Automorphism Groups

Lecture Notes

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Contents

Disclaimer: this is a draft and probably contains many typos and mistakes.
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Recommendations. Chapters 1-4 are suitable for a 3rd year bachelor course or the beginning of a master course. Chapters 5-9 are for a master course.

Notes concerning text book literature. We use material from the following text books:

- Cameron's *Permutation groups* [**40**] and *Oligomorphic permutation groups* [**39**],
- Dixon and Mortimer's *Permutation groups* [**48**],
- The collection *Notes on Infinite Permutation Groups* [**12**] by Macpherson, Möller, and Neumann,
- Kechris' *Classical descriptive set theory* [**88**],
- Gao's *Invariant descriptive set theory* [**60**], and
- Hodges' *Model theory* [**72**].

Oligomorphic permutation groups are the topic of the short and stimulating book by Peter Cameron [**39**]. The book on permutation groups by the same author [**40**] is mostly about finite permutation groups, but Section 5 covers oligomorphic permutation groups. There is even less material on infinite permutation groups in [**48**]. The notes in [**12**] are on infinite permutation groups, but they neglect the topological aspect of the topic. Hodges [**73**] is titled *'model theory'*, but covers many fundamental things for permutation groups on the way, including topological aspects. Relevant facts about Polish groups can be found in [**60**, **88**].

Prerequisites. The text is essentially self-contained. Signatures, structures, substructures, etc. are introduced, even though many readers may be familiar with these concepts. However, we do not introduce first-order logic, but rather refer to logic bachelor course notes [**18**]; the same applies for axiomatic set theory (we assume familiarity for instance with Zorn's lemma).

The present notes contains a few theorems that are just stated but not proved, for instance

- the theorem of Ryll-Nardzewski (Theorem 3.2.3); however, we have extracted all the consequences of the (proof of the) Ryll-Nardzewski theorem that we need in this text into Theorem 3.1.1 which we do prove. The full proof of Theorem 3.2.3 can be found in many text books on model theory.
- the theorem of Birkhoff-Kakutani (Theorem 4.2.8); this is covered for example in [**60**, **88**].
- a consequence of a theorem of Lusin-Sierpiński (Theorem 6.3.12); this is again covered in [**88**] (21.6).

Exercises.
The text contains 155 exercises; some of them are graded using the Mandala scale.

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CHAPTER 1

Permutation Groups

A *permutation* of a set X is a bijection between X and X. We use cycle notation for permutations: e.g., $(12)(345)$ denotes the permutation g such that $g(1) = 2$, $g(2) = 1$, $g(3) = 4$, $g(4) = 5$, and $g(5) = 3$. A *permutation group* G on a set X is a set of permutations of X such that the following three conditions hold.

(1) G contains the identity permutation id_X.

(2) G contains for every permutation $u \in G$ also its inverse u^{-1} defined by

$$u^{-1}(u(x)) = x \text{ for all } x \in X.$$

(3) G contains for all $u, v \in G$ their composition $u \circ v$, defined by

$$(u \circ v)(x) = u(v(x)) \text{ for all } x \in X.$$

Examples.

- The set of all permutations of X, denoted by $\mathrm{Sym}(X)$. We use S_n as a shortcut for $\mathrm{Sym}(\{1, \ldots, n\})$ and S_ω as a shortcut for $\mathrm{Sym}(\mathbb{N})$.
- The set of functions $\{t_a \colon \mathbb{Z} \to \mathbb{Z} \mid t_a(x) = x + a, a \in \mathbb{Z}\} \subset \mathrm{Sym}(\mathbb{Z})$.
- The set of all order-preserving permutations of $\mathbb{Q}$.

All of the three examples of permutation groups given above are *transitive*, i.e., for any $a, b \in X$ there exists $u \in G$ such that $u(a) = b$.

Inspiration "Erlanger Programm" of Felix Klein: Understanding structure (in his case, geometry) by understanding its symmetry (via permutation groups).

Definition 1.0.1. *Let G be a permutation group on a set A and H a permutation group on a set B. Then a bijection i between A and B is called a* (permutation group) isomorphism *if there exists a bijection ξ between G and H such that for all $g \in G$ and $a \in A$ we have*

$$i(g(a)) = \xi(g)(i(a)).$$

1.1. Structures

A *signature* τ is a set of relation and function symbols. Each symbol is equipped with an arity $k \in \mathbb{N}$. A *τ-structure* $\underline{A}$ is a set A (the *domain* of $\underline{A}$) together with

- a relation $R^{\underline{A}} \subseteq A^k$ for each k-ary relation symbol in τ, and
- a function $f^{\underline{A}} \colon A^k \to A$ for each k-ary function symbol in τ; here we allow the case $k = 0$ to model constant symbols.

Unless stated otherwise, $A, B, C, \ldots$ denote the domains of the structures $\underline{A}$, $\underline{B}$, $\underline{C}$, $\ldots$, respectively. We sometimes write $(A; R_1^{\underline{A}}, R_2^{\underline{A}}, \ldots, f_1^{\underline{A}}, f_2^{\underline{A}}, \ldots)$ for the relational structure $\underline{A}$ with relations $R_1^{\underline{A}}, R_2^{\underline{A}}, \ldots$ and functions $f_1^{\underline{A}}, f_2^{\underline{A}}, \ldots$ When there is no danger of confusion, we use the same symbol for a function and its function symbol, and for a relation and its relation symbol. We say that a structure is infinite if its domain is infinite.

EXAMPLE 1. A *directed graph* (or, short, *digraph* is a relational structure over the signature that contains a single binary relation symbol for the edge relation of the graph. △

EXAMPLE 2. A *(simple, undirected) graph* is a pair (V, E) consisting of a set of *vertices* V and a set of *edges* $E \subseteq \binom{V}{2}$, that is, E is a set of 2-element subsets of V. Graphs can be modelled using relational structures $\underline{G}$ using a signature that contains a single binary relation symbol R, putting $R^{\underline{G}} := E$. If we insist that a structure with this signature satisfies $(x, y) \in R^{\underline{G}} \Rightarrow (y, x) \in R^{\underline{G}}$ and not $(x, x) \in R^{\underline{G}}$, then we can associate to such a structure an undirected graph and obtain a bijective correspondence between undirected graphs and structures $\underline{G}$ as described above. △

1.1.1. Extensions and substructures. A τ-structure $\underline{A}$ is a *substructure* of a τ-structure $\underline{B}$ iff

- $A \subseteq B$,
- for each $R \in \tau$, and for all tuples $\bar{a}$ from A, $\bar{a} \in R^{\underline{A}}$ iff $\bar{a} \in R^{\underline{B}}$, and
- for each $f \in \tau$ we have that $f^{\underline{A}}(\bar{a}) = f^{\underline{B}}(\bar{a})$.

In this case, we also say that $\underline{B}$ is an *extension* of $\underline{A}$. Substructures $\underline{A}$ of $\underline{B}$ and extensions $\underline{B}$ of $\underline{A}$ are called *proper* if $A \neq B$.

Note that for every subset S of the domain of $\underline{B}$ there is a unique smallest substructure of $\underline{B}$ whose domain contains S, which is called the *substructure of $\underline{B}$ generated by S*, and which is denoted by $\underline{B}[S]$.

EXAMPLE 3. A *group* is a structure $\underline{G}$ with a binary function symbol $\cdot$ for multiplication, a unary function symbol $^{-1}$ for taking the inverse, and a constant denoted by 1, called the *neutral element*, satisfying the sentences

- $\forall x, y, z.\, x \cdot (y \cdot z) = (x \cdot y) \cdot z$,
- $\forall x.\, x \cdot x^{-1} = 1$,
- $\forall x.\, 1 \cdot x = x$, and $\forall x.\, x \cdot 1 = x$.

In this signature, the subgroups of $\underline{G}$ are precisely the substructures $\underline{G}$ as defined above; we also write $\underline{H} \leq \underline{G}$ if $\underline{H}$ is a subgroup of $\underline{G}$. Every group has the *trivial* subgroup which is the subgroup $\{1\}$ that just contains the identity element. To distinguish groups from permutation groups, we might also refer to a group as an *abstract group*. Clearly, every permutation group G gives rise to an abstract group $\underline{G}$ where $\circ$ takes the role of multiplication. △

EXAMPLE 4. Let G be a permutation group on a set A. Let $\underline{A}$ be a structure with domain A whose signature only contains unary function symbols that denote permutations of A. Note that every τ-term must have exactly one variable, and every τ-term $t(x)$ defines over $\underline{A}$ a permutation $t^{\underline{A}}$. Then $\underline{A}$ is called a *G-set* if the set of all permutations obtained in this way is precisely G. (We do allow that several function symbols denote the same element of G.) △

1.1.2. Homomorphisms. In the following, let $\underline{A}$ and $\underline{B}$ be τ-structures. A *homomorphism* h from $\underline{A}$ to $\underline{B}$ is a mapping from A to B that *preserves* each function and each relation for the symbols in τ; that is,

- if $(a_1, \dots, a_k)$ is in $R^{\underline{A}}$, then $(h(a_1), \dots, h(a_k))$ must be in $R^{\underline{B}}$; and
- $f^{\underline{B}}(h(a_1), \dots, h(a_k)) = h(f^{\underline{A}}(a_1, \dots, a_k))$.

