Functional A Posteriori Error Control for Conforming Mixed Approximations of Parabolic Problems

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Dual variational technique

Exposed in detail in

- P. Neittaanmäki and S. Repin. *Reliable methods for computer simulation. Error control and a posteriori estimates*. Elsevier, New York, 2004.

Method of integral identities

Exposed in detail in

- S. Repin. *A posteriori estimates for partial differential equations*. Walter de Gruyter, Berlin, 2008

In this talk we derive error equalities and estimates for mixed approximations. The error is measured in a combined norm taking account both the error in the primal and dual variables.

1) Derive an *error equality* for a problem with a lower order term
2) Use this result to obtain a *two-sided estimate* for the case without a lower order term

Notation and definitions

Let $\Omega \in \mathbb{R}^d$, $d \geq 1$ be a bounded domain with boundary $\Gamma = \partial\Omega$. We denote the inner product and norm in $\mathsf{L}^2(\Omega)$ by

$$\langle v, w \rangle_\Omega := \int_\Omega v\, w \, \mathrm{d}x \qquad \text{and} \qquad |v|_\Omega := \sqrt{\langle v, v \rangle_\Omega}$$

The space with square summable gradients and divergences are denoted by

$$\mathsf{H}^1(\Omega) := \{\varphi \in \mathsf{L}^2(\Omega) \mid \nabla\varphi \in \mathsf{L}^2(\Omega)\},$$
$$\mathsf{D}(\Omega) := \{\varphi \in \mathsf{L}^2(\Omega) \mid \operatorname{div}\varphi \in \mathsf{L}^2(\Omega)\},$$

with the inner products $\langle\cdot,\cdot\rangle_{\mathsf{H}^1(\Omega)}, \langle\cdot,\cdot\rangle_{\mathsf{D}(\Omega)}$ and induced norms $|\cdot|_{\mathsf{H}^1(\Omega)}, |\cdot|_{\mathsf{D}(\Omega)}$, respectively. Functions from $\mathsf{H}^1(\Omega)$ vanishing on the boundary we denote by $\mathsf{H}^1_\Gamma(\Omega)$.

Partial integration

$$\forall\, \varphi \in \mathsf{H}^1_\Gamma(\Omega) \quad \forall\, \psi \in \mathsf{D}(\Omega) \qquad \langle \nabla\varphi, \psi \rangle_\Omega = -\langle \varphi, \operatorname{div}\psi \rangle_\Omega.$$

Friedrichs inequality

$$\forall w \in \mathsf{H}^1_\Gamma(\Omega) \qquad |w|_\Omega \leq C_{\mathrm{F}} |\nabla w|_\Omega,$$

Reaction-diffusion problem

Find the primal variable $u \in \mathsf{H}^1(\Omega)$ and the dual variable $p \in \mathsf{D}(\Omega)$ such that

$$
\begin{aligned}
p &= \nabla u && \text{in } \Omega, \\
-\operatorname{div} p + u &= f && \text{in } \Omega, \\
u &= 0 && \text{on } \Gamma,
\end{aligned}
$$

where $f \in \mathsf{L}^2(\Omega)$.

Reaction-diffusion problem

Find the primal variable $u \in \mathsf{H}^1(\Omega)$ and the dual variable $p \in \mathsf{D}(\Omega)$ such that

$$\begin{aligned} p &= \nabla u && \text{in } \Omega, \\ -\operatorname{div} p + u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma, \end{aligned}$$

where $f \in \mathsf{L}^2(\Omega)$.

Let $(\tilde{u}, \tilde{p}) \in \mathsf{H}^1_\Gamma(\Omega) \times \mathsf{D}(\Omega)$ be arbitrary. Then

$$|f - \tilde{u} + \operatorname{div} \tilde{p}|^2_\Omega + |\tilde{p} - \nabla \tilde{u}|^2_\Omega = |u - \tilde{u} + \operatorname{div}(\tilde{p} - p)|^2_\Omega + |\tilde{p} - p + \nabla(u - \tilde{u})|^2_\Omega$$

Reaction-diffusion problem

Find the primal variable $u \in \mathsf{H}^1(\Omega)$ and the dual variable $p \in \mathsf{D}(\Omega)$ such that

$$\begin{aligned} p &= \nabla u && \text{in } \Omega, \\ -\operatorname{div} p + u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma, \end{aligned}$$

where $f \in \mathsf{L}^2(\Omega)$.

Let $(\tilde{u}, \tilde{p}) \in \mathsf{H}^1_\Gamma(\Omega) \times \mathsf{D}(\Omega)$ be arbitrary. Then

$$\begin{aligned} |f - \tilde{u} + \operatorname{div} \tilde{p}|^2_\Omega + |\tilde{p} - \nabla \tilde{u}|^2_\Omega &= |u - \tilde{u} + \operatorname{div}(\tilde{p} - p)|^2_\Omega + |\tilde{p} - p + \nabla(u - \tilde{u})|^2_\Omega \\ &= |u - \tilde{u}|^2_\Omega + |\operatorname{div}(\tilde{p} - p)|^2_\Omega + 2\langle u - \tilde{u}, \operatorname{div}(\tilde{p} - p)\rangle_\Omega \\ &\quad + |\tilde{p} - p|^2_\Omega + |\nabla(u - \tilde{u})|^2_\Omega + 2\langle \tilde{p} - p, \nabla(u - \tilde{u})\rangle_\Omega \end{aligned}$$

Find the primal variable $u \in \mathsf{H}^1(\Omega)$ and the dual variable $p \in \mathsf{D}(\Omega)$ such that

$$\begin{aligned} p &= \nabla u && \text{in } \Omega, \\ -\operatorname{div} p + u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma, \end{aligned}$$

where $f \in \mathsf{L}^2(\Omega)$.

Let $(\tilde{u}, \tilde{p}) \in \mathsf{H}^1_\Gamma(\Omega) \times \mathsf{D}(\Omega)$ be arbitrary. Then

$$\begin{aligned} |f - \tilde{u} + \operatorname{div} \tilde{p}|^2_\Omega + |\tilde{p} - \nabla \tilde{u}|^2_\Omega &= |u - \tilde{u} + \operatorname{div}(\tilde{p} - p)|^2_\Omega + |\tilde{p} - p + \nabla(u - \tilde{u})|^2_\Omega \\ &= |u - \tilde{u}|^2_\Omega + |\operatorname{div}(\tilde{p} - p)|^2_\Omega + \cancel{2\langle u - \tilde{u}, \operatorname{div}(\tilde{p} - p)\rangle_\Omega} \\ &\quad + |\tilde{p} - p|^2_\Omega + |\nabla(u - \tilde{u})|^2_\Omega + \cancel{2\langle \tilde{p} - p, \nabla(u - \tilde{u})\rangle_\Omega} \\ &= |u - \tilde{u}|^2_{\mathsf{H}^1(\Omega)} + |p - \tilde{p}|^2_{\mathsf{D}(\Omega)} \end{aligned}$$

So we have an *error equality* for the reaction-diffusion problem.

