# On the helicity of a bounded domain and the Biot–Savart operator

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Some “variations on a theme” related to the paper

- A. Alonso Rodríguez, J. Camaño, R. Rodríguez, A. Valli and P. Venegas, *Finite element approximation of the spectrum of the curl operator in a multiply-connected domain*, Found. Comput. Math., 18 (2018), 1493–1533.

## Outline

# Introduction and physical remarks

## Physical framework

By the Lorentz law the density of the magnetic force is given by $\mathbf{F} = \mathbf{J} \times \mathbf{B}$, where $\mathbf{J}$ is the current density and $\mathbf{B}$ is the magnetic induction.

- Linear isotropic media: $\mathbf{B} = \mu \mathbf{H}$ (the scalar function $\mu$ being the magnetic permeability).
- Eddy current or static approximation: $\mathbf{J} = \mathbf{curl}\, \mathbf{H}$.

If $\mathbf{curl}\, \mathbf{H} = \lambda \mathbf{H}$ ($\lambda$ a scalar function) the magnetic force vanishes:

$$\mathbf{F} = \mathbf{curl}\, \mathbf{H} \times \mu \mathbf{H} = \lambda \mathbf{H} \times \mu \mathbf{H} = \mathbf{0}\,.$$

## Force-free fields

- Fields satisfying $\mathbf{curl}\,\mathbf{H} = \lambda\mathbf{H}$ are called force-free fields. If $\lambda$ is a constant they are called linear force-free fields; in particular, the eigenvectors of the curl operator (defined on a suitable domain) are linear force-free fields.

In fluid dynamics, force-free fields are called Beltrami fields, and a Beltrami field $\mathbf{u}$ that is divergence-free and tangential to the boundary is a steady solution of the Euler equations for incompressible inviscid flows (with pressure given by $p = -\frac{|\mathbf{u}|^2}{2}$).

# Linear force-free fields

Some interesting physical examples and remarks:

- Arnold–Beltrami-Childress fields (a well-known example of chaotic flows) are linear force-free fields:

$$\mathbf{u}(x,y,z) = (C\sin kz + B\cos ky, A\sin kx + C\cos kz, B\sin ky + A\cos kx);$$

- linear force-free fields are the asymptotic configurations (they are the only resistive magnetohydrostatic force-free fields that remain force-free as time changes) [Jette (1970)];
- a field which is divergence-free and tangential to the boundary (e.g., the magnetic field) and which maximizes the helicity with fixed energy is a linear force-free field [Woltjer (1958)].

## The helicity of a vector field

Let us explain better this last result.

The helicity of a vector field $\mathbf{v}$, a concept introduced by Woltjer (1958) and named by Moffatt (1969), is given by

$$H(\mathbf{v}) = \frac{1}{4\pi} \int_\Omega \int_\Omega \mathbf{v}(\mathbf{x}) \times \mathbf{v}(\mathbf{y}) \cdot \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|^3} \, d\mathbf{x} \, d\mathbf{y} \, .$$

It is a "measure of the extent to which the field lines wrap and coil around one another" [Cantarella et al. (2000a), Cantarella et al. (2001)].

Helicity is particularly interesting for divergence-free vector fields that are tangential to the boundary ("closed" or "confined" vector fields).

# The helicity of a vector field (cont'd)

Focusing on the physical meaning, "it is widely recognized that the key property of turbulence that is most conducive to dynamo action is its helicity" [Moffatt (2016)]. [Dynamo action is the physical mechanism through which a rotating, convecting, and electrically conducting fluid is able to maintain a magnetic field.]

Summing up:

- linear force-free fields are important physical objects in fluid dynamics, turbulence, electromagnetism and plasma physics
- the maximum of the helicity with fixed energy is realized by a linear force-free field tangential to the boundary.

# Geometrical preliminaries

## Basic notations

We assume that $\Omega$ is a bounded domain in $\mathbb{R}^3$, with Lipschitz boundary $\partial\Omega$.

The unit outward normal vector on $\partial\Omega$ will be denoted by **n**.

We define

$$H(\mathrm{curl}\,;\Omega) = \{\mathbf{w} \in (L^2(\Omega))^3 \,|\mathrm{curl}\,\mathbf{w} \in (L^2(\Omega))^3\}\,,$$

endowed with the norm

$$\|\mathbf{w}\|_{\mathrm{curl}\,;\Omega} = \{\|\mathbf{w}\|^2_{0,\Omega} + \|\mathrm{curl}\,\mathbf{w}\|^2_{0,\Omega}\}^{1/2}\,.$$

## Homological tools

We also recall some geometrical results (see, e.g., Cantarella et al. (2002); see also Benedetti et al. (2012)).

Suppose that the first Betti number of $\overline{\Omega}$ is not zero, say, $g > 0$; then the first Betti number of $\partial\Omega$ is equal to $2g$ and it is possible to consider $2g$ non-bounding cycles on $\partial\Omega$, $\{\gamma_j\}_{j=1}^g \cup \{\gamma'_j\}_{j=1}^g$, that are (representative of) the generators of the first homology group of $\partial\Omega$.