A homomorphism from $\underline{A}$ to $\underline{B}$ is called a *strong homomorphism* if it also preserves the complements of the relations from $\underline{A}$. Injective strong homomorphisms are called *embeddings*, and we write $e \colon \underline{A} \hookrightarrow \underline{B}$ if e is an embedding of $\underline{A}$ into $\underline{B}$.

EXAMPLE 5. If G and H are groups and $h \colon G \to H$ is a map, then in order to verify that h is a homomorphism it suffices to prove that h preserves $\circ$ (Why?).

Moreover, note that injective homomorphisms are embeddings (this is not true for structures whose signature contains relation symbols! Find a counterexample!). △

1.1.3. Isomorphisms. Surjective embeddings are called *isomorphisms*. The following is immediate from the definitions.

PROPOSITION 1.1.1. *G and H are isomorphic (as permutation groups, Definition 1.0.1) if and only if there exists a G-set $\underline{A}$ and an H-set $\underline{B}$ such that $\underline{A}$ and $\underline{B}$ are isomorphic as structures (as introduced above).*

Note that if G and H are isomorphic (as permutation groups), then the corresponding abstract groups are isomorphic as well, but the converse need not be true (see Corollary 5.1.2 for a characterisation of permutation groups that are isomorphic as abstract groups).

1.1.4. Automorphisms. Homomorphisms and isomorphisms from $\underline{B}$ to itself are called *endomorphisms* and *automorphisms*, respectively. When $f \colon A \to B$ and $g \colon B \to C$, then $g \circ f$ denotes the composed function $x \mapsto g(f(x))$. Clearly, the composition of two homomorphisms (embeddings, automorphisms) is again a homomorphism (embedding, automorphism). Let $\mathrm{Aut}(\underline{A})$ and $\mathrm{End}(\underline{A})$ be the sets of automorphisms and endomorphisms, respectively, of $\underline{A}$. The set $\mathrm{Aut}(\underline{A})$ can be viewed as a group, and $\mathrm{End}(\underline{A})$ as a monoid with respect to composition.

1.1.5. Expansions and reducts. Let σ, τ be signatures with $\sigma \subseteq \tau$. If $\underline{A}$ is a σ-structure and $\underline{B}$ is a τ-structure, both with the same domain, such that $R^{\underline{A}} = R^{\underline{B}}$ for all relations $R \in \sigma$ and $f^{\underline{A}} = f^{\underline{B}}$ for all functions and constants $f \in \sigma$, then $\underline{A}$ is called a *reduct* of $\underline{B}$, and $\underline{B}$ is called an *expansion* of $\underline{A}$.

1.1.6. Disjoint unions. Let τ be a relational signature. A *disjoint union* $\underline{A} \uplus \underline{B}$ of two τ-structures $\underline{A}$ and $\underline{B}$ is the union of isomorphic copies of $\underline{A}$ and $\underline{B}$ with disjoint domains. That is, for all $R \in \tau$ we have $R^{\underline{A} \uplus \underline{B}} = R^{\underline{A}} \cup R^{\underline{B}}$. As disjoint unions are unique up to isomorphism, we usually speak of *the* disjoint union of $\underline{A}$ and $\underline{B}$. The disjoint union of a set of τ-structures $\mathcal{C}$ is defined analogously (and the disjoint union of an empty set of structures is the τ-structure with empty domain). A relational structure is called *connected* if it is not the disjoint union of two nonempty structures.

EXAMPLE 6. For digraphs (see Example 1), connectivity in the sense we have just introduced corresponds to *weak connectivity* in graph theory. The definition of *strong connectivity* for digraphs can be found in Exercise 5. For undirected graphs, connectivity in the sense introduced above coincides with the notion of connectivity from graph theory. △

1.1.7. Direct products. Let $\underline{A}$ and $\underline{B}$ be τ-structures. Then the (*direct*, or *categorical*) *product* $\underline{A} \times \underline{B}$ is the τ-structure $\underline{C}$ with domain $A \times B$ such that

- for each k-ary $R \in \tau$ we have $((a_1, b_1), \ldots, (a_k, b_k)) \in R^{\underline{C}}$ if and only if $(a_1, \ldots, a_k) \in R^{\underline{A}}$ and $(b_1, \ldots, b_k) \in R^{\underline{B}}$;
- for each k-ary $f \in \tau$

$$f^{\underline{C}}((a_1, b_1), \ldots, (a_k, b_k)) = (f(a_1, \ldots, a_k), f(b_1, \ldots, b_k)).$$

The direct product $\underline{A} \times \underline{A}$ is also denoted by $\underline{A}^2$, and the k-fold product $\underline{A} \times \cdots \times \underline{A}$, defined analogously, by $\underline{A}^k$ (it is straightforward to define this also for infinite k).

1.1.8. Congruences. Let $\underline{A}$ be a structure with a purely functional signature τ. A *congruence* of $\underline{A}$ is an equivalence relation E on A such that for every k-ary $f \in \tau$ the function $f^{\underline{A}}$ is a homomorphism from $(A; E)^k$ to $(A; E)$.

EXAMPLE 7. Let G be a permutation group on a set A, and let $\underline{A}$ be a G-set. Recall that $\underline{A}$ is a structure with a purely functional signature (all functions are unary). Then the congruences of $\underline{A}$ are equivalence relations on A that are preserved by all permutations in G. Such equivalence relations are also called *congruences of* G, and an important topic of this chapter (see in particular Section 1.4). $\triangle$

EXAMPLE 8. Let $\underline{G}$ be a group. Then there is a natural bijection between the congruences of $\underline{G}$ and the normal subgroups of $\underline{G}$; this is treated in Section 4.6. $\triangle$

Exercises.

(1) Show that a function $f : A \to A$ preserves $h : A \to A$ if and only if f preserves the *graph of* h, i.e., the binary relation

$$\{(a, h(a)) \mid a \in A\}.$$

(2) Show that isomorphic structures have isomorphic automorphism groups, but that the converse is false.

(3) Prove Proposition 1.1.1.

(4) Consider the following structures.

$$\Gamma_1 := (\mathbb{Q}; \{(x, y) : x = y + 1\})$$
$$\Gamma_2 := \big(\mathbb{Q}; \big\{(x, y, u, v) : x - y = u - v \in \{1, -1\}\big\}\big)$$
$$\Gamma_3 := (\mathbb{Q}; \{(x, y) : |x - y| = 1\})$$

Show that

$$\{\mathrm{id}_{\mathbb{Q}}\} \subsetneq \mathrm{Aut}(\Gamma_1) \subsetneq \mathrm{Aut}(\Gamma_2) \subsetneq \mathrm{Aut}(\Gamma_3) \subsetneq \mathrm{Sym}(\mathbb{Q}).$$

(5) A directed graph $\underline{G}$ (Example 2) is called *strongly connected* if for any $a, b \in G$ there exists a sequence $c_0, c_1, \dots, c_n$ with $c_0 = a$, $c_n = b$, and $(c_i, c_{i+1}) \in E^{\underline{G}}$ for all $i \in \{0, \dots, n-1\}$. Show that if $\underline{G}$ is finite and connected (in the sense of Example 6) and $\mathrm{Aut}(\underline{G})$ is transitive, then $\underline{G}$ is strongly connected.

(6) Show that the previous exercise is false in general for infinite digraphs $\underline{G}$.

1.2. Automorphism Groups

Let G be a permutation group on a set A. When is G the automorphism group of a structure with domain A? This has the following elegant answer.

DEFINITION 1.2.1. *We say that* $S \subseteq \mathrm{Sym}(A)$ *is* (locally) closed *(or* closed in $\mathrm{Sym}(A)$*) if it contains all* $f \in \mathrm{Sym}(A)$ *with the property that for all finite* $F \subseteq A$ *there exists a* $g \in S$ *such that* $f(x) = g(x)$ *for all* $x \in F$.

PROPOSITION 1.2.2. *A permutation group* G *on a set* A *is the automorphism group of a relational structure with domain* A *if and only if* G *is closed in* $\mathrm{Sym}(A)$.

In the proof of this proposition, the following concept is useful. We write $\mathrm{sInv}(G)$ for the set of all relations over A that are strongly preserved by all permutations $g \in G$, i.e., both g and g^{-1} preserve the relation.