Diffusion problem

Find the primal variable $u \in \mathsf{H}^1(\Omega)$ and the dual variable $p \in \mathsf{D}(\Omega)$ such that

$$\begin{aligned} p &= \nabla u && \text{in } \Omega, \\ -\operatorname{div} p &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma, \end{aligned}$$

where $f \in \mathsf{L}^2(\Omega)$.

Diffusion problem

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\end{aligned}
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where $f \in \mathsf{L}^2(\Omega)$.

Diffusion problem

Find the primal variable $u \in \mathsf{H}^1(\Omega)$ and the dual variable $p \in \mathsf{D}(\Omega)$ such that

$$\begin{aligned} p &= \nabla u && \text{in } \Omega, \\ -\operatorname{div} p + u &= f + u && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma, \end{aligned}$$

where $f \in \mathsf{L}^2(\Omega)$.

Let $(\tilde{u}, \tilde{p}) \in \mathsf{H}^1_\Gamma(\Omega) \times \mathsf{D}(\Omega)$ be arbitrary. Then

$$|u - \tilde{u}|^2_{\mathsf{H}^1(\Omega)} + |p - \tilde{p}|^2_{\mathsf{D}(\Omega)} = |f + u - \tilde{u} + \operatorname{div} \tilde{p}|^2_\Omega + |\tilde{p} - \nabla \tilde{u}|^2_\Omega$$

Find the primal variable $u \in \mathsf{H}^1(\Omega)$ and the dual variable $p \in \mathsf{D}(\Omega)$ such that

$$\begin{aligned} p &= \nabla u && \text{in } \Omega, \\ -\operatorname{div} p + u &= f + u && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma, \end{aligned}$$

where $f \in \mathsf{L}^2(\Omega)$.

Let $(\tilde{u}, \tilde{p}) \in \mathsf{H}^1_\Gamma(\Omega) \times \mathsf{D}(\Omega)$ be arbitrary. Then

$$\begin{aligned} |u - \tilde{u}|^2_{\mathsf{H}^1(\Omega)} + |p - \tilde{p}|^2_{\mathsf{D}(\Omega)} &= |f + u - \tilde{u} + \operatorname{div} \tilde{p}|^2_\Omega + |\tilde{p} - \nabla \tilde{u}|^2_\Omega \\ &= |f + \operatorname{div} \tilde{p}|^2_\Omega + |u - \tilde{u}|^2_\Omega + 2\langle f + \operatorname{div} \tilde{p}, u - \tilde{u}\rangle_\Omega + |\tilde{p} - \nabla \tilde{u}|^2_\Omega \end{aligned}$$

Diffusion problem

Find the primal variable $u \in \mathsf{H}^1(\Omega)$ and the dual variable $p \in \mathsf{D}(\Omega)$ such that

$$\begin{aligned} p &= \nabla u && \text{in } \Omega, \\ -\operatorname{div} p + u &= f + u && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma, \end{aligned}$$

where $f \in \mathsf{L}^2(\Omega)$.

Let $(\tilde{u}, \tilde{p}) \in \mathsf{H}^1_\Gamma(\Omega) \times \mathsf{D}(\Omega)$ be arbitrary. Then

$$\begin{aligned} |u - \tilde{u}|^2_{\cancel{\mathsf{H}^1(\Omega)}} + |p - \tilde{p}|^2_{\mathsf{D}(\Omega)} &= |f + u - \tilde{u} + \operatorname{div} \tilde{p}|^2_\Omega + |\tilde{p} - \nabla \tilde{u}|^2_\Omega \\ &= |f + \operatorname{div} \tilde{p}|^2_\Omega + \cancel{|u - \tilde{u}|^2_\Omega} + 2\langle f + \operatorname{div} \tilde{p}, u - \tilde{u} \rangle_\Omega + |\tilde{p} - \nabla \tilde{u}|^2_\Omega \end{aligned}$$

Find the primal variable $u \in \mathsf{H}^1(\Omega)$ and the dual variable $p \in \mathsf{D}(\Omega)$ such that

$$\begin{aligned} p &= \nabla u && \text{in } \Omega, \\ -\operatorname{div} p + u &= f + u && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma, \end{aligned}$$

where $f \in \mathsf{L}^2(\Omega)$.

Let $(\tilde{u}, \tilde{p}) \in \mathsf{H}^1_\Gamma(\Omega) \times \mathsf{D}(\Omega)$ be arbitrary. Then

$$|\nabla(u - \tilde{u})|^2_\Omega + |p - \tilde{p}|^2_{\mathsf{D}(\Omega)} = |f + \operatorname{div} \tilde{p}|^2_\Omega + 2\langle f + \operatorname{div} \tilde{p}, u - \tilde{u}\rangle_\Omega + |\tilde{p} - \nabla \tilde{u}|^2_\Omega$$

Find the primal variable $u \in \mathsf{H}^1(\Omega)$ and the dual variable $p \in \mathsf{D}(\Omega)$ such that

$$\begin{aligned} p &= \nabla u && \text{in } \Omega, \\ -\operatorname{div} p + u &= f + u && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma, \end{aligned}$$

where $f \in \mathsf{L}^2(\Omega)$.

Let $(\tilde{u}, \tilde{p}) \in \mathsf{H}^1_\Gamma(\Omega) \times \mathsf{D}(\Omega)$ be arbitrary. Then

$$\begin{aligned} |\nabla(u - \tilde{u})|^2_\Omega + |p - \tilde{p}|^2_{\mathsf{D}(\Omega)} &= |f + \operatorname{div} \tilde{p}|^2_\Omega + 2\langle f + \operatorname{div} \tilde{p}, u - \tilde{u}\rangle_\Omega + |\tilde{p} - \nabla\tilde{u}|^2_\Omega \\ &\leq |f + \operatorname{div} \tilde{p}|^2_\Omega + 2|f + \operatorname{div} \tilde{p}|_\Omega \, |u - \tilde{u}|_\Omega + |\tilde{p} - \nabla\tilde{u}|^2_\Omega \\ &\leq |f + \operatorname{div} \tilde{p}|^2_\Omega + 2C_\mathrm{F}|f + \operatorname{div} \tilde{p}|_\Omega \, |\nabla(u - \tilde{u})|_\Omega + |\tilde{p} - \nabla\tilde{u}|^2_\Omega \\ &\leq |f + \operatorname{div} \tilde{p}|^2_\Omega + \gamma C^2_\mathrm{F}|f + \operatorname{div} \tilde{p}|^2_\Omega + \frac{1}{\gamma}|\nabla(u - \tilde{u})|^2_\Omega + |\tilde{p} - \nabla\tilde{u}|^2_\Omega \\ &\leq |f + \operatorname{div} \tilde{p}|^2_\Omega + 2C^2_\mathrm{F}|f + \operatorname{div} \tilde{p}|^2_\Omega + \frac{1}{2}|\nabla(u - \tilde{u})|^2_\Omega + |\tilde{p} - \nabla\tilde{u}|^2_\Omega \end{aligned}$$

Diffusion problem

Find the primal variable $u \in \mathsf{H}^1(\Omega)$ and the dual variable $p \in \mathsf{D}(\Omega)$ such that

$$\begin{aligned} p &= \nabla u && \text{in } \Omega, \\ -\operatorname{div} p + u &= f + u && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma, \end{aligned}$$

where $f \in \mathsf{L}^2(\Omega)$.

