})$

- Refine towards singularity N times
- Quadrature order on level i: $\frac{N(1+\max(m-1,1))}{i} - 1$
- Evaluate quadrature rule $Q_{\text{hp}}(f)$

Analysis:

$$\left| \int_K f - Q_{\text{hp}}(f) \right| \leq C 2^{-mN} \qquad m \geq 1$$

Observation:
Enriched FEM/DG methods *absolutely* need very accurate quadrature.

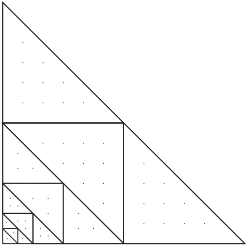

Figure: hp-quadrature rule for $N = 4, s = 2$

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- Efficient and accurate (numerical) quadrature is *a must*
- Linear Dependency

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Thank you for you attention.