Definition 1.2.3 (Canonical structure). *A relational structure with domain A whose relations are exactly the relations from* sInv(G) *is called a* canonical structure *for G.*

Proof of Proposition 1.2.2. For the forwards implication, suppose that $G = \mathrm{Aut}(\underline{A})$ and that $f \in \mathrm{Sym}(A) \setminus G$. Then f or f^{-1} does not preserve a relation R or function from $\underline{A}$. Suppose that $(a_1, \ldots, a_n) \in R^{\underline{A}}$ but $(f(a_1), \ldots, f(a_n)) \notin R^{\underline{A}}$. Then there is no $g \in G$ such that $g(x) = f(x)$ for all $x \in \{a_1, \ldots, a_n\}$. The proof for f^{-1} is analogous.

For the reverse implication, let $\underline{A}$ be a canonical structure for G. Clearly, every $g \in G$ is an automorphism of $\underline{A}$. Conversely, let $f \in \mathrm{Aut}(\underline{A})$. By assumption, to show that $f \in G$, it suffices to show that for every finite tuple $(a_1, \ldots, a_n)$ of elements from X there exists an $g \in G$ such that $f(x) = g(x)$ for all $x \in \{a_1, \ldots, a_n\}$. The relation $R := \{(g(a_1), \ldots, g(a_n)) \mid g \in G\}$ is preserved by all operations in G and hence belongs to the relations of $\underline{A}$. Thus, f preserves R. Also id $\in G$, and therefore $(a_1, \ldots, a_n) \in R$, and so R contains $(f(a_1), \ldots, f(a_n)) = (g(a_1), \ldots, g(a_n))$ for some $g \in G$. We therefore have $G = \mathrm{Aut}(\underline{A})$ as desired. □

1.2.1. The topology of pointwise convergence. The word *closed* suggests a topology, and indeed there is corresponding topology on G, called the *topology of pointwise convergence*. Topological aspects will be treated properly in our chapter on topological groups, Chapter 4. However, we already give some of the basic topological definitions now, specialised to the topology of pointwise convergence on $\mathrm{Sym}(A)$, which is the topology we will be working with in the following sections. A *basic open set* is a subset of $\mathrm{Sym}(A)$ of the form $S(a, b) := \{g \in \mathrm{Sym}(A) \mid g(a) = b\}$ where $a, b \in A^n$ for some $n \geq 1$, and $g(a) := (g(a_1), \ldots, g(a_n))$. A subset of $\mathrm{Sym}(A)$ is *open* if it is a union of basic open sets.

Proposition 1.2.4. *The open subsets of* $\mathrm{Sym}(A)$ *define a topology. A subset of* $\mathrm{Sym}(A)$ *is closed (in the sense of Definition 1.2.1) if and only if it is the complement of an open set.*

Proof. The empty set and $\mathrm{Sym}(A)$ are clearly open. The intersection of two basic open $S(a, b)$ and $S(a', b')$ equals $S((a, a'), (b, b'))$, again a basic open set. For the second statement, let $S \subseteq \mathrm{Sym}(A)$. Then S is closed if and only if $C := \mathrm{Sym}(A) \setminus S$ can be written as

$$C = \bigcup_{n \in \mathbb{N}, a, b \in A^n \text{ s.t. } \forall g \in S : g(a) \neq b} S(a, b)$$

and the latter is a union of basic open sets. □

Closed permutation groups on a countable set have at most 2^ω many elements. But they cannot have arbitrary cardinalities smaller than 2^ω, which we can prove even without assuming the continuum hypothesis. To phrase our fundamental result about the cardinalities of permutation groups on a countable set, we need the following definition.

Definition 1.2.5 (point stabiliser). *Let G be a permutation group over the base set A and let $B \subseteq A$. Then the* point stabiliser $G_{(B)}$ *is the subgroup of G consisting of all $\alpha \in G$ such that $\alpha(b) = b$ for all $b \in B$. If $b = (b_1, \ldots, b_n)$ is a sequence of elements of A, then we also write G_b instead of $G_{(\{b_1, \ldots, b_n\})}$.*

Theorem 1.2.6 (Corollary 4.1.5 in [**73**]). *Let $\underline{G} \leq \mathrm{Sym}(\mathbb{N})$ be closed. Then the following are equivalent.*

(1) *There is an $a \in \mathbb{N}^n$, $n \in \mathbb{N}$ such that $|G_a| = 1$.*

(2) $|G| \leq \omega$.
(3) $|G| < 2^\omega$.

PROOF. For the implication from (1) to (2), let $\bar{b}_1, \bar{b}_2, \ldots$ be an enumeration of the tuples $\bar{b} \in \mathbb{N}^n$ such that there exists $g_{\bar{b}} \in G$ with $g_{\bar{b}}(\bar{a}) = \bar{b}$. Then every $g \in G$ can be written as $g_{\bar{b}_i} \circ h$ for some $i \in \mathbb{N}$ and some $h \in G_{\bar{a}}$.

The implication from (2) to (3) is trivial.

For the implication from (3) to (1), suppose that $\neg(1)$. We define inductively sequences $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ of tuples of elements of $\mathbb{N}$ and a sequence $g_0, g_1, g_2, \ldots$ of elements of G such that for every $i \in \mathbb{N}$

(1) $g_i(b_i) = b_i$,
(2) $g_i(a_i) \neq a_i$,
(3) a_i contains i as an entry,
(4) b_{i+1} is the concatenation of all sequences $k_i \circ \cdots \circ k_0(a_0, \ldots, a_i)$ where $k_j \in \{g_j, 1\}$ for all $j \in \{0, \ldots, k\}$.

Initially, $b_0 := ()$. If b_i has been chosen for $i \in \mathbb{N}$, we have by assumption that $|G_{b_i}| \geq 2$ and hence there is some $g_i \in G$ which fixes b_i but is not the identity, so there is a tuple a_i such that $g_i(a_i) \neq a_i$, which gives us (1) and (2). We may add the entry i to a_i to ensure (3). Then item (4) determines b_{i+1} and this concludes the construction of the sequences.

For any subset $S \subseteq \mathbb{N} \setminus \{0\}$ and $i \in \mathbb{N}$, define

$$g_i^S := \begin{cases} g_i & i \in S \\ 1 & i \notin S \end{cases}$$
$$f_i^S := g_i^S \circ \cdots \circ g_1^S \circ g_0^S.$$

For each $j \geq i$ we have $f_j^S(a_i) = f_i^S(a_i)$ by the properties (1) and (4). In particular, (3) implies that $f_j^S(i) = f_i^S(i)$. Define $h^S \colon \mathbb{N} \to \mathbb{N}$ by $h^S(i) := f_i^S(i)$ for every $i \in \mathbb{N}$. To see that h^S is surjective, let $i \in \mathbb{N}$ and put $j := (f_i^S)^{-1}(i)$. If $j \leq i$ then

$$h^S(j) = f_j^S(j) = f_i^S(j) = f_i^S((f_i^S)^{-1}(i)) = i.$$

If $j > i$, then

$$h^S(j) = f_j^S((f_i^S)^{-1}(i)) = g_j^S \circ \cdots \circ g_{i+1}^S(i) = i.$$

We claim that for every finite $F \subseteq \mathbb{N}$ there exists $g \in G$ such that $h^S(x) = g(x)$ for all $x \in F$. Let $i := max(F)$. Then note that for every $x \in F$ we have $h^S(x) = f_x^S(x) = f_i^S(x)$ and $f_i^S \in G$. In particular, h^S is injective. Since G is closed in S_ω, we have that $h^S \in G$.

It remains to show that $h^S \neq h^T$ whenever $S \neq T$. Let $i > 0$ be smallest which is, say, in S but not in T. Let $j \geq i$ be larger than all the entries in $(f_{i-1}^S)^{-1}(a_i)$.

Then

$$\begin{aligned}
h^S((f^S_{i-1})^{-1}(a_i)) &= f^S_j((f^S_{i-1})^{-1}(a_i)) \\
&= g^S_j \circ \cdots \circ g^S_{i+1} \circ g^S_i(a_i) \\
&= g^S_i(a_i) && \text{(by (4) and (1)} \\
&= g_i(a_i) && \text{(since } i \in S) \\
&\neq a_i && \text{(by (2)} \\
&= g^T_j \circ \cdots \circ g^T_{i+1} \circ g^T_i(a_i) && \text{(since } i \notin T) \\
&= f^T_j(f^T_{i-1})^{-1}(a_i) \\
&= f^T_j(f^S_{i-1})^{-1}(a_i) && \text{(by the choice of } i) \\
&= h^T((f^S_{i-1})^{-1}(a_i)).
\end{aligned}$$

□

1.2.2. Aut-sInv. Recall that the automorphism group of a relational structure $\underline{A}$, i.e., the set of all automorphisms of $\underline{A}$, is denoted by $\mathrm{Aut}(\underline{A})$. In the following it will be convenient to define the operator Aut also on sets $\mathcal{R}$ of relations over the same domain A, in which case $\mathrm{Aut}(\mathcal{R})$ denotes the set of all permutations p of A such that p and its inverse p^{-1} preserve all relations form $\mathcal{R}$.