They are such that $\{\gamma_j\}_{j=1}^g$ are (representative of) the generators of the first homology group of $\overline{\Omega}$ (the tangent vector on $\gamma_j$ is denoted by $\mathbf{t}_j$), while $\{\gamma'_j\}_{j=1}^g$ are (representative of) the generators of the first homology group of $\Omega' = B \setminus \overline{\Omega}$, being $B$ an open ball containing $\overline{\Omega}$ (the tangent vector on $\gamma'_j$ is denoted by $\mathbf{t}'_j$).

## Homological tools (cont'd)

It is also known that

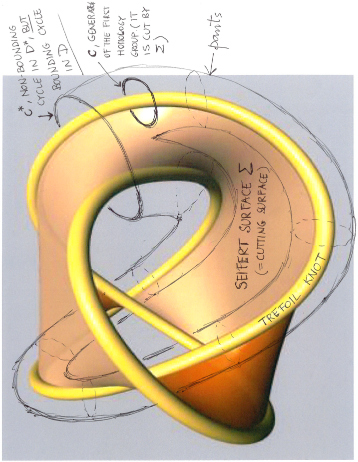
- in $\Omega$ there exist $g$ 'cutting' surfaces $\{\Sigma_j\}_{j=1}^g$, that are connected orientable Lipschitz surfaces satisfying $\Sigma_j \subset \Omega$ and $\partial\Sigma_j \subset \partial\Omega$, such that every curl-free vector in $\Omega$ has a global potential in the 'cut' domain $\Omega^0 := \Omega \setminus \bigcup_{j=1}^g \Sigma_j$; each surface $\Sigma_j$ satisfies $\partial\Sigma_j = \gamma'_j$, 'cuts' the corresponding cycle $\gamma_j$ and does not intersect the other cycles $\gamma_i$ for $i \neq j$;
- in $\Omega'$ there exist $g$ 'cutting' surfaces $\{\Sigma'_j\}_{j=1}^g$, that are connected orientable Lipschitz surfaces satisfying $\Sigma'_j \subset \Omega'$ and $\partial\Sigma'_j \subset \partial\Omega$, such that every curl-free vector in $\Omega'$ has a global potential in the 'cut' domain $(\Omega')^0 := \Omega' \setminus \bigcup_{j=1}^g \Sigma'_j$; each surface $\Sigma'_j$ satisfies $\partial\Sigma'_j = \gamma_j$, 'cuts' the corresponding cycle $\gamma'_j$, and does not intersect the other cycles $\gamma'_i$ for $i \neq j$.

## Homological tools (cont'd)

[Looking back at the literature on this topic, where some misunderstanding appears, it is interesting to make clear that:

- the statement concerning the 'cutting' surfaces $\Sigma_j$ does not mean that the 'cut' domain $\Omega^0$ is simply-connected nor that it is homologically trivial: an example in this sense is furnished by $\Omega = Q \setminus K$, where $Q$ is a cube and $K$ is the trefoil knot.]

# The trefoil knot and its Seifert surface




$D$ = cube − trefoil knot

$D^*$ = cube − (trefoil knot + cutting surface) = $D$ − cutting surface

## The space of harmonic fields

We need to introduce the space of harmonic Neumann vector fields

$$\mathcal{H}(m) = \{\boldsymbol{\rho} \in (L^2(\Omega))^3 \,|\, \mathrm{curl}\, \boldsymbol{\rho} = \mathbf{0} \text{ in } \Omega, \\ \mathrm{div}\, \boldsymbol{\rho} = 0 \text{ in } \Omega, \boldsymbol{\rho} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}\,.$$

This space has dimension $g$, and a basis for it is given by $\{\boldsymbol{\rho}_j\}_{j=1}^{g}$, where $\boldsymbol{\rho}_j$ satisfies $\oint_{\gamma_k} \boldsymbol{\rho}_j \cdot \mathbf{t}_k = \delta_{jk}$ (see, e.g., Alonso Rodríguez et al. (2018)).

A similar result holds also for the space of harmonic Neumann vector fields defined in $\Omega'$ with normal component equal to zero on $\partial\Omega' = \partial B \cup \partial\Omega$, whose basis functions are denoted by $\boldsymbol{\rho}'_i$, $i = 1, \ldots, g$.

# The Biot–Savart operator

## Helicity and the Biot–Savart operator

The Biot–Savart operator is defined by means of the gradient of the Newtonian kernel.

In the following we furnish a variational characterization of its orthogonal projection over the space of divergence-free vector fields that are tangential to the boundary, opening the way to devise efficient finite element numerical approximation schemes.

Since this projected Biot–Savart operator is shown to be compact, its spectrum is discrete, and there is an eigenvalue with maximum absolute value. The computation of this eigenvalue furnishes a simple characterization of the helicity of a bounded domain, without restriction on its topological shape.