Let $(\tilde{u}, \tilde{p}) \in \mathsf{H}^1_\Gamma(\Omega) \times \mathsf{D}(\Omega)$ be arbitrary. Then

$$\frac{1}{2}|\nabla(u - \tilde{u})|^2_\Omega + |p - \tilde{p}|^2_{\mathsf{D}(\Omega)} \leq (1 + 2C_F^2)|f + \operatorname{div} \tilde{p}|^2_\Omega + |\tilde{p} - \nabla \tilde{u}|^2_\Omega$$

Diffusion problem

Find the primal variable $u \in \mathsf{H}^1(\Omega)$ and the dual variable $p \in \mathsf{D}(\Omega)$ such that

$$\begin{aligned} p &= \nabla u && \text{in } \Omega, \\ -\operatorname{div} p + u &= f + u && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma, \end{aligned}$$

where $f \in \mathsf{L}^2(\Omega)$.

Let $(\tilde{u}, \tilde{p}) \in \mathsf{H}^1_\Gamma(\Omega) \times \mathsf{D}(\Omega)$ be arbitrary. Then

$$|\nabla(u - \tilde{u})|^2_\Omega + |p - \tilde{p}|^2_{\mathsf{D}(\Omega)} \leq (1 + 4C_F^2)|f + \operatorname{div} \tilde{p}|^2_\Omega + 2|\tilde{p} - \nabla \tilde{u}|^2_\Omega$$

Diffusion problem

Find the primal variable $u \in \mathsf{H}^1(\Omega)$ and the dual variable $p \in \mathsf{D}(\Omega)$ such that

$$\begin{aligned} p &= \nabla u && \text{in } \Omega, \\ -\operatorname{div} p + u &= f + u && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma, \end{aligned}$$

where $f \in \mathsf{L}^2(\Omega)$.

Let $(\tilde{u}, \tilde{p}) \in \mathsf{H}^1_\Gamma(\Omega) \times \mathsf{D}(\Omega)$ be arbitrary. Then

$$|\nabla(u - \tilde{u})|^2_\Omega + |p - \tilde{p}|^2_{\mathsf{D}(\Omega)} \leq (1 + 4C_{\mathrm{F}}^2)|f + \operatorname{div} \tilde{p}|^2_\Omega + 2|\tilde{p} - \nabla \tilde{u}|^2_\Omega$$

On the other hand

$$|\tilde{p} - \nabla \tilde{u}|^2_\Omega = |\tilde{p} - p + \nabla(u - \tilde{u})|^2_\Omega \leq 2\left(|\tilde{p} - p|^2_\Omega + |\nabla(u - \tilde{u})|^2_\Omega\right)$$

Find the primal variable $u \in \mathsf{H}^1(\Omega)$ and the dual variable $p \in \mathsf{D}(\Omega)$ such that

$$\begin{aligned} p &= \nabla u && \text{in } \Omega, \\ -\operatorname{div} p + u &= f + u && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma, \end{aligned}$$

where $f \in \mathsf{L}^2(\Omega)$.

Let $(\tilde{u}, \tilde{p}) \in \mathsf{H}^1_\Gamma(\Omega) \times \mathsf{D}(\Omega)$ be arbitrary. Then

$$|\nabla(u - \tilde{u})|^2_\Omega + |p - \tilde{p}|^2_{\mathsf{D}(\Omega)} \leq (1 + 4C_F^2)|f + \operatorname{div} \tilde{p}|^2_\Omega + 2|\tilde{p} - \nabla\tilde{u}|^2_\Omega$$

So we have

$$\frac{1}{2}|\tilde{p} - \nabla\tilde{u}|^2_\Omega \leq |\tilde{p} - p|^2_\Omega + |\nabla(u - \tilde{u})|^2_\Omega$$

Diffusion problem

Find the primal variable $u \in \mathsf{H}^1(\Omega)$ and the dual variable $p \in \mathsf{D}(\Omega)$ such that

$$\begin{aligned} p &= \nabla u && \text{in } \Omega, \\ -\operatorname{div} p + u &= f + u && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma, \end{aligned}$$

where $f \in \mathsf{L}^2(\Omega)$.

Let $(\tilde{u}, \tilde{p}) \in \mathsf{H}^1_\Gamma(\Omega) \times \mathsf{D}(\Omega)$ be arbitrary. Then

$$|\nabla(u - \tilde{u})|^2_\Omega + |p - \tilde{p}|^2_{\mathsf{D}(\Omega)} \leq (1 + 4C_\mathrm{F}^2)|f + \operatorname{div} \tilde{p}|^2_\Omega + 2|\tilde{p} - \nabla\tilde{u}|^2_\Omega$$

So we have

$$\frac{1}{2}|\tilde{p} - \nabla\tilde{u}|^2_\Omega + |f + \operatorname{div} \tilde{p}|^2_\Omega \leq |\tilde{p} - p|^2_\Omega + |\operatorname{div}(\tilde{p} - p)|^2_\Omega + |\nabla(u - \tilde{u})|^2_\Omega$$

Diffusion problem

Find the primal variable $u \in \mathsf{H}^1(\Omega)$ and the dual variable $p \in \mathsf{D}(\Omega)$ such that

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$$|\nabla(u - \tilde{u})|^2_\Omega + |p - \tilde{p}|^2_{\mathsf{D}(\Omega)} \leq (1 + 4C_{\mathrm{F}}^2)|f + \operatorname{div} \tilde{p}|^2_\Omega + 2|\tilde{p} - \nabla \tilde{u}|^2_\Omega$$

So we have

$$\frac{1}{2}|\tilde{p} - \nabla \tilde{u}|^2_\Omega + |f + \operatorname{div} \tilde{p}|^2_\Omega \leq |p - \tilde{p}|^2_{\mathsf{D}(\Omega)} + |\nabla(u - \tilde{u})|^2_\Omega$$

Diffusion problem

Find the primal variable $u \in \mathsf{H}^1(\Omega)$ and the dual variable $p \in \mathsf{D}(\Omega)$ such that

$$\begin{aligned} p &= \nabla u && \text{in } \Omega, \\ -\operatorname{div} p + u &= f + u && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma, \end{aligned}$$

where $f \in \mathsf{L}^2(\Omega)$.