For $P \subseteq \mathrm{Sym}(A)$, and sets $\mathcal{R}$ of relations over the domain A, we present a description of the closure operator $P \mapsto \mathrm{Aut}(\mathrm{sInv}(P))$; the closure operator $\mathcal{R} \mapsto \mathrm{sInv}(\mathrm{Aut}(\mathcal{R}))$ will be described in Section 3.1.

DEFINITION 1.2.7. *For $P \subseteq \mathrm{Sym}(A)$, we define*

- $\langle P\rangle$*, the* permutation group generated by P*, to be the smallest permutation group on A that contains P.*
- $\overline{P}$*, the* closure of P in $\mathrm{Sym}(A)$*, to be the smallest closed subset of* $\mathrm{Sym}(A)$ *that contains P.*

EXAMPLE 9. Let P be the set of permutations f of $\mathbb{N}$ that have *finite support*, that is, the set $\{i \in \mathbb{N} \mid f(i) \neq i\}$ is finite. Then $P \subsetneq \overline{P} = \mathrm{Sym}(\mathbb{N})$. △

PROPOSITION 1.2.8. *Let $P \subseteq \mathrm{Sym}(A)$ be arbitrary. Then* $\mathrm{Aut}(\mathrm{sInv}(P)) = \overline{\langle P\rangle}$ *equals the smallest permutation group that contains P and is closed in* $\mathrm{Sym}(A)$*.*

PROOF. Let P' be the smallest permutation group that contains P and is closed in $\mathrm{Sym}(A)$. Since $P \subseteq P'$ and P' is a permutation group, we must have $\langle P\rangle \subseteq P'$, and therefore also $\overline{\langle P\rangle} \subseteq P'$ since P' is closed in $\mathrm{Sym}(A)$. To show the converse inclusion $P' \subseteq \overline{\langle P\rangle}$, it suffices to verify that $\overline{\langle P\rangle}$ is a closed subgroup of $\mathrm{Sym}(A)$. Since $\overline{\langle P\rangle}$ is clearly closed in $\mathrm{Sym}(A)$ we only have to show that $\overline{\langle P\rangle}$ contains compositions and inverses. We do the verification for closure under compositions on finite subsets F of A. Indeed, when $f, g \in \overline{\langle P\rangle}$, then there are $f', g' \in \langle P\rangle$ such that $f(x) = f'(x)$ for all $x \in F$ and $g(x) = g'(x)$ for all $x \in f(F)$. We therefore have $g(f(x)) = g'(f'(x))$ for all $x \in F$, and hence $g \circ f \in \overline{\langle P\rangle}$, as desired.

We now show that $\overline{\langle P\rangle} \subseteq \mathrm{Aut}(\mathrm{sInv}(P))$. Let $p \in \overline{\langle P\rangle}$ be arbitrary, and let R be from $\mathrm{sInv}(P)$. We have to show that p and p^{-1} preserve R. Let $t \in R$; we have that $p(t) = q_1 \circ \cdots \circ q_k(t)$ for some permutations $q_1, \ldots, q_k \in P \cup P^{-1}$. Since $q_1, \ldots, q_k$ preserve R, we have that $q(t) \in R$. The argument for p^{-1} is analogous.

Finally, we show $\mathrm{Aut}(\mathrm{sInv}(P)) \subseteq \overline{\langle P\rangle}$. Let p be from $\mathrm{Aut}(\mathrm{sInv}(P))$. It suffices to show that for every finite subset $\{a_1, \ldots, a_n\}$ of A there is a $q \in \langle P\rangle$ such that $p(a_i) = q(a_i)$ for all $i \leq n$. Consider the relation $\{(q(a_1), \ldots, q(a_n)) \mid q \in \langle P\rangle\}$. It is preserved by all permutations in P. Therefore, p preserves this relation, and so there exists $q \in \langle P\rangle$ as required. □

Exercises.

(7) Let G be the permutation group on $\mathbb{Z}$ that consists of all shift operations $\{x \mapsto x + c \mid c \in \mathbb{Z}\}$. Is G closed? 

(8) Let G be the permutation group on $\mathbb{Z}$ that is generated by the transpositions $\tau_i := (i, -i)$, for $i \in \mathbb{Z}$. What is the cardinality of G, and what is the cardinality of $\overline{G}$? 

(9) Let $P = \{f, g\} \subseteq \mathrm{Sym}(\mathbb{Z})$ where f is a transposition and g is $x \mapsto x + 1$. Determine the cardinalities of $\langle P \rangle$, $\overline{P}$, and $\overline{\langle P \rangle}$. 

(10) The *finitary alternating group* A on $\mathbb{N}$ is the set of all permutations of $\mathbb{N}$ that can be written as a composition of an even number of transpositions. Determine $\overline{A}$. 

(11) Let $G, H \leq \mathrm{Sym}(A)$. Show that if H is closed, then $G \leq H$ if and only if every relation $R \subseteq A^n$ which is preserved by H is also preserved by G. 

1.3. Group Actions

We now consider *abstract groups*, that is, algebraic structures $\underline{G}$ over a set G of group elements, with a function symbol for multiplication of group elements, a function for the inverse of a group element, and the constant for the identity (see Example 3). The link to permutation groups is given by the concept of an *action* of such a group on a set, which is described below.

Definition 1.3.1. *Let $\underline{G}$ be a group and X a set. An* action of $\underline{G}$ on X *is a homomorphism ϕ from $\underline{G}$ to* $\mathrm{Sym}(X)$. *An action ϕ is called* faithful *if ϕ is injective.*

Example 10 (The componentwise action). If $\underline{G}$ is a permutation group on a set X and $n \in \mathbb{N}$, then *the componentwise action of $\underline{G}$ on X^n* is given by

$$\xi(g)(x_1, \dots, x_n) := (g(x_1), \dots, g(x_n)).$$

Note that this action is faithful unless $n = 0$. △

Example 11. If $\underline{G}$ is a permutation group on a set X and $n \in \mathbb{N}$, then *the setwise action of $\underline{G}$ on $\binom{X}{n}$* is given by

$$\xi(g)(\{x_1, \dots, x_n\}) := \{g(x_1), \dots, g(x_n)\}.$$

If and $n > 0$, then this action is faithful; this follows e.g. from the argument given in Example 76 in Chapter 5. △

Clearly, to every action of $\underline{G}$ on X we can associate a permutation group as considered before, namely the image of the action in $\mathrm{Sym}(X)$. Conversely, to every permutation group G on a set X we can associate an abstract group $\underline{G}$ whose domain is G (the permutations), where composition and inverse are defined in the obvious way, and which acts on X faithfully by $\phi(g) := g$.

In this way we can also use other terminology introduced for permutation groups (such a transitivity, congruences, primitivity, etc.) for group actions. For instance, we say that an action $\xi \colon \underline{G} \to \mathrm{Sym}(X)$ is *transitive* if the permutation group $\xi(G) \leq \mathrm{Sym}(X)$ is transitive. We give an alternative characterisation of action which in many texts is taken to be the official definition.

Proposition 1.3.2. *Let $\underline{G}$ be a group and X a set. The $\phi \colon \underline{G} \to \mathrm{Sym}(X)$ is an action of $\underline{G}$ on X if and only if the map $\cdot \colon G \times X \to X$ defined by $g \cdot x := \phi(g)(x)$ satisfies*

- *$(gh) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G$ and $x \in X$, and*
- *$1 \cdot x = x$ for every $x \in X$.*

The action ϕ is faithful if and only if for any two distinct $g, h \in G$ there exists an $x \in X$ such that $g \cdot x \neq h \cdot x$.

PROOF. The proof is just moving symbols. $\square$

If $x \in X$ then the *orbit* of x with respect to an action of $\underline{G}$ on X is the set $G \cdot x := \{g \cdot x \mid g \in G\}$. An *orbit of k-tuples* is an orbit of the componentwise action of $\underline{G}$ on X^k (Example 10).

Cayley's theorem states that every group has a representation as a permutation group.

THEOREM 1.3.3 (Cayley's theorem). *Let $\underline{G}$ be any group. Then $\underline{G}$ has a faithful action on G.*

PROOF. The action ξ on G is by *left translation*: for $g \in G$, we define $\xi(g)$ by

$$\xi(g)(h) := gh$$

for all $h \in G$. It is straightforward to verify that this map is an injective group homomorphism. $\square$

We close this section with some important examples of group actions.