## The Biot–Savart operator

Let us consider the Hilbert space

$$\boldsymbol{\mathcal{V}} = \{\mathbf{v} \in (L^2(\Omega))^3 \,|\operatorname{div}\mathbf{v} = 0 \text{ in } \Omega, \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}\,.$$

The Biot–Savart operator $BS$ is defined in $\boldsymbol{\mathcal{V}}$ as

$$BS(\mathbf{v})(\mathbf{x}) = \frac{1}{4\pi}\int_\Omega \mathbf{v}(\mathbf{y}) \times \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}\, d\mathbf{y}\,. \tag{1}$$

Since $\mathbf{v}\cdot\mathbf{n} = 0$ on $\partial\Omega$ and $\operatorname{div}\mathbf{v} = 0$ in $\Omega$, the vector field

$$\widetilde{\mathbf{v}} = \begin{cases} \mathbf{v} & \text{in } \Omega \\ \mathbf{0} & \text{in } \mathbb{R}^3 \setminus \overline{\Omega} \end{cases}$$

satisfies $\operatorname{div}\widetilde{\mathbf{v}} = 0$ in $\mathbb{R}^3$, and $BS(\mathbf{v})$ can be rewritten as

$$BS(\mathbf{v})(\mathbf{x}) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \widetilde{\mathbf{v}}(\mathbf{y}) \times \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}\, d\mathbf{y}\,.$$

## The Biot–Savart operator (cont'd)

It is well-known that $BS(\mathbf{v}) \in (H^1(\mathbb{R}^3))^3$ and satisfies in $\mathbb{R}^3$ the relations $\operatorname{curl} BS(\mathbf{v}) = \widetilde{\mathbf{v}}$ and $\operatorname{div} BS(\mathbf{v}) = 0$. Hence we have $BS(\mathbf{v}) \in (H^1(\Omega))^3$ and

$$\begin{cases} \operatorname{curl} BS(\mathbf{v}) = \mathbf{v} & \text{in } \Omega \\ \operatorname{div} BS(\mathbf{v}) = 0 & \text{in } \Omega\,. \end{cases}$$

Let us introduce the scalar function $\phi_{\mathbf{v}} \in H^1(\Omega)$, solution to the Neumann problem

$$\begin{cases} \Delta \phi_{\mathbf{v}} = 0 & \text{in } \Omega \\ \mathbf{grad}\, \phi_{\mathbf{v}} \cdot \mathbf{n} = BS(\mathbf{v}) \cdot \mathbf{n} & \text{on } \partial\Omega \\ \int_\Omega \phi_{\mathbf{v}} = 0\,, \end{cases}$$

whose existence is guaranteed by the fact that

$$\int_{\partial\Omega} BS(\mathbf{v}) \cdot \mathbf{n} = \int_\Omega \operatorname{div} BS(\mathbf{v}) = 0\,.$$

## The projected Biot–Savart operator

The modified (projected) Biot–Savart operator is defined in $\boldsymbol{\mathcal{V}}$ as follows:

$$\widehat{BS}(\mathbf{v}) = BS(\mathbf{v}) - \mathbf{grad}\, \phi_{\mathbf{v}} \,. \tag{2}$$

Clearly, $\widehat{BS}(\mathbf{v})$ is the $(L^2(\Omega))^3$-orthogonal projection of $BS(\mathbf{v})$ over $\boldsymbol{\mathcal{V}}$, and satisfies

$$\begin{cases} \operatorname{curl} \widehat{BS}(\mathbf{v}) = \mathbf{v} & \text{in } \Omega \\ \operatorname{div} \widehat{BS}(\mathbf{v}) = 0 & \text{in } \Omega \\ \widehat{BS}(\mathbf{v}) \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \,. \end{cases} \tag{3}$$

## Vanishing line integrals

Another important property of both standard and projected Biot–Savart fields is the following:

### Proposition

*It holds*

$$\oint_{\gamma_j} BS(\mathbf{v}) \cdot \mathbf{t}_j = 0 \; \textit{and} \; \oint_{\gamma_j} \widehat{BS}(\mathbf{v}) \cdot \mathbf{t}_j = 0 \quad \forall \; j = 1, \ldots g \, .$$

## Vanishing line integrals (cont'd)

**Proof.** Let us recall that $BS(\mathbf{v})$ is indeed defined in $\mathbb{R}^3$, hence we can apply the Stokes theorem on the surface $\Sigma'_j \subset \Omega'$, which satisfies $\partial\Sigma'_j = \gamma_j$. We have

$$\oint_{\gamma_j} BS(\mathbf{v}) \cdot \mathbf{t}_j = \int_{\Sigma'_j} \mathrm{curl}\, BS(\mathbf{v}) \cdot \mathbf{n} = 0\,,$$

as $\mathrm{curl}\, BS(\mathbf{v}) = \widetilde{\mathbf{v}}$ in $\mathbb{R}^3$, hence $\mathrm{curl}\, BS(\mathbf{v}) = \mathbf{0}$ in $\Omega'$. The same result holds for $\widehat{BS}(\mathbf{v})$, as it differs from $BS(\mathbf{v})$ by $\mathbf{grad}\, \phi_{\mathbf{v}}$. $\square$

## A characterization of the projected Biot–Savart operator

In conclusion, the projected Biot–Savart field $\widehat{BS}(\mathbf{v})$ satisfies

$$\begin{cases} \operatorname{curl} \widehat{BS}(\mathbf{v}) = \mathbf{v} & \text{in } \Omega \\ \operatorname{div} \widehat{BS}(\mathbf{v}) = 0 & \text{in } \Omega \\ \widehat{BS}(\mathbf{v}) \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \\ \oint_{\gamma_j} \widehat{BS}(\mathbf{v}) \cdot \mathbf{t}_j = 0 & \forall\, j = 1, \dots g\,. \end{cases} \tag{4}$$

It is well-known that this problem has a unique solution (and here we will prove this result by showing that problem (4) is equivalent to a well-posed saddle-point variational problem).