Let $(\tilde{u}, \tilde{p}) \in \mathsf{H}^1_\Gamma(\Omega) \times \mathsf{D}(\Omega)$ be arbitrary. Then

$$\begin{aligned} |f + \operatorname{div} \tilde{p}|^2 + \frac{1}{2}|\tilde{p} - \nabla\tilde{u}|^2 \\ \leq |\nabla(u - \tilde{u})|^2 + |p - \tilde{p}|^2_{\mathsf{D}} \\ \leq (1 + 4C_F^2)|f + \operatorname{div} \tilde{p}|^2 + 2|\tilde{p} - \nabla\tilde{u}|^2 \end{aligned}$$

Summary: static problems

(Let Ω be a bounded Lipschitz domain and the material parameters α, ρ be uniformly positive definite, bounded, and self-adjoint.)

- I. Anjam and D. Pauly. *Functional a posteriori error control for conforming mixed approximations of coercive problems with lower order terms*, 2016 (accepted)

Reaction-diffusion: error equality

$$\begin{aligned} p &= \alpha\nabla u && \text{in } \Omega, \\ -\operatorname{div} p + \rho u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_D, \\ p \cdot n &= 0 && \text{on } \Gamma_N. \end{aligned}$$

$$\begin{aligned} &|u - \tilde{u}|_\rho^2 + |\nabla(u - \tilde{u})|_\alpha^2 + |p - \tilde{p}|_{\alpha^{-1}}^2 + |\operatorname{div}(p - \tilde{p})|_{\rho^{-1}}^2 \\ &\quad = |f - \rho\tilde{u} + \operatorname{div}\tilde{p}|_{\rho^{-1}}^2 + |\tilde{p} - \alpha\nabla\tilde{u}|_{\alpha^{-1}}^2 \end{aligned}$$

Complex valued "reaction-diffusion": two-sided estimate
(i is the imaginary unit, w belongs to $\mathbb{R} \setminus \{0\}$)

$$\begin{aligned} p &= \alpha\nabla u && \text{in } \Omega, \\ -\operatorname{div} p + i\omega\rho u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_D, \\ p \cdot n &= 0 && \text{on } \Gamma_N. \end{aligned}$$

$$\begin{aligned} &\tfrac{\sqrt{2}}{\sqrt{2}+1}\left(|f - i\omega\rho\tilde{u} + \operatorname{div}\tilde{p}|_{(|\omega|\rho)^{-1}}^2 + |\tilde{p} - \alpha\nabla\tilde{u}|_{\alpha^{-1}}^2\right) \\ &\quad \leq |u - \tilde{u}|_{|\omega|\rho}^2 + |\nabla(u - \tilde{u})|_\alpha^2 + |p - \tilde{p}|_{\alpha^{-1}}^2 + |\operatorname{div}(p - \tilde{p})|_{(|\omega|\rho)^{-1}}^2 \\ &\quad \leq \tfrac{\sqrt{2}}{\sqrt{2}-1}\left(|f - i\omega\rho\tilde{u} + \operatorname{div}\tilde{p}|_{(|\omega|\rho)^{-1}}^2 + |\tilde{p} - \alpha\nabla\tilde{u}|_{\alpha^{-1}}^2\right) \end{aligned}$$

- P. Neittaanmäki and S. Repin. *Reliable methods for computer simulation. Error control and a posteriori estimates*. Elsevier, New York, 2004.
- Z. Cai and S. Zhang. *Flux recovery and a posteriori error estimators: Conforming elements for scalar elliptic equations*. SIAM J. Numer. Anal., 48(2): 578–602, 2010

Summary: static problems

(Let Ω be a bounded Lipschitz domain and the material parameters α, ρ be uniformly positive definite, bounded, and self-adjoint.)

- I. Anjam. *Functional a posteriori error control for conforming mixed approximations of the reaction-convection-diffusion equations*, 2016 (arXiv:1601.05310)

Reaction-convection-diffusion: depending on the divergence of the convection vector b an error estimate or two-sided bound is obtained. For example, if $\rho - \operatorname{div} b > 0$, we have the error equality

$$\begin{aligned} p &= \alpha\nabla u && \text{in } \Omega, \\ -\operatorname{div} p + b\cdot\nabla u + \rho u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma. \end{aligned}$$

$$\begin{aligned} &|u-\tilde{u}|^2_{\rho-\operatorname{div} b} + |\nabla(u-\tilde{u})|^2_\alpha \\ &+|p-\tilde{p}|^2_{\alpha^{-1}} + |b\cdot\nabla(u-\tilde{u}) - \operatorname{div}(p-\tilde{p})|^2_{\rho^{-1}} \\ &\qquad = |f-\rho\tilde{u} - b\cdot\nabla\tilde{u} + \operatorname{div}\tilde{p}|^2_{\rho^{-1}} + |\tilde{p}-\alpha\nabla\tilde{u}|^2_{\alpha^{-1}} \end{aligned}$$

- And in this talk (joint work with D. Pauly)

Diffusion: two-sided estimate

$$\begin{aligned} p &= \alpha\nabla u && \text{in } \Omega, \\ -\operatorname{div} p &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_D, \\ p\cdot n &= 0 && \text{on } \Gamma_N. \end{aligned}$$

$$\begin{aligned} &|f+\operatorname{div}\tilde{p}|^2 + \tfrac{1}{2}|\tilde{p}-\alpha\nabla\tilde{u}|^2_{\alpha^{-1}} \\ &\qquad \le |\nabla(u-\tilde{u})|^2_\alpha + |p-\tilde{p}|^2_{\alpha^{-1}} + |\operatorname{div}(p-\tilde{p})|^2 \\ &\qquad\qquad \le (1+4C)|f+\operatorname{div}\tilde{p}|^2 + 2|\tilde{p}-\alpha\nabla\tilde{u}|^2_{\alpha^{-1}} \end{aligned}$$

- S. Repin, S. Sauter, and A. Smolianski. *Two-sided a posteriori error estimates for mixed formulations of elliptic problems*. SIAM J. Numer. Anal., 45(3): 928–945, 2007

We extend the notation of the previous section:

We denote by $\Omega_T := \Omega \times (0, T)$ the space-time domain, where $(0, T)$ is a time-interval with $T > 0$. Its mantle boundary is denoted by $\Gamma_T := \partial\Omega \times (0, T)$.

$$\langle \cdot , \cdot \rangle_{\Omega_T} := \int_0^T \langle \cdot , \cdot \rangle_\Omega \, \mathrm{d}t \qquad \text{and} \qquad | \cdot |_{\Omega_T} := \sqrt{\langle \cdot , \cdot \rangle_{\Omega_T}}$$

the inner product and norm for scalar or vector valued functions in $\mathsf{L}^2(\Omega_T)$.
We define the following Sobolev spaces

$$\begin{aligned}
\mathsf{H}^{1,0}(\Omega_T) &:= \{\varphi \in \mathsf{L}^2(\Omega_T) \mid \nabla\varphi \in \mathsf{L}^2(\Omega_T)\}, \\
\mathsf{H}^{1,1}(\Omega_T) &:= \{\varphi \in \mathsf{H}^{1,0}(\Omega_T) \mid \partial_t\varphi \in \mathsf{L}^2(\Omega_T)\}, \\
\mathsf{W}(\Omega_T) &:= \{\varphi \in \mathsf{H}^{1,1}(\Omega_T) \mid \Delta\varphi \in \mathsf{L}^2(\Omega_T)\}, \\
\mathsf{D}(\Omega_T) &:= \{\psi \in \mathsf{L}^2(\Omega_T) \mid \operatorname{div}\psi \in \mathsf{L}^2(\Omega_T)\},
\end{aligned}$$

where ∇, div, and Δ are spatial differential operators, and ∂_t is the derivative with respect to time. These are Hilbert spaces equipped with their respective graph norms.