EXAMPLE 12 (Action by left translation). A *left coset* of a subgroup $\underline{H}$ of $\underline{G}$ is a set of the form $\{gh \mid h \in H\}$ for $g \in G$, also written gH. Clearly, the set of all left cosets of $\underline{H}$ partitions G, and is denoted by G/H. The cardinality of G/H is called the *index* of $\underline{H}$ in $\underline{G}$. We define an action ξ of $\underline{G}$ on G/H by setting $\xi(f)(gH) := (fg)H$. This action is also called the *action of $\underline{G}$ on G/H by left translation.* It is transitive since for any $g_1H, g_2H \in G/H$ the map $\xi(g_2 g_1^{-1})$ takes g_1H to g_2H. Note that this example generalises the construction in the proof of Cayley's theorem since we may take $\underline{H} = \{1\}$. $\triangle$

EXAMPLE 13 (Action by conjugation). The map $\xi \colon G \to G$ given by

$$\xi(g)(h) := ghg^{-1}$$

is called the *action of $\underline{G}$ on G by conjugation.* First note that for every $g \in \underline{G}$ the operation $\xi(g)$ is in fact an automorphism of $\underline{G}$, because it preserves the group structure:

$$\xi(g)(h_1h_2) = g(h_1h_2)g^{-1} = gh_1g^{-1}gh_2g^{-1} = \xi(g)(h_1) \cdot \xi(g)(h_2).$$

In fact, ξ is a homomorphism from $\underline{G}$ to $\mathrm{Aut}(\underline{G})$, because for all $g_1, g_2 \in G$ and $h \in G$ we have

$$\begin{aligned}\xi(g_1g_2)(h) &= (g_1g_2)h(g_1g_2)^{-1} \\ &= g_1(g_2hg_2^{-1})g_2^{-1} \\ &= \xi(g_1)(\xi(g_2)h).\end{aligned}$$

The orbits of this action are called the *conjugacy classes*, and the stabiliser of $h \in G$ with respect to this action is the centraliser $C_G(h)$ of h:

$$C_G(h) = \{g \in G \mid gh = hg\}. \qquad \triangle$$

Exercises.

(12) Show that if $(a_1a_2\ldots)(b_1b_2\ldots)\ldots$ is the cycle representation of $g \in S_n$, and $f \in S_n$, then $(f(a_1)f(a_2)\ldots)(f(b_1)f(b_2)\ldots)\ldots$ is the cycle representation of $f^{-1}gf$.

(13) Let $\underline{H}_1$ and $\underline{H}_2$ be subgroups of $\mathrm{Sym}(X)$. Show that $\underline{H}_1$ and $\underline{H}_2$ are isomorphic as permutation groups if and only if there exists $f \in \mathrm{Sym}(X)$ such that

$$H_1 = \{fhf^{-1} \mid h \in H_2\}.$$

(14) Let $\underline{G}$ be a group with an action ϕ on A, and let $\underline{A}$ be the corresponding G-set, i.e., the structure with domain A which contains a unary operation for every permutation in the image of ϕ. Show that the domains of the substructures of $\underline{A}$ are precisely the orbits of the action of $\underline{G}$ on A.

(15) (Exercise 1 on page 9 of [**39**]) Let H a subgroup of G. When is the the action ϕ of G on G/H by left translation faithful? **Hint.** Show that the kernel of ϕ is $\bigcap_{g \in G} g^{-1}Hg$.

(16) Let $k \geq 1$ and $n > k$. Show that the setwise action of S_n on k-element subsets of $\{1, \ldots, n\}$ is faithful.

In the following we review the classical theory how permutation groups can be built from simpler permutation groups forming various forms of products. The *direct product* of a sequence of groups $(\underline{G}_i)_{i \in I}$ is the product of this sequence as defined in general in Section 1.1.6; note that the product is again a group. Products appear in several ways when studying permutation groups; the first is when we want to describe the relation between a permutation group and its 'transitive constituents', described in the following.

1.3.1. The intransitive action of the direct product. If $\underline{G}$ acts on a set X and $O \subset X$ is an orbit with respect to this action, then $\underline{G}$ naturally acts transitively on O by restriction; we call the corresponding group $\underline{H}$ the *group induced by* O, or a *transitive constituent*.

PROPOSITION 1.3.4 (see [**39**]). *Let $\underline{G}$ be a permutation group on X and let I be the set of orbits of $\underline{G}$. Then for every $O \in I$ let $\underline{G}[O] := \{g|_O : g \in G\}$ be the permutation group induced by G on O. Then $\underline{G}$ is isomorphic to a subgroup of $\prod_{O \in I} \underline{G}_O$, and $g \mapsto g|_O$ is a surjective homomorphism from $\underline{G}$ to $\underline{G}[O]$ for each $O \in I$.*

PROOF. Straightforward from the definitions. □

DEFINITION 1.3.5. *Let $\underline{G}_1$ and $\underline{G}_2$ be groups acting on disjoint sets X and Y, respectively. Then the action of $\underline{G}_1 \times \underline{G}_2$ on $X \cup Y$ defined by $(g_1, g_2) \cdot z = g_1 z$ if $z \in X$, and $g_2 z$ if $y \in Y$, is called the* natural intransitive action *of $\underline{G}_1 \times \underline{G}_2$ on $X \cup Y$.*

If $\underline{G}_1$ and $\underline{G}_2$ are the automorphism groups of relational structures $\underline{A}$ and $\underline{B}$ with disjoint domains A and B and signatures τ_1 and τ_2, respectively, then the image of the natural intransitive action on $A \cup B$ (as a homomorphism from $\underline{G}_1 \times \underline{G}_2$ to $\mathrm{Sym}(A \cup B)$) can also be described as the automorphism group of a relational structure $\underline{C}$: we can take for $\underline{C}$ the structure with domain $A \cup B$ and with signature $\tau_1 \cup \tau_2 \cup \{P\}$, where $P^{\underline{C}} = A$, $R^{\underline{C}} = R^{\underline{A}}$ for $R \in \tau_1 \setminus \tau_2$, and $R^{\underline{C}} = R^{\underline{B}}$ for $R \in \tau_2 \setminus \tau_1$, and $R^{\underline{C}} = R^{\underline{A}} \cup R^{\underline{B}}$ if $R \in \tau_1 \cap \tau_2$.

Exercises.

(17) Give an example of two structures $\underline{A}$ and $\underline{B}$ that illustrates why we need the extra unary predicate in the definition of the structure $\underline{C}$ above. In other words: show that if we view $\underline{A}$ and $\underline{B}$ as $(\tau_1 \cup \tau_2)$-structures, then the automorphism group of the disjoint union $\underline{A} \uplus \underline{B}$ is in general not the same as the image of the intransitive action of $\underline{G}_1 \times \underline{G}_2$ on $A \cup B$.

(18) Prove Proposition 1.3.4.

1.3.2. The product action. When $\underline{G}_1$ is a group acting on a set X, and $\underline{G}_2$ a group acting on a set Y, there is another important natural action of $\underline{G} := \underline{G}_1 \times \underline{G}_2$ besides the intransitive natural action of $\underline{G}$, which is called the *product action* of $\underline{G}$. In this action, $\underline{G}$ acts on $X \times Y$ by $(g_1, g_2) \cdot (x, y) = (g_1 x, g_1 y)$. If the actions of $\underline{G}_1$ and $\underline{G}_2$ are transitive, then the product action is clearly transitive, too.

When $\underline{G}_1$ and $\underline{G}_2$ are the automorphism groups of structures $\underline{A}$ and $\underline{B}$, then the image of the product action of $\underline{G}$ in $\mathrm{Sym}(A \times B)$ is the automorphism group of the following structure, which we call the *full product structure* of two relational structures $\underline{A}$ and $\underline{B}$, and denote by $\underline{A} \boxtimes \underline{B}$. Let σ be the signature of $\underline{A}$, and τ be the signature of $\underline{B}$; we assume that σ and τ are disjoint, otherwise we rename the relations so that the assumption is satisfied. For each k-ary $R \in \sigma$, the structure $\underline{A} \boxtimes \underline{B}$ contains the relation $\{((a_1, b_1), \dots, (a_k, b_k)) \mid (a_1, \dots, a_k) \in R^{\underline{A}}, b_1, \dots, b_k \in B\}$, and for each k-ary $R \in \tau$, it contains the relation $\{((a_1, b_1), \dots, (a_k, b_k)) \mid (b_1, \dots, b_k) \in R^{\underline{B}}, a_1, \dots, a_k \in A\}$. Finally, we also add the relations $P_1 = \{((a_1, b_1), (a_2, b_2)) \mid a_1 = a_2\}$ and $P_2 = \{((a_1, b_1), (a_2, b_2)) \mid b_1 = b_2\}$ to $\underline{A} \boxtimes \underline{B}$.

PROPOSITION 1.3.6. *The automorphism group of* $\underline{C} := \underline{A} \boxtimes \underline{B}$ *is* $\underline{G}_1 \times \underline{G}_2$ *in its product action on* $A \times B$.