A consequence is that the projected Biot–Savart operator is completely characterized as the solution operator to problem (4).

# Variational theory

## Function spaces

Let us introduce some function spaces that will be useful in the sequel:

$$\boldsymbol{\mathcal{X}} = \{\mathbf{w} \in H(\mathrm{curl}\,;\Omega) \,|\, \mathrm{curl}\,\mathbf{w}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}\,,$$

$$\boldsymbol{\mathcal{Z}} = \{\mathbf{w} \in \boldsymbol{\mathcal{X}} \,|\, \oint_{\gamma_j} \mathbf{w}\cdot\mathbf{t}_j = 0 \text{ for } j = 1,\dots,g\}\,,$$

$$\boldsymbol{\mathcal{H}} = \mathbf{grad}\, H^1(\Omega)\,.$$

Note that $\boldsymbol{\mathcal{V}} = \boldsymbol{\mathcal{H}}^\perp$.

## A variational formulation

A suitable variational formulation of problem (4) is the following constrained least-square formulation.

For $\mathbf{v} \in \boldsymbol{\mathcal{V}}$, the couple $(\widehat{BS}(\mathbf{v}), \mathbf{0})$ is the solution $(\mathbf{u}, \mathbf{q}) \in \boldsymbol{\mathcal{Z}} \times \boldsymbol{\mathcal{H}}$ of the problem

$$\begin{array}{rcl} \int_\Omega \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \mathbf{w} + \int_\Omega \mathbf{q} \cdot \mathbf{w} & = & \int_\Omega \mathbf{v} \cdot \operatorname{curl} \mathbf{w} \\ \int_\Omega \mathbf{u} \cdot \mathbf{p} & = & 0 \end{array} \quad (5)$$

for each $(\mathbf{w}, \mathbf{p}) \in \boldsymbol{\mathcal{Z}} \times \boldsymbol{\mathcal{H}}$.

We will see that this problem has a unique solution. For the moment let us show that problem (4) and (5) are equivalent.

## Equivalence of strong and variational problems

### Proposition

*The couple* $(\widehat{BS}(\mathbf{v}), \mathbf{0})$ *is a solution to* (5).

**Proof.** The first equation in (5) is clearly satisfied. From the first equation in (4) it follows at once that $\widehat{BS}(\mathbf{v}) \in H(\mathrm{curl}\,;\Omega)$ and that $\mathrm{curl}\,\widehat{BS}(\mathbf{v}) \cdot \mathbf{n} = 0$ on $\partial\Omega$. From the last equation in (4) it follows that $\widehat{BS}(\mathbf{v}) \in \mathcal{Z}$. Finally, due to the second and third equations in (4) $\widehat{BS}(\mathbf{v})$ is orthogonal to $\mathcal{H}$, namely, the second equation in (5) is satisfied. □

## Equivalence of strong and variational problems (cont'd)

Before coming to the reciprocal result we need some preliminary results. The following lemma is proved in Alonso Rodríguez et al. (2018).

### Lemma (orthogonality)

*Assume that* $\vartheta, \varphi \in H^1(\Omega)$ *and* $1 \leq k, i \leq g$. *Then*

$$\int_{\partial\Omega} \mathbf{grad}\,\varphi \cdot (\mathbf{n} \times \mathbf{grad}\,\vartheta) = 0 \quad , \quad \int_{\partial\Omega} \mathbf{grad}\,\varphi \cdot (\mathbf{n} \times \boldsymbol{\rho}'_i) = 0$$

$$\int_{\partial\Omega} \boldsymbol{\rho}_k \cdot (\mathbf{n} \times \mathbf{grad}\,\vartheta) = 0 \quad , \quad \int_{\partial\Omega} \boldsymbol{\rho}_k \cdot (\mathbf{n} \times \boldsymbol{\rho}'_i) = \delta_{ki} \, .$$

## Equivalence of strong and variational problems (cont'd)

Then we are in a condition to prove:

### Proposition

*Let* $(\mathbf{u}, \mathbf{q})$ *be a solution to* (5)*. Then* $\mathbf{q} = \mathbf{0}$ *and* $\mathbf{u}$ *is a solution to* (4)*.*

**Proof.** Since $\mathcal{H} \subset \mathcal{Z}$, we can choose $\mathbf{w} = \mathbf{q}$ in the first equation of (5) and from $\mathrm{curl}\, \mathbf{q} = \mathbf{0}$ we find at once $\mathbf{q} = \mathbf{0}$.
The fourth equation in (4) comes from $\mathbf{u} \in \mathcal{Z}$, and the second equation in (5) gives $\mathrm{div}\, \mathbf{u} = 0$ in $\Omega$ and $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$.
Knowing $\mathbf{q} = \mathbf{0}$, the first equation implies $\mathrm{curl}\, (\mathrm{curl}\, \mathbf{u} - \mathbf{v}) = \mathbf{0}$ in $\Omega$. Moreover, integrating by parts we also find for each $\mathbf{w} \in \mathcal{Z}$

$$\int_{\partial\Omega} (\mathrm{curl}\, \mathbf{u} - \mathbf{v}) \cdot \mathbf{n} \times \mathbf{w} = 0 \,.$$