Functions fanishing on the mantle boundary are denoted by

$$\mathsf{H}^{1,0}_{\Gamma_T}(\Omega_T), \qquad \mathsf{H}^{1,1}_{\Gamma_T}(\Omega_T), \qquad \text{and} \qquad \mathsf{W}_{\Gamma_T}(\Omega_T).$$

Time-dependent reaction-diffusion problem

Find the primal variable $u \in \mathsf{H}^{1,1}(\Omega_T)$ and the dual variable $p \in \mathsf{D}(\Omega_T)$ such that

$$\begin{aligned} p &= \nabla u && \text{in } \Omega_T, \\ \partial_t u - \operatorname{div} p + u &= f && \text{in } \Omega_T, \\ u &= 0 && \text{on } \Gamma_T, \\ u(\cdot, 0) &= u_0 && \text{in } \Omega, \end{aligned}$$

where $f = f(x,t) \in \mathsf{L}^2(\Omega_T)$ and the initial value $u_0 = u_0(x) \in \mathsf{L}^2(\Omega)$.

Find the primal variable $u \in \mathsf{H}^{1,1}(\Omega_T)$ and the dual variable $p \in \mathsf{D}(\Omega_T)$ such that

$$\begin{aligned} p &= \nabla u && \text{in } \Omega_T, \\ \partial_t u - \operatorname{div} p + u &= f && \text{in } \Omega_T, \\ u &= 0 && \text{on } \Gamma_T, \\ u(\cdot, 0) &= u_0 && \text{in } \Omega, \end{aligned}$$

where $f = f(x,t) \in \mathsf{L}^2(\Omega_T)$ and the initial value $u_0 = u_0(x) \in \mathsf{L}^2(\Omega)$.

Let $(\tilde{u}, \tilde{p}) \in \mathsf{H}^{1,1}_{\Gamma_T}(\Omega_T) \times \mathsf{D}(\Omega_T)$ be arbitrary. Then

$$\begin{aligned} &|f - \tilde{u} + \operatorname{div} \tilde{p} - \partial_t \tilde{u}|^2_{\Omega_T} + |\tilde{p} - \nabla \tilde{u}|^2_{\Omega_T} \\ &= |u - \tilde{u} + \operatorname{div}(\tilde{p} - p) + \partial_t (u - \tilde{u})|^2_{\Omega_T} + |\tilde{p} - p + \nabla (u - \tilde{u})|^2_{\Omega_T} \end{aligned}$$

Find the primal variable $u \in \mathsf{H}^{1,1}(\Omega_T)$ and the dual variable $p \in \mathsf{D}(\Omega_T)$ such that

$$\begin{aligned} p &= \nabla u && \text{in } \Omega_T, \\ \partial_t u - \operatorname{div} p + u &= f && \text{in } \Omega_T, \\ u &= 0 && \text{on } \Gamma_T, \\ u(\cdot, 0) &= u_0 && \text{in } \Omega, \end{aligned}$$

where $f = f(x,t) \in \mathsf{L}^2(\Omega_T)$ and the initial value $u_0 = u_0(x) \in \mathsf{L}^2(\Omega)$.

Let $(\tilde{u}, \tilde{p}) \in \mathsf{H}^{1,1}_{\Gamma_T}(\Omega_T) \times \mathsf{D}(\Omega_T)$ be arbitrary. Then

$$\begin{aligned} &|f - \tilde{u} + \operatorname{div} \tilde{p} - \partial_t \tilde{u}|^2_{\Omega_T} + |\tilde{p} - \nabla \tilde{u}|^2_{\Omega_T} \\ &= |u - \tilde{u} + \operatorname{div}(\tilde{p} - p) + \partial_t (u - \tilde{u})|^2_{\Omega_T} + |\tilde{p} - p + \nabla (u - \tilde{u})|^2_{\Omega_T} \\ &= |u - \tilde{u}|^2_{\Omega_T} + |\operatorname{div}(\tilde{p} - p) + \partial_t (u - \tilde{u})|^2_{\Omega_T} \\ &\qquad + 2\langle u - \tilde{u}, \operatorname{div}(\tilde{p} - p)\rangle_{\Omega_T} + 2\langle u - \tilde{u}, \partial_t (u - \tilde{u})\rangle_{\Omega_T} \\ &\qquad + |\tilde{p} - p|^2_{\Omega_T} + |\nabla (u - \tilde{u})|^2_{\Omega_T} + 2\langle \tilde{p} - p, \nabla (u - \tilde{u})\rangle_{\Omega_T} \end{aligned}$$

Find the primal variable $u \in \mathsf{H}^{1,1}(\Omega_T)$ and the dual variable $p \in \mathsf{D}(\Omega_T)$ such that

$$\begin{aligned} p &= \nabla u && \text{in } \Omega_T, \\ \partial_t u - \operatorname{div} p + u &= f && \text{in } \Omega_T, \\ u &= 0 && \text{on } \Gamma_T, \\ u(\cdot, 0) &= u_0 && \text{in } \Omega, \end{aligned}$$

where $f = f(x,t) \in \mathsf{L}^2(\Omega_T)$ and the initial value $u_0 = u_0(x) \in \mathsf{L}^2(\Omega)$.

Let $(\tilde{u}, \tilde{p}) \in \mathsf{H}^{1,1}_{\Gamma_T}(\Omega_T) \times \mathsf{D}(\Omega_T)$ be arbitrary. Then

$$\begin{aligned} & |f - \tilde{u} + \operatorname{div} \tilde{p} - \partial_t \tilde{u}|^2_{\Omega_T} + |\tilde{p} - \nabla \tilde{u}|^2_{\Omega_T} \\ &= |u - \tilde{u} + \operatorname{div}(\tilde{p} - p) + \partial_t(u - \tilde{u})|^2_{\Omega_T} + |\tilde{p} - p + \nabla(u - \tilde{u})|^2_{\Omega_T} \\ &= |u - \tilde{u}|^2_{\Omega_T} + |\operatorname{div}(\tilde{p} - p) + \partial_t(u - \tilde{u})|^2_{\Omega_T} \\ &\qquad + \cancel{2\langle u - \tilde{u}, \operatorname{div}(\tilde{p} - p)\rangle_{\Omega_T}} + 2\langle u - \tilde{u}, \partial_t(u - \tilde{u})\rangle_{\Omega_T} \\ &\qquad + |\tilde{p} - p|^2_{\Omega_T} + |\nabla(u - \tilde{u})|^2_{\Omega_T} + \cancel{2\langle \tilde{p} - p, \nabla(u - \tilde{u})\rangle_{\Omega_T}} \end{aligned}$$