PROOF. Let h be the product action of $\underline{G} = \underline{G}_1 \times \underline{G}_2$ on $A \times B$, viewed as a homomorphism from $\underline{G}$ to $\mathrm{Sym}(A \times B)$. Let (g_1, g_2) be an element of $\underline{G}$. Then $h((g_1, g_2))$ is the permutation $(x, y) \mapsto (g_1 x, g_2 y)$ of $A \times B$, and this map preserves $\underline{C}$: when $((a_1, b_1), \dots, (a_k, b_k)) \in R^{\underline{C}}$, for $R \in \sigma$, then $(a_1, \dots, a_k) \in R^{\underline{A}}$, and so $(g_1 a_1, \dots, g_1 a_k) \in R^{\underline{A}}$. Therefore, $((g_1 a_1, g_2 b_1), \dots, (g_1 a_k, g_2 b_k)) \in R^{\underline{C}}$. The proof for the relation symbols $R \in \tau$ is analogous.

We now show that conversely, every automorphism g of $\underline{C}$ is in the image of h. Let Note that P_1 and P_2 are congruences of the automorphism group of $\underline{C}$. Fix elements $a_0 \in A, b_0 \in B$. Let g_1 be the permutation of A that maps $a \in A$ to the point a' such that $g((a, b_0)) = (a', b')$. Similarly, let g_2 be the permutation of B that maps $b \in B$ to the point b' such that $g((a_0, b)) = (a', b')$. Since g preserves P_1, P_2, the definition of g_1 and g_2 does not depend on the choice of a_0 and b_0. Moreover, g_1 is from $\underline{G}_1$, since g preserves the relations for the symbols from σ. Similarly, g_2 is from $\underline{G}_2$. Then $g' := h((g_1, g_2))$ equals g, since $g'((a, b) = (g_1 a, g_2 b) = g(a, b)$. Hence, g is a permutation of $A \times B$ that lies in the image of h. □

Note that Proposition 1.3.6 becomes false in general when we omit the relations P_1 and P_2 in $\underline{A} \boxtimes \underline{B}$ (consider for example the countably infinite structure without structure $\underline{B}$ (that is, $\underline{B}$ has the empty signature).

Finally we remark that $\mathrm{Aut}((\underline{A} \boxtimes \underline{B}) \boxtimes \underline{C})$ and $\mathrm{Aut}(\underline{A} \boxtimes (\underline{B} \boxtimes \underline{C}))$ are isomorphic as permutation groups. We explicitly define the d-fold full product as follows.

DEFINITION 1.3.7 (Full product of d structures). *Let* $\underline{B}_1, \dots, \underline{B}_d$ *be structures with disjoint relational signatures* $\tau_1, \dots, \tau_d$. *We denote by* $\underline{B}_1 \boxtimes \cdots \boxtimes \underline{B}_d$ *the structure with domain* $B := B_1 \times \cdots \times B_d$ *that contains for every* $i \leq d$, *and every* m*-ary* $R \in (\tau_i \cup \{=\})$ *an* m*-ary relation defined by*

$$\{((x_1^1, \dots, x_1^d), \dots, (x_m^1, \dots, x_m^d)) \in B^m \mid (x_1^i, \dots, x_m^i) \in R^{\underline{B}_i}\} .$$

If $\underline{B} := \underline{B}_1 = \cdots = \underline{B}_k$, *then we first rename* $R \in \tau_i$ *into* R_i *so that the factors have pairwise disjoint signatures, and then write* $\underline{B}^{[d]}$ *for* $\underline{B}_1 \boxtimes \cdots \boxtimes \underline{B}_d$.

Exercises.

(19) Let $\underline{G} = \underline{K} \times \underline{H}$. Prove that $\underline{G}$ has a subgroup $\underline{K}^*$ isomorphic to $\underline{K}$ and a subgroup $\underline{H}^*$ isomorphic to $\underline{H}$ such that
   (a) $G = K^*H^* := \{kh \mid k \in K^*, h \in H^*\}$
   (b) $K^* \cap H^* = \{1\}$;
   (c) $kh = hk$ for all $k \in K^*$ and $h \in H^*$.

(20) Prove that the previous exercise provides a characterisation of groups that are direct products: if $\underline{G}$ is a group with subgroups $\underline{K}$ and $\underline{H}$, then the map $K \times H \to G$ given by $(k,h) \mapsto kh$ is an isomorphism between $\underline{K} \times \underline{H}$ and $\underline{G}$ if and only if the three items from the previous exercise hold for $\underline{K} = \underline{K}^*$ and $\underline{H} = \underline{H}^*$.

1.4. Congruences and Primitivity

Recall from Section 1.1.8 that a *congruence* of a permutation group G on a set D is an equivalence relation on D that is preserved by all permutations in G. The equivalence classes of a congruence are also called *congruence classes*.

Definition 1.4.1. *Let G be a permutation group on a set D. A subset S of D is called a* block *of G if $g(S) = S$ or $g(S) \cap S = \emptyset$ for every $g \in G$.*

Lemma 1.4.2. *Let G be a permutation group on a set D. Then $S \subseteq D$ is a block of G if and only if S is a congruence class of a congruence of G.*

Proof. Suppose that S is an equivalence class of the congruence C, and suppose that $g(S) \cap S$ contains an element t. That is, there is an element $s \in S$ such that $g(s) = t \in S$. We will show that $g(S) = S$. Arbitrarily choose $r \in S$. Then $(r,s) \in C$, and hence $(g(r), g(s)) = (g(r), t) \in C$. Since $t \in S$, it follows that $g(r) \in S$. Hence, $g(S) \subseteq S$. But also $(r,t) \in C$, and $(g^{-1}(r), g^{-1}(t)) = (g^{-1}(r), s) \in C$. Since $s \in S$, it follows that $g^{-1}(r) \in S$ and $r \in g(S)$. Hence, $S \subseteq g(S)$.

Now suppose that $S \subseteq D$ is a block of G. Define

$$C := \{(x,y) \mid \exists g \in G : g(x), g(y) \in S\} \cup \{(x,x) \mid x \in D\}\,.$$

This relation is clearly reflexive, symmetric, and preserved by G. In order to prove that C is a congruence it remains to verify transitivity. Let $(x,y),(y,z) \in C$. Then there are $g_1, g_2 \in G$ such that $g_1(x), g_1(y), g_2(y), g_2(z) \in S$. Thus, $S \cap g_2(g_1^{-1}(S))$ contains $g_2(y)$, and $(g_2 \circ g_1^{-1})(S) = S$ because S is a block. Since $g_1(x) \in S$ this implies that $g_2(g_1^{-1}(g_1(x))) \in S$. So $g_2(x) \in S$ and $g_2(z) \in S$, and $(x,z) \in C$. $\square$

The relation $\{(x,x) \mid x \in D\}$ is a congruence of every $\underline{G} \leq \mathrm{Sym}(D)$, and called the *trivial* congruence. A congruence is called *proper* if it is distinct from the equivalence relation that has only one equivalence class.

Definition 1.4.3. *A transitive permutation group G is called* primitive *if every proper congruence of G is trivial, and* imprimitive *otherwise.*

Note that if every proper congruence of G is trivial, then it is necessarily transitive, except in the case where G is a permutation group on a two-element set.

Definition 1.4.4. *An* orbital *is an orbit of pairs, that is, a set of the form $\{(g(a), g(b)) \mid g \in G\}$ for $a, b \in D$. If O is an orbital, the* orbital digraph of O *is the directed graph with vertex set D and edges O.*

Example 14. Let G be the permutation group on $\{1,2,3,4\}$ generated by (1234). Then G has four orbitals, depicted in Figure 1.1. $\triangle$

In a transitive permutation group on D, the *trivial* orbital is the orbital

$$\Delta_D := \{(a,a) \mid a \in D\}.$$

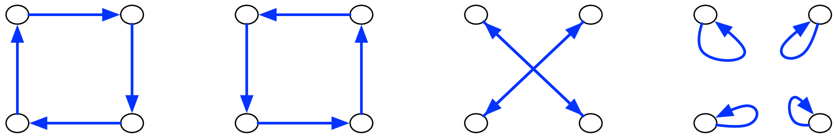


FIGURE 1.1. The four orbitals of the transitive permutation group generated by (1234).

THEOREM 1.4.5 (Higman's theorem). *A transitive permutation group G on a (not necessarily finite) set X is primitive if and only if the orbital digraph of all non-trivial orbitals is connected.*

PROOF. First suppose that G is primitive. Let O be a non-trivial orbital and let $S \subseteq X$ be a connected component of the orbital digraph of O. Clearly, $g(S) \cap S = \emptyset$ or $g(S) \cap S = S$ for every $g \in G$, because g preserves O, and hence S is a block. If S is a singleton set, then by the transitivity of G all components of X are singletons, contrary to the assumption that O is non-trivial. Hence, the primitivity of G implies that $S = X$ (Lemma 1.4.2), which shows that the orbital digraph of O is connected.

