Time-Harmonic Maxwell equations with sign-changing coefficients

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Reference : "T-coercivity for the Maxwell problem with sign-changing coefficients", A-S BBD, P. Ciarlet Jr and L. Chesnel, in Communications in PDE, 2014

Here for simplicity, we suppose:

- $\mu = 1$
- Ω topologically trivial

but a more general case is treated in the paper.

We consider the Maxwell problem, in a bounded domain $\Omega \subset \mathbb{R}^3$ (topologically trivial) at a pulsation $\omega > 0$:

$$(P) \begin{cases} \text{Find } E \in H(\mathrm{rot}, \Omega) \text{ such that} \\ \mathrm{rot}\,\mathrm{rot}\, E - \omega^2 \varepsilon E = J & (\Omega) \\ E \times n = 0 & (\partial\Omega) \end{cases}$$

assuming $J \in L^2(\Omega)$ with $\mathrm{div}\, J = 0$ in Ω.

We suppose $\varepsilon \in L^\infty(\Omega)$ and $\frac{1}{\varepsilon} \in L^\infty(\Omega)$, so that ε can change its sign but cannot vanish.

The properties of (P) are strongly related to the properties of the following scalar problem:

$$(P_\varphi) \begin{cases} \text{Find } \varphi \in H^1_0(\Omega) \text{ such that} \\ \mathrm{div}(\varepsilon \nabla \varphi) = f & (\Omega) \end{cases}$$

for $f \in H^{-1}(\Omega)$.

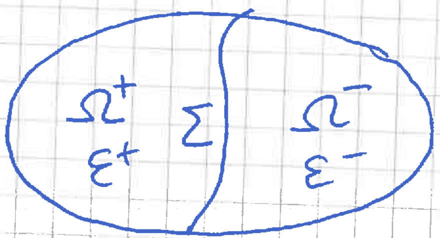
1 – Summary of some known results concerning (P_φ)

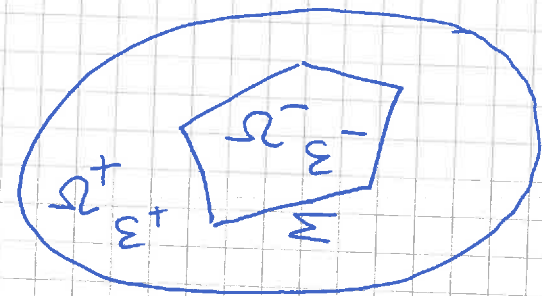
We will present the results for the specific configuration of a transmission problem between a "positive" medium and a "negative" one. More precisely we suppose:

$$\overline{\Omega} = \overline{\Omega^+} \cup \overline{\Omega^-} \qquad \overline{\Omega^+} \cap \overline{\Omega^-} = \Sigma$$

$$\varepsilon = \varepsilon^+ \text{ constant in } \Omega^+$$

$$\varepsilon = \varepsilon^- \text{ constant in } \Omega^-$$





Many people have contributed to the understanding of this problem: Costabel & Stefan, Dauge & Texier, Nicaise & Vehel, Ramdani, Nguyen ... and my team.

The main results are:

1) If $\frac{\varepsilon_+}{\varepsilon_-} = -1$, (P_φ) is not Fredholm.

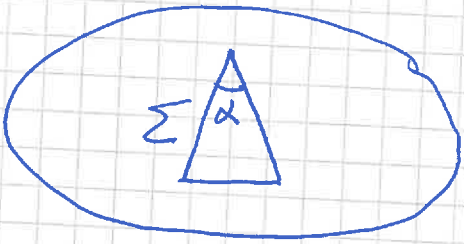
In particular, in the symmetric case: if $\varepsilon_+ = -\varepsilon_-$, there is a kernel of ∞ dimension!

(If $\varepsilon_+ \neq -\varepsilon_-$, the problem (P_φ) is well-posed.)

2) If $\left|\frac{\varepsilon_+}{\varepsilon_-}\right|$ is large enough or small enough, (P_φ) is well-posed.