## Equivalence of strong and variational problems (cont'd)

Since $\operatorname{curl} \mathbf{u} - \mathbf{v}$ is curl-free, it is well-known that it can be written as

$$\operatorname{curl} \mathbf{u} - \mathbf{v} = \mathbf{grad}\, \varphi + \sum_{k=1}^{g} \beta_k \boldsymbol{\rho}_k \, .$$

Moreover, we recall from Buffa (2001), Hiptmair et al. (2012) that the tangential trace of $\mathbf{w} \in \boldsymbol{\mathcal{X}}$ can be written on $\partial\Omega$ as

$$\mathbf{n} \times \mathbf{w} = \mathbf{n} \times \mathbf{grad}\, \vartheta + \sum_{j=1}^{g} \zeta_j \, \mathbf{n} \times \boldsymbol{\rho}_j + \sum_{i=1}^{g} \eta_i \, \mathbf{n} \times \boldsymbol{\rho}'_i \, ,$$

for $\vartheta \in H^1(\Omega)$, where $\zeta_j = \oint_{\gamma_j} \mathbf{w} \cdot \mathbf{t}_j$. Knowing that $\mathbf{w} \in \boldsymbol{\mathcal{Z}}$, this representation formula reduces to

$$\mathbf{n} \times \mathbf{w} = \mathbf{n} \times \mathbf{grad}\, \vartheta + \sum_{i=1}^{g} \eta_i \, \mathbf{n} \times \boldsymbol{\rho}'_i \, .$$

## Equivalence of strong and variational problems (cont'd)

Thus from the orthogonality lemma we easily obtain

$$0 = \int_{\partial\Omega} (\mathrm{curl}\,\mathbf{u} - \mathbf{v}) \cdot (\mathbf{n} \times \mathbf{w}) = \sum_{k=1}^{g} \beta_k \eta_k \,.$$

Since $\eta_k$ are arbitrary, it follows that $\beta_k = 0$ for $k = 1, \ldots, g$. As a consequence, we can write $\mathrm{curl}\,\mathbf{u} - \mathbf{v} = \mathbf{grad}\,\varphi$ in $\Omega$.
Since $\mathbf{u} \in \boldsymbol{\mathcal{Z}}$, it follows $\mathrm{curl}\,\mathbf{u} \in \boldsymbol{\mathcal{V}}$ and thus $\mathbf{grad}\,\varphi \in \boldsymbol{\mathcal{V}} = \boldsymbol{\mathcal{H}}^{\perp}$.
Hence we conclude that $\mathbf{grad}\,\varphi = \mathbf{0}$ and $\mathrm{curl}\,\mathbf{u} = \mathbf{v}$ in $\Omega$. □

## Existence and uniqueness

The existence and uniqueness theory for problem (5) is based on classical results for saddle-point problems.

Let us start by introducing the Hilbert space

$$H_0(\mathrm{div}\,;\Omega) = \{\mathbf{v} \in (L^2(\Omega))^3 \,|\, \mathrm{div}\,\mathbf{v} \in L^2(\Omega), \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\} \,.$$

The well-posedness of problem (5) is a consequence of the following lemmas, that are adapted from Alonso Rodríguez et al. (2018).

## Existence and uniqueness (cont'd)

### Lemma (Friedrichs)

*Let the Hilbert space* $\mathcal{X} \cap H_0(\mathrm{div}\,;\Omega)$ *be endowed with the norm*

$$\|\mathbf{w}\|_\star := \{\|\mathbf{w}\|^2_{0,\Omega} + \|\mathrm{div}\,\mathbf{w}\|^2_{0,\Omega} + \|\mathrm{curl}\,\mathbf{w}\|^2_{0,\Omega}\}^{1/2} .$$

*In* $\mathcal{X} \cap H_0(\mathrm{div}\,;\Omega)$ *the seminorm*

$$|||\mathbf{w}||| := \left\{\|\mathrm{curl}\,\mathbf{w}\|^2_{0,\Omega} + \|\mathrm{div}\,\mathbf{w}\|^2_{0,\Omega} + \sum_{j=1}^{g} \left| \oint_{\gamma_j} \mathbf{w} \cdot \mathbf{t}_j \right|^2 \right\}^{1/2}$$

*is indeed a norm equivalent to the norm* $\|\mathbf{w}\|_\star$.