Now suppose that the orbital digraph of every non-trivial orbital is connected. Let C be a congruence such that there are distinct $a, b \in X$ with $(a, b) \in C$. By assumption, the orbital digraph of the orbital that contains (a, b) is connected. Hence, the transitivity of the relation C implies that C has at most one congruence class. We conclude that G is primitive. □

Yet another perspective on congruences, blocks, and primitivity of transitive permutation groups G involves the following definition.

DEFINITION 1.4.6 (set stabiliser). *Let G be a permutation group on X. If $S \subseteq X$ then the subgroup*

$$G_{\{S\}} := \{g \in G \mid g(S) = S\}$$

is called that set stabiliser *of G as S.*

Note that the set stabiliser is a point stabiliser (see Definition 1.2.5) for the action of G on $|S|$-element subsets of X (Example 11). The following is Theorem 1.5A in [**48**].

THEOREM 1.4.7. *Let G be a group which acts transitively on a set D, and let $a \in D$. Let $\mathcal{B}$ be the set of all blocks B of G that contain a, and let $\mathcal{S}$ be the set of all subgroups H of G that contain G_a. Then there is a bijection μ between $\mathcal{B}$ and $\mathcal{S}$ given by $\mu(B) := G_{\{B\}}$; the inverse mapping is given by $\mu^{-1}(H) = \{g(a) \mid g \in H\}$.*

Note that the mapping μ is order-preserving in the sense that if $B_1, B_2 \in \mathcal{B}$ then $B_1 \subseteq B_2 \Leftrightarrow \mu(B_1) \subseteq \mu(B_2)$.

COROLLARY 1.4.8. *Let G be a group acting transitively on a set D with at least two elements. Then G is primitive if and only if each point stabiliser G_a, for $a \in D$, is a maximal subgroup of G.*

PROOF. The statement follows from Theorem 1.4.7. For a self-contained proof, suppose that $a \in D$ is such that G_a is not maximal, i.e., there exists a proper subgroup H of G that properly contains G_a. Then $B := H(a) := \{h(a) \mid h \in H\}$ is a block of G. To see this, let $g \in G$. Suppose that $g(B) \cap B \neq \emptyset$. Then there exist $h_1, h_2 \in H$ such that $g(h_1(a)) = h_2(a)$. Hence, $h_2^{-1} \circ g \circ h_1 \in G_a$, so $g \in h_2 G_a h_1^{-1} \subseteq H$. It follows that $g(B) = B$ for all $g \in G$, showing that B is a block. Next, observe that $G_B = H$ since for every $g \in G$ we have $g(B) = B$ if and only if $g \in H$, as we have seen in the previous paragraph. If $B = D$, then $G_B = G$, contrary to the assumption that $H = G_B$ is a proper subgroup of G. If $B = \{a\}$, then $G_B = G_a$, contrary to the assumption that $H = G_B$ properly contains G_a. Hence, G must have a non-trivial proper congruence by Lemma 1.4.2, and hence is not primitive.

Conversely, if G is not primitive, then by Lemma 1.4.2 it has a block B such that $1 < |B|$ and $B \neq D$. Let $a \in B$. Clearly, G_B is a subgroup of G that contains G_a (see Exercise 22). Also note that $\{g(a) \mid g \in G_B\} = B$ (see Exercise 23). Hence, if $G_B = G_a$ then $B = \{a\}$, contrary to the assumptions on B. If $G_B = G$ then the transitivity of G implies that $B = D$, contrary to the assumptions on B. Thus, G_B shows that G_a is not a maximal subgroup of G. □

Exercises.

(21) Let $G \leq S_n$ be primitive. Show that if G contains a transposition, then $G = S_n$.

(22) Show that if B is a block of the permutation group G and $a \in B$, then G_a is contained in G_B.

(23) Show that if B is a block of a transitive permutation group G, then

$$\{g|_B : g \in G_B\}$$

is transitive as well.

(24) (from [**48**], Exercise 1.5.8) Let $G \leq S_6$ be the group generated by

$$\{(123456), (26)(35)\}.$$

Find all blocks that contain 1.
Find all subgroups of G that contain G_1.

(25) Give examples of permutation groups $G \leq S_{2n}$ which cannot be generated by fewer than n elements.

(26) (from [**48**], Exercise 1.5.14) Suppose that $G \leq S_n$ has r orbits. Show that G can be generated by a subset of size $n - r$ (in particular, every permutation group on n elements can be generated by $n - 1$ elements).

1.5. Semidirect Products

Semidirect products can be seen either as a way to construct new groups from simpler ones (Section 1.5.3), or, equivalently, as a tool to decompose a given group into simpler constituents (Section 1.5.4). They generalise the concept of direct products of groups. We first introduce some fundamental concepts for (abstract) groups.

1.5.1. Normal subgroups. A subgroup $\underline{N}$ of $\underline{G}$ with domain N is called *normal* if $gN = Ng$ for all elements g of $\underline{G}$; in this case, we write $\underline{N} \lhd \underline{G}$.

Example 15. If $\underline{G}_1$ and $\underline{G}_2$ are groups, then the direct product $\underline{G}_1 \times \underline{G}_2$ has a normal subgroup isomorphic to $\underline{G}_1$ with the elements $\{(g_1, 1) \mid g_1 \in G_1\}$. △

Recall the following equivalent characterisations of normality of subgroups.

Proposition 1.5.1. *Let $\underline{G}$ be a group, and $\underline{N}$ be a subgroup of $\underline{G}$. Then the following are equivalent.*

(1) *$\underline{N}$ is normal.*
(2) *$\underline{G}$ has the congruence $E = \{(a, b) \mid ab^{-1} \in N\}$.*
(3) *There is a homomorphism h from $\underline{G}$ to some group such that $N = h^{-1}(0)$.*
(4) *For every $g \in G$ and every $v \in N$ we have $gvg^{-1} \in N$.*

PROOF. (1) $\Rightarrow$ (2): to verify that E is a congruence, we have to show that for all $(a_1, b_1), (a_2, b_2) \in E$, $(a_1a_2, b_1b_2) \in E$. Indeed, $(a_1a_2)(b_1b_2)^{-1} = a_1(a_2b_2^{-1})b_1^{-1} \in a_1Nb_1^{-1} = Na_1b_1^{-1} \subseteq NN = N$.

(2) $\Rightarrow$ (3): $g \mapsto gN$ is a group homomorphism from $\underline{G}$ to $\underline{G}/\underline{N}$.

(3) $\Rightarrow$ (4): For $g \in G$ and $v \in h^{-1}(0)$, we must show that $gvg^{-1} \in h^{-1}(0)$. Indeed, $h(gvg^{-1}) = h(g)h(v)h(g)^{-1} = h(g)0h(g)^{-1} = 0$.

(4) $\Rightarrow$ (1): assume that $gNg^{-1} \subseteq N$ for all $g \in G$. Let $a \in G$ be arbitrary. Applying the assumption for $g = a$ we find that $aN \subseteq Na$. Applying the assumption for $g = a^{-1}$ we find that $a^{-1}N(a^{-1})^{-1} = a^{-1}Na \subseteq N$, and hence $Na \subseteq aN$. We conclude that $aN = Na$. □

EXAMPLE 16. The *alternating group of degree n* is the subgroup $\underline{A}_n$ of $\underline{S}_n$ which consists of all *even* permutations, i.e., the permutations that can be written as a composition of an even number of transpositions. Then the map sgn that sends $g \in \underline{S}_n$ to 0 if $g \in A_n$, and to 1 otherwise, is a homomorphism from $\underline{S}_n$ to $\mathbb{Z}_2$ and A_n is a normal subgroup of S_n. △

Non-trivial groups without non-trivial proper normal subgroups are called *simple*.

Exercises.

(27) Show that the group of permutations of $\mathbb{N}$ with finite support is a normal subgroup of $\mathrm{Sym}(\mathbb{N})$.
(28) Show that A_5 has no proper non-trivial normal subgroups.
(29) Show that $\bigcap_{g \in G} g^{-1}Hg$ is the largest normal subgroup of G which is contained in H (also see Exercise 15).