3) If Σ is "smooth" and if $\frac{\varepsilon_+}{\varepsilon_-} \neq -1$, (P_φ) is Fredholm (with possibly a finite dimensional kernel).

4) In 2D, if Σ is polygonal, (P_φ) is Fredholm if and only if



$$\frac{\varepsilon_+}{\varepsilon_-} \notin \left[-I_\alpha, -\frac{1}{I_\alpha}\right]$$

where $I_\alpha = \frac{2\pi - \alpha}{\alpha}$ and α is the smallest angle of Σ.

For example: if Σ is a square, $\alpha = \frac{\pi}{2}$, $I_\alpha = 3$.

For $\frac{\varepsilon_+}{\varepsilon_-} \in \left]-I_\alpha, -\frac{1}{I_\alpha}\right[$, one can recover Fredholmness, but in a functional space larger than $H^1_0(\Omega)$, including strong singularities at the corners.

5) Partial results exist for a polyhedral Σ in 3D. For instance, for the Fichera cube, (P_φ) is well posed if $\frac{\varepsilon_+}{\varepsilon_-} \notin \left[-7, -\frac{1}{7}\right]$, but this is not optimal.

2 - The Maxwell problem (P) when (P_φ) is well-posed

We suppose in this part that (P_φ) is well-posed.
We introduce some classical notations:

$$H_N(\mathrm{rot}) = \{u \in L^2(\Omega);\ \mathrm{rot}\, u \in L^2(\Omega),\ u \times n|_{\partial\Omega} = 0\}$$

$$X_N(\varepsilon) = \left\{ u \in H_N(\mathrm{rot}) ;\ \int_\Omega \varepsilon u \nabla\varphi = 0 \quad \forall \varphi \in H_0^1(\Omega) \right\}$$

$$\left(u \in X_N(\varepsilon) \Rightarrow \mathrm{div}(\varepsilon u) = 0 \right)$$

$$X_N = X_N(1) = \{ u \in H_N(\mathrm{rot}),\ \mathrm{div}\, u = 0 \}$$

$$X_T = \{ u \in L^2(\Omega),\ \mathrm{rot}\, u \in L^2(\Omega),\ \mathrm{div}\, u = 0,\ u.n|_{\partial\Omega} = 0 \}$$

A natural way to formulate problem (P) is as follows:

$$(P_E) \begin{cases} \text{Find } E \in X_N(\varepsilon) \text{ such that } \forall E' \in X_N(\varepsilon) \\ \int_\Omega \mathrm{rot} E\ \mathrm{rot} E' - \omega^2 \varepsilon E E' = \int_\Omega J E' \end{cases}$$

Then the mathematical analysis consists in 2 steps:

<u>Step 1</u>: Prove $(P) \Leftrightarrow (P_E)$

<u>Step 2</u>: Prove the compactness of the embedding $X_N(\varepsilon) \hookrightarrow L^2(\Omega)$

Then we obtain Fredholm property for (P)...

<u>Remark</u>: notice that in the case of a symmetric domain with $\varepsilon^+ = -\varepsilon^-$, due to the infinite kernel of the scalar problem, the embedding $X_N(\varepsilon) \hookrightarrow L^2(\Omega)$ is not compact!

But here we suppose that (P_φ) is well-posed.

<u>Proof of step 1</u>:

We want to prove that a solution (E) of (P_E)

is a solution of (P).

Let $E' \in H_N(\mathrm{rot})$.