For any $\varphi \in \mathsf{L}^2(\Omega_T)$, for which $\partial_t \varphi \in \mathsf{L}^2(\Omega_T)$, we have

$$\langle \partial_t \varphi, \varphi \rangle_{\Omega_T} = \int_\Omega \int_0^T \partial_t \varphi \, \varphi \, \mathrm{d}t \, \mathrm{d}x = \frac{1}{2} \int_\Omega \left(|\varphi(x,T)|^2 - |\varphi(x,0)|^2 \right) \mathrm{d}x = \frac{1}{2} \left(|\varphi(\cdot, T)|^2_\Omega - |\varphi(\cdot, 0)|^2_\Omega \right).$$

Time-dependent reaction-diffusion problem

Find the primal variable $u \in \mathsf{H}^{1,1}(\Omega_T)$ and the dual variable $p \in \mathsf{D}(\Omega_T)$ such that

$$
\begin{aligned}
p &= \nabla u && \text{in } \Omega_T, \\
\partial_t u - \operatorname{div} p + u &= f && \text{in } \Omega_T, \\
u &= 0 && \text{on } \Gamma_T, \\
u(\cdot, 0) &= u_0 && \text{in } \Omega,
\end{aligned}
$$

where $f = f(x,t) \in \mathsf{L}^2(\Omega_T)$ and the initial value $u_0 = u_0(x) \in \mathsf{L}^2(\Omega)$.

Let $(\tilde{u}, \tilde{p}) \in \mathsf{H}^{1,1}_{\Gamma_T}(\Omega_T) \times \mathsf{D}(\Omega_T)$ be arbitrary. Then

$$
\begin{aligned}
&|f - \tilde{u} + \operatorname{div} \tilde{p} - \partial_t \tilde{u}|^2_{\Omega_T} + |\tilde{p} - \nabla \tilde{u}|^2_{\Omega_T} \\
&= |u - \tilde{u} + \operatorname{div}(\tilde{p} - p) + \partial_t (u - \tilde{u})|^2_{\Omega_T} + |\tilde{p} - p + \nabla(u - \tilde{u})|^2_{\Omega_T} \\
&= |u - \tilde{u}|^2_{\Omega_T} + |\operatorname{div}(\tilde{p} - p) + \partial_t(u - \tilde{u})|^2_{\Omega_T} \\
&\qquad + \cancel{2\langle u - \tilde{u}, \operatorname{div}(\tilde{p} - p)\rangle_{\Omega_T}} + |(u - \tilde{u})(\cdot, T)|^2_{\Omega} - |(u - \tilde{u})(\cdot, 0)|^2_{\Omega} \\
&\qquad + |\tilde{p} - p|^2_{\Omega_T} + |\nabla(u - \tilde{u})|^2_{\Omega_T} + \cancel{2\langle \tilde{p} - p, \nabla(u - \tilde{u})\rangle_{\Omega_T}}
\end{aligned}
$$

For any $\varphi \in \mathsf{L}^2(\Omega_T)$, for which $\partial_t \varphi \in \mathsf{L}^2(\Omega_T)$, we have

$$
\langle \partial_t \varphi, \varphi \rangle_{\Omega_T} = \int_\Omega \int_0^T \partial_t \varphi \, \varphi \, \mathrm{d}t \, \mathrm{d}x = \frac{1}{2} \int_\Omega \left(|\varphi(x,T)|^2 - |\varphi(x,0)|^2 \right) \mathrm{d}x = \frac{1}{2} \left(|\varphi(\cdot, T)|^2_\Omega - |\varphi(\cdot, 0)|^2_\Omega \right).
$$

Time-dependent reaction-diffusion problem

Find the primal variable $u \in \mathsf{H}^{1,1}(\Omega_T)$ and the dual variable $p \in \mathsf{D}(\Omega_T)$ such that

$$\begin{aligned} p &= \nabla u && \text{in } \Omega_T, \\ \partial_t u - \operatorname{div} p + u &= f && \text{in } \Omega_T, \\ u &= 0 && \text{on } \Gamma_T, \\ u(\cdot, 0) &= u_0 && \text{in } \Omega, \end{aligned}$$

where $f = f(x,t) \in \mathsf{L}^2(\Omega_T)$ and the initial value $u_0 = u_0(x) \in \mathsf{L}^2(\Omega)$.

Let $(\tilde{u}, \tilde{p}) \in \mathsf{H}^{1,1}_{\Gamma_T}(\Omega_T) \times \mathsf{D}(\Omega_T)$ be arbitrary. Then

$$\begin{aligned} |u - \tilde{u}|^2_{\mathsf{H}^{1,0}(\Omega_T)} + |p - \tilde{p}|^2_{\Omega_T} + |\operatorname{div}(\tilde{p} - p) + \partial_t(u - \tilde{u})|^2_{\Omega_T} + |(u - \tilde{u})(\cdot, T)|^2_{\Omega} \\ = |f - \tilde{u} + \operatorname{div} \tilde{p} - \partial_t \tilde{u}|^2_{\Omega_T} + |\tilde{p} - \nabla \tilde{u}|^2_{\Omega_T} + |u_0 - \tilde{u}(\cdot, 0)|^2_{\Omega} \end{aligned}$$

Time-dependent reaction-diffusion problem

Find the primal variable $u \in \mathsf{H}^{1,1}(\Omega_T)$ and the dual variable $p \in \mathsf{D}(\Omega_T)$ such that

$$\begin{aligned} p &= \nabla u && \text{in } \Omega_T, \\ \partial_t u - \operatorname{div} p + u &= f && \text{in } \Omega_T, \\ u &= 0 && \text{on } \Gamma_T, \\ u(\cdot, 0) &= u_0 && \text{in } \Omega, \end{aligned}$$

where $f = f(x,t) \in \mathsf{L}^2(\Omega_T)$ and the initial value $u_0 = u_0(x) \in \mathsf{L}^2(\Omega)$.