## Existence and uniqueness (cont'd)

**Proof.** Take $j$ with $1 \leq j \leq g$. Since $\oint_{\gamma_j} \mathbf{w} \cdot \mathbf{t}_j$ can be written as $\oint_{\gamma_j} \mathbf{w} \cdot \mathbf{t}_j = \int_{\partial\Omega} (\mathbf{w} \times \mathbf{n}) \cdot \rho'_j$ (see Alonso Rodríguez et al. (2018)), it follows that $\left| \oint_{\gamma_j} \mathbf{w} \cdot \mathbf{t}_j \right| \leq C_2 \|\mathbf{w}\|_{\mathrm{curl}\,;\Omega}$, thus $|||\mathbf{w}|||^2 \leq C \|\mathbf{w}\|_\star^2$.
The other inequality is proved by contradiction. We suppose that for all $n \in \mathbb{N}$ there exists $\mathbf{v}_n \in \boldsymbol{\mathcal{X}} \cap H_0(\mathrm{div}\,;\Omega)$ such that $\|\mathbf{v}_n\|_\star > n\,|||\mathbf{v}_n|||$. Let $\mathbf{u}_n = \mathbf{v}_n / \|\mathbf{v}_n\|_\star$. It follows that $\|\mathbf{u}_n\|_\star = 1$ and

$$\|\mathrm{curl}\,\mathbf{u}_n\|_{0,\Omega}^2 + \|\mathrm{div}\,\mathbf{u}_n\|_{0,\Omega}^2 + \sum_{j=1}^{g} \left| \oint_{\gamma_j} \mathbf{u}_n \cdot \mathbf{t}_j \right|^2 < \frac{1}{n^2} \quad \forall\, n \in \mathbb{N}. \quad (6)$$

## Existence and uniqueness (cont'd)

The space $\mathcal{X} \cap H_0(\mathrm{div}\,;\Omega)$ is compactly imbedded in $L^2(\Omega)^3$; hence, since the sequence $\{\mathbf{u}_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{X} \cap H_0(\mathrm{div}\,;\Omega)$, there exists a subsequence of $\mathbf{u}_n$ (for simplicity, still denoted by $\mathbf{u}_n$) and a vector field $\mathbf{u} \in \mathcal{X} \cap H_0(\mathrm{div}\,;\Omega)$ such that $\mathbf{u}_n \to \mathbf{u}$ in $L^2(\Omega)^3$. Thus from (6) we obtain that

$$\|\mathbf{u}_n - \mathbf{u}_m\|_\star^2 \le C\Big\{\|\mathbf{u}_n - \mathbf{u}_m\|_{0,\Omega}^2 + \|\mathrm{div}\,\mathbf{u}_n\|_{0,\Omega}^2 + \|\mathrm{div}\,\mathbf{u}_m\|_{0,\Omega}^2 + \|\mathrm{curl}\,\mathbf{u}_n\|_{0,\Omega}^2 + \|\mathrm{curl}\,\mathbf{u}_m\|_{0,\Omega}^2\Big\}\,.$$

Then $\{\mathbf{u}_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete space $\mathcal{X} \cap H_0(\mathrm{div}\,;\Omega)$, which implies that $\mathbf{u}_n \to \mathbf{u}$ in $\mathcal{X} \cap H_0(\mathrm{div}\,;\Omega)$ with $\|\mathbf{u}\|_\star = 1$.

## Existence and uniqueness (cont'd)

From (6) we obtain that $\mathrm{curl}\,\mathbf{u} = \mathbf{0}$ in $\Omega$, $\mathrm{div}\,\mathbf{u} = 0$ in $\Omega$, and that $\oint_{\gamma_j} \mathbf{u} \cdot \mathbf{t}_j = 0$ for each $j = 1, \ldots, g$. Therefore $\mathbf{u} \in \mathcal{H}(m)$, say, $\mathbf{u} = \sum_{k=1}^{g} \alpha_k \boldsymbol{\rho}_k$. In particular, we have

$$0 = \oint_{\gamma_j} \mathbf{u} \cdot \mathbf{t}_j = \sum_{k=1}^{g} \alpha_k \int_{\gamma_j} \boldsymbol{\rho}_k \cdot \mathbf{t}_j = \alpha_j \,.$$

In conclusion, we have found $\mathbf{u} = \mathbf{0}$ in $\Omega$ and a contradiction is produced. □

## Existence and uniqueness (cont'd)

### Lemma (ellipticity in the kernel)

*There exists* $\alpha > 0$ *such that*

$$\int_\Omega |\mathrm{curl}\, \mathbf{w}|^2 \geq \alpha \|\mathbf{w}\|^2_{\mathrm{curl}\,;\Omega} \qquad \forall\, \mathbf{w} \in \boldsymbol{\mathcal{Z}} \cap \boldsymbol{\mathcal{H}}^\perp \,,$$

*being*

$$\boldsymbol{\mathcal{H}}^\perp = \left\{ \mathbf{w} \in (L^2(\Omega))^3 \mid \int_\Omega \mathbf{w} \cdot \mathbf{q} = 0 \ \textit{for all}\ \mathbf{q} \in \boldsymbol{\mathcal{H}} \right\} .$$

**Proof.** We have already seen that $\boldsymbol{\mathcal{H}}^\perp = \boldsymbol{\mathcal{V}}$, hence $\boldsymbol{\mathcal{Z}} \cap \boldsymbol{\mathcal{H}}^\perp = \boldsymbol{\mathcal{Z}} \cap \boldsymbol{\mathcal{V}}$. Then the ellipticity in the kernel $\boldsymbol{\mathcal{Z}} \cap \boldsymbol{\mathcal{V}}$ follows from Friedrichs lemma. □