Since (P_φ) is well-posed:

$$\exists!\, \varphi' \in H^1_0(\Omega) \,/\, \int_\Omega \varepsilon(E' + \nabla\varphi').\nabla\varphi = 0 \quad \forall \varphi \in H^1_0(\Omega)$$

Then:

$$E' + \nabla\varphi' \in X_N(\varepsilon).$$

and we have:

$$\int_\Omega \mathrm{rot}\, E \,.\, \mathrm{rot}\,(E' + \nabla\varphi') - \omega^2 \varepsilon E\,(E' + \nabla\varphi') = \int_\Omega J(E' + \nabla\varphi')$$

One can check easily that all terms with $\nabla\varphi'$ vanish, so that:

$$\int_\Omega \mathrm{rot}\, E . \mathrm{rot}\, E' - \omega^2 \varepsilon E E' = \int_\Omega J E' \qquad \forall E' \in H_N(\mathrm{rot})$$

which shows that E is a solution of (P). ∎

<u>Proof of step 2</u>:

We first prove a

<u>T-coercivity lemma 1</u>:

There exists a bounded operator T on X_T such that

$$\forall u, v \in X_T \quad \int_\Omega \frac{1}{\varepsilon} \mathrm{rot}\, u \;\mathrm{rot}\,(Tv) = \int_\Omega \mathrm{rot}\, u \;\mathrm{rot}\, v$$

<u>Proof of the lemma</u>:

We would like "$\mathrm{rot}\,(Tv) = \varepsilon\, \mathrm{rot}\, v$" but this is not possible because $\mathrm{div}(\varepsilon\, \mathrm{rot}\, v) \neq 0$, so we add a gradient as follows:

$$\mathrm{rot}\,(Tv) = \varepsilon(\mathrm{rot}\, v + \nabla\varphi) \qquad \varphi \in H^1_0(\Omega)$$

First, since (P_φ) is well-posed,

$\exists!\, \varphi \in H^1_0(\Omega)$ such that $\operatorname{div}\left(\varepsilon(\operatorname{rot} v + \nabla\varphi)\right) = 0$.

Then (Amrouche-Bernardi-Dauge):

$\exists!\, w \in V_T$ such that $\operatorname{rot} w = \varepsilon(\operatorname{rot} v + \nabla\varphi)$

Finally we set $Tv = w$. ■

Now we can prove the compact embedding.

Let (u_n) a bounded sequence in $X_N(\varepsilon)$.

Since $\operatorname{div}(\varepsilon u_n) = 0$, again

$\exists!\, w_n \in V_T$ such that $\operatorname{rot} w_n = \varepsilon u_n$

By the T-coercivity lemma $(w_{nm} = w_n - w_m)$:

$$\int_\Omega |\operatorname{rot} w_{nm}|^2 = \int_\Omega \frac{1}{\varepsilon} \operatorname{rot} w_{nm} \operatorname{rot}(T w_{nm})$$

$$= \int_\Omega u_{nm} \operatorname{rot}(T w_{nm})$$

$$= \int_\Omega \operatorname{rot} u_{nm} \, (T w_{nm})$$

- $\operatorname{rot} u_{nm}$ is bounded in $L^2(\Omega)$

- $T w_n$ is bounded in X_T which is compactly embedded in $L^2(\Omega)$, so $T w_{nm}$ is a Cauchy sequence (for an extracted subsequence).

The result follows. ■

3_ The case where (P_φ) is Fredholm and has a finite dimensional kern

For simplicity, we suppose that this kernel has the dimension 1:

$\exists \varphi_\varepsilon \in H_0^1(\Omega)$, $\varphi_\varepsilon \neq 0$ such that

$$\int_\Omega \varepsilon \nabla \varphi_\varepsilon \cdot \nabla \varphi' = 0 \quad \forall \varphi' \in H_0^1(\Omega)$$

Then we denote by S^ε a subspace of $H_0^1(\Omega)$ such that:

$$H_0^1(\Omega) = \operatorname{Span}\{\varphi_\varepsilon\} \oplus S^\varepsilon$$

and the scalar problem is now well-posed in S^ε in the sense that; for all $f \in H^{-1}(\Omega)$:

$$(P_\varphi^\varepsilon) \begin{cases} \exists !\, \varphi \in S^\varepsilon \text{ such that } \forall \varphi' \in S^\varepsilon \\ \int_\Omega \varepsilon \nabla \varphi \cdot \nabla \varphi' = \langle f, \varphi' \rangle \end{cases}$$

Note that in that case, (P_E) is no longer equivalent to (P). Indeed, let us set:

$$E^\varepsilon = \nabla \varphi_\varepsilon$$

Then $E^\varepsilon \in X_N(\varepsilon)$ is solution of (P_E) for $J = 0$, but not of (P) !!!