Let $(\tilde{u}, \tilde{p}) \in \mathsf{H}^{1,1}_{\Gamma_T}(\Omega_T) \times \mathsf{D}(\Omega_T)$ be arbitrary. Then

$$\begin{aligned} |u - \tilde{u}|^2_{\mathsf{H}^{1,0}(\Omega_T)} + |p - \tilde{p}|^2_{\Omega_T} + |\operatorname{div}(\tilde{p} - p) + \partial_t(u - \tilde{u})|^2_{\Omega_T} + |(u - \tilde{u})(\cdot, T)|^2_{\Omega} \\ = |f - \tilde{u} + \operatorname{div} \tilde{p} - \partial_t \tilde{u}|^2_{\Omega_T} + |\tilde{p} - \nabla \tilde{u}|^2_{\Omega_T} + |u_0 - \tilde{u}(\cdot, 0)|^2_{\Omega} \end{aligned}$$

Let $u_0 \in \mathsf{H}^1(\Omega)$ and $\tilde{u} \in \mathsf{W}_{\Gamma_T}(\Omega_T)$ be arbitrary. Then we can set $\tilde{p} = \nabla \tilde{u}$ above, and obtain

$$\begin{aligned} |u - \tilde{u}|^2_{\mathsf{H}^{1,0}(\Omega_T)} + |\nabla(u - \tilde{u})|^2_{\Omega_T} + |\Delta(\tilde{u} - u) + \partial_t(u - \tilde{u})|^2_{\Omega_T} + |(u - \tilde{u})(\cdot, T)|^2_{\Omega} \\ = |f - \tilde{u} + \Delta \tilde{u} - \partial_t \tilde{u}|^2_{\Omega_T} + |u_0 - \tilde{u}(\cdot, 0)|^2_{\Omega} \end{aligned}$$

$$\mathsf{W}(\Omega_T) := \{\varphi \in \mathsf{H}^{1,1}(\Omega_T) \mid \Delta\varphi \in \mathsf{L}^2(\Omega_T)\}$$

$$\forall \varphi \in \mathsf{W}(\Omega_T) \qquad |-\Delta\varphi + \partial_t \varphi|^2_{\Omega_T} = |\Delta\varphi|^2_{\Omega_T} + |\partial_t\varphi|^2_{\Omega_T} + |\nabla\varphi(\cdot, T)|^2_{\Omega} - |\nabla\varphi(\cdot, 0)|^2_{\Omega}$$

Find the primal variable $u \in \mathsf{H}^{1,1}(\Omega_T)$ and the dual variable $p \in \mathsf{D}(\Omega_T)$ such that

$$\begin{aligned} p &= \nabla u && \text{in } \Omega_T, \\ \partial_t u - \operatorname{div} p + u &= f && \text{in } \Omega_T, \\ u &= 0 && \text{on } \Gamma_T, \\ u(\cdot, 0) &= u_0 && \text{in } \Omega, \end{aligned}$$

where $f = f(x,t) \in \mathsf{L}^2(\Omega_T)$ and the initial value $u_0 = u_0(x) \in \mathsf{L}^2(\Omega)$.

Let $(\tilde{u}, \tilde{p}) \in \mathsf{H}^{1,1}_{\Gamma_T}(\Omega_T) \times \mathsf{D}(\Omega_T)$ be arbitrary. Then

$$\begin{aligned} |u - \tilde{u}|^2_{\mathsf{H}^{1,0}(\Omega_T)} + |p - \tilde{p}|^2_{\Omega_T} + |\operatorname{div}(\tilde{p} - p) + \partial_t(u - \tilde{u})|^2_{\Omega_T} + |(u - \tilde{u})(\cdot, T)|^2_{\Omega} \\ = |f - \tilde{u} + \operatorname{div} \tilde{p} - \partial_t \tilde{u}|^2_{\Omega_T} + |\tilde{p} - \nabla \tilde{u}|^2_{\Omega_T} + |u_0 - \tilde{u}(\cdot, 0)|^2_{\Omega} \end{aligned}$$

Let $u_0 \in \mathsf{H}^1(\Omega)$ and $\tilde{u} \in \mathsf{W}_{\Gamma_T}(\Omega_T)$ be arbitrary. Then we can set $\tilde{p} = \nabla \tilde{u}$ above, and obtain

$$\begin{aligned} |u - \tilde{u}|^2_{\mathsf{H}^{1,1}(\Omega_T)} + |\nabla(u - \tilde{u})|^2_{\mathsf{D}(\Omega_T)} + |(u - \tilde{u})(\cdot, T)|^2_{\mathsf{H}^1(\Omega)} \\ = |f - \tilde{u} + \Delta \tilde{u} - \partial_t \tilde{u}|^2_{\Omega_T} + |u_0 - \tilde{u}(\cdot, 0)|^2_{\mathsf{H}^1(\Omega)} \end{aligned}$$

Find the primal variable $u \in \mathsf{H}^{1,1}(\Omega_T)$ and the dual variable $p \in \mathsf{D}(\Omega_T)$ such that

$$\begin{aligned} p &= \nabla u && \text{in } \Omega_T, \\ \partial_t u - \operatorname{div} p &= f && \text{in } \Omega_T, \\ u &= 0 && \text{on } \Gamma_T, \\ u(\cdot, 0) &= u_0 && \text{in } \Omega, \end{aligned}$$

where $f = f(x,t) \in \mathsf{L}^2(\Omega_T)$ and the initial value $u_0 = u_0(x) \in \mathsf{L}^2(\Omega)$.

Heat equation

Find the primal variable $u \in \mathsf{H}^{1,1}(\Omega_T)$ and the dual variable $p \in \mathsf{D}(\Omega_T)$ such that

$$\begin{aligned} p &= \nabla u && \text{in } \Omega_T, \\ \partial_t u - \operatorname{div} p + u &= f + u && \text{in } \Omega_T, \\ u &= 0 && \text{on } \Gamma_T, \\ u(\cdot, 0) &= u_0 && \text{in } \Omega, \end{aligned}$$

where $f = f(x,t) \in \mathsf{L}^2(\Omega_T)$ and the initial value $u_0 = u_0(x) \in \mathsf{L}^2(\Omega)$.

Let $(\tilde{u}, \tilde{p}) \in \mathsf{H}^{1,1}_{\Gamma_T}(\Omega_T) \times \mathsf{D}(\Omega_T)$ be arbitrary. Then

$$\begin{aligned} |u - \tilde{u}|^2_{\mathsf{H}^{1,0}(\Omega_T)} + |p - \tilde{p}|^2_{\Omega_T} + |\operatorname{div}(\tilde{p} - p) + \partial_t(u - \tilde{u})|^2_{\Omega_T} + |(u - \tilde{u})(\cdot, T)|^2_{\Omega} \\ = |f + u - \tilde{u} + \operatorname{div} \tilde{p} - \partial_t \tilde{u}|^2_{\Omega_T} + |\tilde{p} - \nabla \tilde{u}|^2_{\Omega_T} + |u_0 - \tilde{u}(\cdot, 0)|^2_{\Omega} \end{aligned}$$

Find the primal variable $u \in \mathsf{H}^{1,1}(\Omega_T)$ and the dual variable $p \in \mathsf{D}(\Omega_T)$ such that

$$\begin{aligned} p &= \nabla u && \text{in } \Omega_T, \\ \partial_t u - \operatorname{div} p + u &= f + u && \text{in } \Omega_T, \\ u &= 0 && \text{on } \Gamma_T, \\ u(\cdot, 0) &= u_0 && \text{in } \Omega, \end{aligned}$$

where $f = f(x,t) \in \mathsf{L}^2(\Omega_T)$ and the initial value $u_0 = u_0(x) \in \mathsf{L}^2(\Omega)$.