## Existence and uniqueness (cont'd)

### Lemma (inf–sup condition)

*There exists* $\beta > 0$ *such that*

$$\sup_{\mathbf{w}\in\mathcal{Z}\setminus\{\mathbf{0}\}} \frac{\left|\int_\Omega \mathbf{w}\cdot\mathbf{p}\right|}{\|\mathbf{w}\|_{\mathrm{curl}\,,\Omega}} \geq \beta\,\|\mathbf{p}\|_{0,\Omega}\,, \qquad \forall\ \mathbf{p} \in \mathcal{H}.$$

**Proof.** The inf–sup condition follows by taking $\mathbf{w} = \mathbf{p} \in \mathcal{H} \subset \mathcal{Z}$ (thus $\mathrm{curl}\,\mathbf{w} = \mathbf{0}$ in $\Omega$). □

By virtue of the ellipticity in the kernel and the inf–sup condition, problem (5) is a well-posed problem, as the Babuška–Brezzi conditions for saddle-point problems are satisfied.

## The projected Biot–Savart operator revisited

We have thus characterized the projected Biot–Savart operator $\widehat{BS}$ in the following way.

### Theorem

*Let* $\mathbf{T} : \boldsymbol{\mathcal{V}} \to \boldsymbol{\mathcal{Z}} \cap \boldsymbol{\mathcal{V}}$ *be the solution operator* $\mathbf{T}\mathbf{v} = \mathbf{u}$, *where* $(\mathbf{u}, \mathbf{q}) \in \boldsymbol{\mathcal{Z}} \times \boldsymbol{\mathcal{H}}$ *is the solution to problem* (5) *(thus* $\mathbf{u} \in \boldsymbol{\mathcal{Z}} \cap \boldsymbol{\mathcal{V}}$, $\mathbf{q} = \mathbf{0}$*). Then* $\mathbf{T}$ *is the projected Biot–Savart operator* $\widehat{BS}$.

This characterization opens the way to efficient finite element numerical approximations. Since the projected Biot–Savart operator is self-adjoint and compact in $\boldsymbol{\mathcal{V}}$ (see, e.g., Cantarella et al. (2001)), its spectrum is discrete and can be efficiently approximated (this has been done for the operator $\mathbf{T}$ in Alonso Rodríguez et al. (2018) by means of edge finite elements).

# Helicity

## Back to the helicity

Let us go back to the helicity of a vector field $\mathbf{v} \in (L^2(\Omega))^3$ defined as

$$H(\mathbf{v}) = \frac{1}{4\pi} \int_\Omega \int_\Omega (\mathbf{v}(\mathbf{x}) \times \mathbf{v}(\mathbf{y})) \cdot \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|^3} \, d\mathbf{x} \, d\mathbf{y} \, .$$

This can be clearly rewritten as

$$H(\mathbf{v}) = \int_\Omega \mathbf{v} \cdot BS(\mathbf{v}) \, .$$

If the vector field $\mathbf{v}$ satisfies the additional assumption $\mathbf{v} \in \boldsymbol{\mathcal{V}}$, an easy consequence of the fact that $\boldsymbol{\mathcal{V}} = \boldsymbol{\mathcal{H}}^\perp$ is that

$$H(\mathbf{v}) = \int_\Omega \mathbf{v} \cdot \widehat{BS}(\mathbf{v}) \, . \qquad (7)$$

## Back to the helicity (cont'd)

### Remark

*For a vector field* $\mathbf{v} \in \boldsymbol{\mathcal{V}} \cap \mathcal{H}(m)^{\perp}$ *the helicity could be defined as*

$$H(\mathbf{v}) = \int_{\Omega} \mathbf{v} \cdot \mathbf{A} \,,$$

*where* $\operatorname{curl} \mathbf{A} = \mathbf{v}$*, namely,* $\mathbf{A}$ *is a vector potential of* $\mathbf{v}$ *(see Moffatt (1969)). In fact, for any other vector field* $\mathbf{A}_\sharp$ *with* $\operatorname{curl} \mathbf{A}_\sharp = \mathbf{v}$ *it holds* $\operatorname{curl} (\mathbf{A} - \mathbf{A}_\sharp) = \mathbf{0}$ *in* $\Omega$*, thus* $(\mathbf{A} - \mathbf{A}_\sharp) \in \boldsymbol{\mathcal{H}} \oplus \mathcal{H}(m)$*.*
*Therefore* $\mathbf{v}$ *is orthogonal to* $\mathbf{A} - \mathbf{A}_\sharp$*, and the helicity turns out to be the same for any vector potential of* $\mathbf{v}$*.*
*However, this is not the case if* $\mathbf{v}$ *belongs to* $\boldsymbol{\mathcal{V}}$ *but not to* $\mathcal{H}(m)^{\perp}$*.*
*Since the most interesting physical cases are associated to a vector field* $\mathbf{v} \in \boldsymbol{\mathcal{V}}$ *(for instance, an inviscid incompressible flow, or the magnetic field), we refer to definition (7).* ■

## The helicity of a domain

The helicity of a domain $\Omega$ is defined by

$$H_\Omega = \sup_{\mathbf{v}\in\boldsymbol{\mathcal{V}},\|\mathbf{v}\|_{L^2(\Omega)}=1} |H(\mathbf{v})| \,. \qquad (8)$$

As a consequence of the fact that the projected Biot–Savart operator $\widehat{BS}$ is self-adjoint and compact, the helicity of $\Omega$ can be represented as

$$H_\Omega = |\lambda^\Omega_{\max}| \,,$$

where $\lambda^\Omega_{\max}$ is the eigenvalue of $\widehat{BS}$ in $\Omega$ of maximum absolute value.