So we have to modify formulation (P_E) as follows:

We set

$$\tilde{X}_N(\varepsilon) = \{ u \in H_N(\mathrm{rot}) ; \int_\Omega \varepsilon u \nabla \varphi = 0 \quad \forall \varphi \in S^\varepsilon \}$$

Clearly $\tilde{X}_N(\varepsilon) \supset X_N(\varepsilon)$.

More precisely :

$$\tilde{X}_N(\varepsilon) = X_N(\varepsilon) \oplus \operatorname{span}\{u_\varepsilon\}$$

where u_ε is such that

$$\begin{cases} u_\varepsilon \in H_N(\mathrm{rot}) \\ \int_\Omega \varepsilon u_\varepsilon \nabla\varphi' = 0 \qquad \forall \varphi' \in S^\varepsilon \\ \int_\Omega \varepsilon u_\varepsilon \nabla\varphi_\varepsilon = 1 \end{cases}$$

And we consider the new problem :

$$(\tilde{P}_E) \quad \begin{cases} E \in \tilde{X}_N(\varepsilon) \text{ such that } \forall E' \in \tilde{X}_N(\varepsilon) \\ \int_\Omega \mathrm{rot}\, E \,\mathrm{rot}\, E' - \omega^2 \varepsilon E E' = \int_\Omega J E' \end{cases}$$

Again there are 2 steps in the analysis :

Step 1 : Prove $(P) \Longleftrightarrow (\tilde{P}_E)$

Step 2 : Prove the compactness of the embedding $\tilde{X}_N(\varepsilon) \hookrightarrow L^2(\Omega)$, or equivalently of $X_N(\varepsilon) \hookrightarrow L^2(\Omega)$.

For the Step 1, we proceed as in the well-posed case, using the well-posedness of (P_φ^ε) now.

For the step 2, the main point is to generalize the regularity lemma.

T-coercivity lemma 2:

There exist a bounded operator T on X_T and two bounded linear forms ℓ_1^ε and ℓ_2^ε on X_T such that:

$$\forall u, v \in X_T \quad \int_\Omega \frac{1}{\varepsilon} \operatorname{rot} u \operatorname{rot}(Tv) = \int_\Omega \operatorname{rot} u \operatorname{rot} v + \ell_1^\varepsilon(u)\, \ell_2^\varepsilon(v)$$

Proof:

This time, we look for Tv such that:

$\operatorname{rot}(Tv) = \varepsilon(\operatorname{rot} v + \nabla\varphi + \alpha U_\varepsilon)$, $\varphi \in S^\varepsilon$, $\alpha \in \mathbb{R}$

We need: $\displaystyle\int_\Omega \varepsilon(\operatorname{rot} v + \nabla\varphi + \alpha U_\varepsilon)\,\varphi' = 0 \quad \forall \varphi' \in H_0^1(\Omega)$

Taking $\varphi' = \varphi_\varepsilon$, we get:

$$\alpha = -\int_\Omega \varepsilon(\operatorname{rot} v)\,\varphi_\varepsilon \overset{\text{def}}{=} \ell_2^\varepsilon(v)$$

Taking $\varphi' \in S^\varepsilon$, we prove the existence and uniqueness of φ.

Finally:

$$\int_\Omega \frac{1}{\varepsilon} \operatorname{rot} u \operatorname{rot}(Tv) = \int_\Omega \operatorname{rot} u\,(\operatorname{rot} v + \nabla\varphi + \alpha U_\varepsilon)$$

$$= \int_\Omega \operatorname{rot} u \operatorname{rot} v + \alpha \underbrace{\int_\Omega \operatorname{rot} u\, U_\varepsilon}_{= \ell_1^\varepsilon(u)}$$