Let $(\tilde{u}, \tilde{p}) \in \mathsf{H}^{1,1}_{\Gamma_T}(\Omega_T) \times \mathsf{D}(\Omega_T)$ be arbitrary. Then

$$\begin{aligned} &|f + \operatorname{div} \tilde{p} - \partial_t \tilde{u}|^2_{\Omega_T} + \frac{1}{2} |\tilde{p} - \nabla \tilde{u}|^2_{\Omega_T} \\ &\quad \leq |\nabla(u - \tilde{u})|^2_{\Omega_T} + |p - \tilde{p}|^2_{\Omega_T} + |\operatorname{div}(\tilde{p} - p) + \partial_t(u - \tilde{u})|^2_{\Omega_T} + |(u - \tilde{u})(\cdot, T)|^2_{\Omega} \\ &\quad \leq \left(1 + 4C_{\mathrm{F}}^2\right) |f + \operatorname{div} \tilde{p} - \partial_t \tilde{u}|^2_{\Omega_T} + 2\left(|\tilde{p} - \nabla \tilde{u}|^2_{\Omega_T} + |u_0 - \tilde{u}(\cdot, 0)|^2_{\Omega}\right). \end{aligned}$$

Heat equation

Find the primal variable $u \in \mathsf{H}^{1,1}(\Omega_T)$ and the dual variable $p \in \mathsf{D}(\Omega_T)$ such that

$$\begin{aligned} p &= \nabla u && \text{in } \Omega_T, \\ \partial_t u - \operatorname{div} p + u &= f + u && \text{in } \Omega_T, \\ u &= 0 && \text{on } \Gamma_T, \\ u(\cdot, 0) &= u_0 && \text{in } \Omega, \end{aligned}$$

where $f = f(x,t) \in \mathsf{L}^2(\Omega_T)$ and the initial value $u_0 = u_0(x) \in \mathsf{L}^2(\Omega)$.

Let $(\tilde{u}, \tilde{p}) \in \mathsf{H}^{1,1}_{\Gamma_T}(\Omega_T) \times \mathsf{D}(\Omega_T)$ be arbitrary. Then

$$\begin{aligned} &|f + \operatorname{div} \tilde{p} - \partial_t \tilde{u}|^2_{\Omega_T} + \frac{1}{2}|\tilde{p} - \nabla \tilde{u}|^2_{\Omega_T} \\ &\quad \leq |\nabla(u - \tilde{u})|^2_{\Omega_T} + |p - \tilde{p}|^2_{\Omega_T} + |\operatorname{div}(\tilde{p} - p) + \partial_t(u - \tilde{u})|^2_{\Omega_T} + |(u - \tilde{u})(\cdot, T)|^2_{\Omega} \\ &\qquad \leq \left(1 + 4C_{\mathrm{F}}^2\right) |f + \operatorname{div} \tilde{p} - \partial_t \tilde{u}|^2_{\Omega_T} + 2\left(|\tilde{p} - \nabla \tilde{u}|^2_{\Omega_T} + |u_0 - \tilde{u}(\cdot, 0)|^2_{\Omega}\right). \end{aligned}$$

Let $u_0 \in \mathsf{H}^1(\Omega)$ and $\tilde{u} \in \mathsf{W}_{\Gamma_T}(\Omega_T)$ be arbitrary. Then

$$\begin{aligned} &|\partial_t(u - \tilde{u})|^2_{\Omega_T} + |\Delta(u - \tilde{u})|^2_{\Omega_T} + |\nabla(u - \tilde{u})(\cdot, T)|^2_{\Omega} \\ &\qquad = |f + \Delta \tilde{u} - \partial_t \tilde{u}|^2_{\Omega_T} + |\nabla(u_0 - \tilde{u}(\cdot, 0))|^2_{\Omega}. \end{aligned}$$

$$\mathsf{W}(\Omega_T) := \{\varphi \in \mathsf{H}^{1,1}(\Omega_T) \mid \Delta\varphi \in \mathsf{L}^2(\Omega_T)\}$$

$$\forall \varphi \in \mathsf{W}(\Omega_T) \qquad |-\Delta\varphi + \partial_t\varphi|^2_{\Omega_T} = |\Delta\varphi|^2_{\Omega_T} + |\partial_t\varphi|^2_{\Omega_T} + |\nabla\varphi(\cdot, T)|^2_{\Omega} - |\nabla\varphi(\cdot, 0)|^2_{\Omega}$$

- This talk (joint work with D. Pauly)

Time-dependent reaction-diffusion: error equality

$$\begin{aligned} p &= \nabla u && \text{in } \Omega_T, \\ \partial_t u - \operatorname{div} p + u &= f && \text{in } \Omega_T, \\ u &= 0 && \text{on } \Gamma_T, \\ u(\cdot,0) &= u_0 && \text{in } \Omega \end{aligned}$$

$$\begin{aligned} &|u-\tilde{u}|^2_{\mathsf{H}^{1,0}(\Omega_T)} + |p-\tilde{p}|^2_{\Omega_T} \\ &+|\operatorname{div}(\tilde{p}-p) + \partial_t(u-\tilde{u})|^2_{\Omega_T} + |(u-\tilde{u})(\cdot,T)|^2_{\Omega} \\ &\quad = |f-\tilde{u}+\operatorname{div}\tilde{p}-\partial_t\tilde{u}|^2_{\Omega_T} + |\tilde{p}-\nabla\tilde{u}|^2_{\Omega_T} + |u_0-\tilde{u}(\cdot,0)|^2_{\Omega} \end{aligned}$$

Heat equation: two-sided estimate (and error equality)

$$\begin{aligned} p &= \nabla u && \text{in } \Omega_T, \\ \partial_t u - \operatorname{div} p &= f && \text{in } \Omega_T, \\ u &= 0 && \text{on } \Gamma_T, \\ u(\cdot,0) &= u_0 && \text{in } \Omega \end{aligned}$$

$$\begin{aligned} &|f+\operatorname{div}\tilde{p}-\partial_t\tilde{u}|^2_{\Omega_T} + \tfrac{1}{2}|\tilde{p}-\nabla\tilde{u}|^2_{\Omega_T} \\ &\quad \le |\nabla(u-\tilde{u})|^2_{\Omega_T} + |p-\tilde{p}|^2_{\Omega_T} \\ &\qquad +|\operatorname{div}(\tilde{p}-p)+\partial_t(u-\tilde{u})|^2_{\Omega_T} + |(u-\tilde{u})(\cdot,T)|^2_{\Omega} \\ &\qquad \le \left(1+4C_{\mathsf{F}}^2\right)|f+\operatorname{div}\tilde{p}-\partial_t\tilde{u}|^2_{\Omega_T} \\ &\qquad\quad +2\left(|\tilde{p}-\nabla\tilde{u}|^2_{\Omega_T} + |u_0-\tilde{u}(\cdot,0)|^2_{\Omega}\right) \end{aligned}$$

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Thank You for your attention!