## The helicity of a domain (cont'd)

The proof of this result follows a well-known argument. Since it is self-adjoint, the projected Biot–Savart operator has a complete system of eigenfunctions $\{\boldsymbol{\omega}_k\}_{k=1}^{\infty}$, which are orthonormal in $\boldsymbol{\mathcal{V}}$ (or, equivalently, in $(L^2(\Omega)^3)$. Associated to these eigenfunctions there is a sequence of (real) eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$. Therefore, writing $\mathbf{v} = \sum_{k=1}^{\infty} v_k \boldsymbol{\omega}_k$, it follows that $\|\mathbf{v}\|^2_{L^2(\Omega)} = \sum_{k=1}^{\infty} v_k^2$ and

$$H(\mathbf{v}) = \sum_{k,j=1}^{\infty} \int_{\Omega} v_k \boldsymbol{\omega}_k \cdot v_j \widehat{BS}(\boldsymbol{\omega}_j) = \sum_{k,j=1}^{\infty} \int_{\Omega} v_k \boldsymbol{\omega}_k \cdot v_j \lambda_j \boldsymbol{\omega}_j = \sum_{k=1}^{\infty} v_k^2 \lambda_k \,.$$

## The helicity of a domain (cont'd)

Moreover, for $\|\mathbf{v}\|_{L^2(\Omega)} = 1$, we have

$$|H(\mathbf{v})| = \left| \sum_{k=1}^{\infty} v_k^2 \lambda_k \right| \leq |\lambda_{\max}^{\Omega}| \sum_{k=1}^{\infty} v_k^2 = |\lambda_{\max}^{\Omega}| \,,$$

and also, being $\boldsymbol{\omega}_{\max}$ the eigenfunction associated to $\lambda_{\max}^{\Omega}$,

$$|H(\boldsymbol{\omega}_{\max})| = \left| \int_{\Omega} \boldsymbol{\omega}_{\max} \cdot \widehat{BS}(\boldsymbol{\omega}_{\max}) \right| = |\lambda_{\max}^{\Omega}| \int_{\Omega} |\boldsymbol{\omega}_{\max}|^2 = |\lambda_{\max}^{\Omega}| \,,$$

hence $H_{\Omega} = |\lambda_{\max}^{\Omega}|$.

# Explicit value of the helicity

The domains for which the eigenvalue of maximum absolute value of the projected Biot–Savart operator is known are quite a few: to our knowledge, only the ball and the spherical shell (see Cantarella et al. (2000a)).

We remind that for the ball of radius $b$ the result is $|\lambda_{\max}| \approx \frac{b}{4.49341}$ (the approximation is due to the fact that the correct denominator is the first positive solution of the equation $x = \tan x$, that approximately is 4.49341).

## Numerical calculation of the helicity

Due to this lack of explicit results, it is important that an efficient approximation method for the computation of the eigenvalues is available.

In Alonso Rodríguez et al. (2018) edge finite elements are used for the approximation of the spectrum of the operator **T**, for any type of bounded domains $\Omega$.

## The "isoperimetric" problem

A geometrical question now arises:

- for which bounded domain the helicity is the maximum among all the bounded domains with the same volume?

This is an open problem. We have not a theoretical answer, but we can present some numerical computations.

## The "isoperimetric" problem (cont'd)

- If $\Omega$ is a torus of radii $r_1 = 1$ and $r_2 = 0.5$ one has $|\lambda_{\max}| \approx \frac{1}{4.89561} \approx 0.20426$. The helicity of a ball $B$ having the same volume of this torus is $H_B \approx 0.23505$, a larger value.
- If $\Omega$ is a perforated cylinder (topologically, a torus) with rectangular cross section given by $[0.005, 1] \times [-0.5, 0.5]$ one has $H_\Omega \approx 0.20175$, while for the ball $B$ with the same volume it holds $H_B \approx 0.20219$, a larger but very close value.
- If $\Omega$ is a torus of radii $r_1 = 0.505$ and $r_2 = 0.5$ one has $H_\Omega \approx 0.19073$, a larger value than that of the helicity of the ball $B$ with the same volume, given by $H_B \approx 0.18718$.

## The “isoperimetric” problem (cont’d)

This goes in the direction of confirming a conjecture in Cantarella et al. (2000b), who suggested that the domain with maximum helicity among all the domains with the same volume is not the sphere, but a sort of “extreme solid torus, in which the hole has been shrunk to a point”.

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