1.5.2. Semidirect products: motivation. Let $\underline{G}$ be a group and let $\underline{K}$ and $\underline{H}$ be subgroups of $\underline{G}$. We have already defined the set-product $KH := \{kh \mid h \in K, k \in H\}$ in Exercise 19. Note that KH might not be a subgroup (Exercise 30). However, if $\underline{H}$ or $\underline{K}$ is a normal subgroup, then KH is a subgroup. For instance, if $\underline{K}$ is a normal subgroup, then

$$\begin{aligned} (kh)(k'h') &= (khkh^{-1})(hh') \in KH \qquad (1)\\ \text{and } (kh)^{-1} &= h^{-1}k^{-1} = (h^{-1}k^{-1}h)h^{-1} \in KH. \end{aligned}$$

EXAMPLE 17. Let $\underline{G} = \underline{S}_n$, for $n \geq 3$, let $\underline{N}$ be the normal subgroup $\underline{A}_n$ of $\underline{S}_n$ (see Example 16), and let $\underline{H}$ be the subgroup of $\underline{G}$ generated by the transposition (12), i.e., $H = \{\mathrm{id}, (12)\}$. Then $G = NH$, because every element $g \in G$ is either in $\underline{A}_n$ or can be written as $(g(12))(12)$, which is in HN since $g(12) \in A_n$ and $(12) \in H$. Clearly, $N \cap H = \{\mathrm{id}\}$. However, $\underline{S}_n$ is *not* isomorphic to $\underline{A}_n \times \mathbb{Z}_n$: for $n \geq 3$, we have

$$(123) \circ (12) = (132) \neq (32) = (12) \circ (123),$$

while

$$((123), 1)(1, (12)) = ((123), (12)) = (1, (12))((123), 1)$$

(see property (c) in Exercise 19). △

Note the appearance of hkh^{-1} in (1), which defines a group action of $\underline{H}$ on K (generalising Example 13). This group action is in fact a homomorphism from $\underline{H}$ to $\mathrm{Aut}(\underline{K})$. Such homomorphisms from $\underline{H}$ to $\mathrm{Aut}(\underline{K})$ will be the starting point of our first definition of semidirect products in the next section, which is in the setting where we do not require that $\underline{K}$ and $\underline{H}$ are subgroups of the same group $\underline{G}$ (and which are therefore called *outer* semidirect products).

Exercises.

(30) Find an example of a group $\underline{G}$ and two subgroups $\underline{H}$ and $\underline{K}$ such that HK is not a subgroup.

1.5.3. The outer semidirect product. Let $\underline{H}$ and $\underline{N}$ be groups and let $\theta \colon \underline{H} \to \mathrm{Aut}(\underline{N})$ be a homomorphism.

EXAMPLE 18. In our running example, $\underline{H}$ is $\mathbb{Z}_2$ and $\underline{N}$ is $\underline{A}_n$ for $n \geq 3$ (see Example 16). Note that for every $t \in S_n$ the map $\alpha_t \colon N \to N$ given by $g \mapsto tgt^{-1}$ (conjugation) is from $\mathrm{Aut}(\underline{N})$. Pick any $t \in S_n \setminus A_n$ such that $t^2 = 1$. Then the map that sends $1 \in \mathbb{Z}_2$ to α_t and that sends 0 to id_N is a homomorphism from $\underline{H}$ to $\mathrm{Aut}(\underline{N})$. △

DEFINITION 1.5.2. *The* semidirect product of $\underline{N}$ by $\underline{H}$ with respect to θ, *denoted by $\underline{N} \rtimes_\theta \underline{H}$ (or $\underline{H} \ltimes_\theta \underline{N}$), is the group $\underline{G}$ with the elements $N \times H$ and group multiplication defined by*

$$(u, x)(v, y) := (u\theta(x)(v), xy)$$

for all $(u, x), (v, y) \in G$. If the reference to θ is clear, we use $\rtimes$ without the subscript.

Note that if θ is the trivial homomorphism that maps every element of H to $\mathrm{id}_N \in \mathrm{Aut}(\underline{N})$, then the semidirect product equals the direct product (we will see in Exercise 31 that a converse of this statement is true as well). Definition 1.5.2 contains some claims that we still have to verify. Multiplication is indeed associative:

$$\begin{aligned}
((u, x)(v, y))(w, z) &= (u\theta(x)(v), xy)(w, z) \\
&= (u\theta(x)(v)\theta(xy)(w), (xy)z) \\
&= (u\theta(x)(v)\theta(x)(\theta(y)(w)), x(yz)) && \text{(since } \theta \text{ is a homomorphism)} \\
&= (u\theta(x)(v\theta(y)(w)), x(yz)) && \text{(since } \theta(x) \in \mathrm{Aut}(\underline{N})\text{)} \\
&= (u, x)(v\theta(y)(w), yz) \\
&= (u, x)((v, y)(w, z)).
\end{aligned}$$

In the following, we write $x(v)$ instead of $\theta(x)(v)$ for better readability. Clearly, $(1, 1)$ is a neutral element, and the inverse of (u, x) is $(x^{-1}(u^{-1}), x^{-1})$:

$$\begin{aligned}
(u, x)(x^{-1}(u^{-1}), x^{-1}) &= (ux(x^{-1}(u^{-1})), xx^{-1}) \\
&= (uu^{-1}, 1) = (1, 1)
\end{aligned}$$

Note that $\underline{H}^* := \{(1, x) \mid x \in H\}$ is a subgroup of $\underline{G}$ that is isomorphic to $\underline{H}$, and that $\underline{N}^* := \{(u, 1) \mid u \in N\}$ is a subgroup of $\underline{G}$ isomorphic to $\underline{N}$. The next proposition collects some further important properties of semidirect products (compare them with the properties of direct products in Exercise 19!).

PROPOSITION 1.5.3. *Let $\underline{G} = \underline{N} \rtimes \underline{H}$. Then*

- $G = N^* H^*$,
- $N^* \cap H^* = \{(1, 1)\}$, *and*
- $\underline{N}^* \lhd \underline{G}$,

PROOF. To see that $G = N^* H^*$ it suffices to observe that (u, x) can be written as $(u, 1)(1, x)$, and obviously $N^* \cap H^* = \{(1, 1)\}$. Finally, for $(u, x) \in N^*$ and $(v, y) \in G$ we have

$$\begin{aligned}
(u, x)(v, 1)(x^{-1}(u^{-1}), x^{-1}) &= (ux(v), x)(x^{-1}(u^{-1}, x^{-1}) \\
&= (ux(v)x(x^{-1}(u^{-1})), xx^{-1}) \\
&= (ux(v)u^{-1}, 1) \in N
\end{aligned}$$

which implies that $\underline{N}^*$ is a normal subgroup (Proposition 1.5.1). $\square$

Note that the action of H^* on N^* by conjugation in $\underline{G}$ reflects the original action of $\underline{H}$ on $\underline{N}$, that is,

$$\begin{aligned}(1,x)(u,1)(1,x)^{-1} &= (x(u),x)(x^{-1}(1),x^{-1})\\ &= (x(u)x(x^{-1}(1)),xx^{-1})\\ &= (x(u),1).\end{aligned}$$

Usually, H^* and N^* are identified with $\underline{H}$ and $\underline{N}$, so we then consider $\underline{H}$ and $\underline{N}$ as subgroups of the semidirect product $\underline{G} = \underline{N} \rtimes \underline{H}$.

1.5.4. The inner direct product. This section provides a characterisation of the groups $\underline{G}$ that can be obtained as a semidirect product of two proper subgroups of $\underline{G}$.

A sequence of groups $\underline{N}, \underline{G}, \underline{H}$ with homomorphisms $\alpha\colon \underline{N} \to \underline{G}$ and $\beta\colon \underline{G} \to \underline{H}$ is called *exact at $\underline{G}$* if the kernel of β equals the image of α. A sequence of groups $\underline{G}_1, \underline{G}_2, \ldots$ with homomorphisms $\alpha_i\colon \underline{G}_i \to \underline{G}_{i+1}$ is called *exact* if it is exact at $\underline{G}_i$ for all $i \geq 2$. A *short exact sequence* is an exact sequence of the form

$$1 \longrightarrow \underline{N} \xrightarrow{\alpha} \underline{G} \xrightarrow{\beta} \underline{H} \longrightarrow 1.$$

Note that in this case, being exact at $\underline{N}$ implies that α is injective, and being exact at $\underline{H}$ implies that β is surjective. Hence, $\underline{N}$ can be considered as a normal subgroup of $\underline{G}$ and $\underline{H}$ is isomorphic to $\underline{G}/\underline{N}$. In this case $\underline{G}$ is called a *group extension* of $\underline{N}$ (by $\underline{H}$).

EXAMPLE 19. Let $\underline{G} = \underline{S}_n$, $\underline{N} = \underline{A}_n$, and $\underline{H} = \mathbb{Z}_2$. The inclusion map from A_n to S_n is an injective homomorphism $\alpha\colon \underline{A}_n \to \underline{S}_n$. Let β be the map sgn from Example 16, which is a surjective homomorphism from $\underline{S}_n$ to $\mathbb{Z}_2$. $\triangle$

PROPOSITION 1.5.4. *Let $\underline{G}$ be a group, $\underline{H} \leq \underline{G}$, and $\underline{N} \triangleleft \underline{G}$. Then the following are equivalent.*

(1) *H is a* complement for N in G, *i.e., $G = NH := \{nh \mid n \in N, h \in H\}$ and $N \cap H = \{1_G\}$.*
(2) *For every $g \in G$ there exists a unique $n \in N$ and $h \in H$ such that $g = nh$.*
(3) *There is a homomorphism $\mu\colon \underline{G} \to \underline{H}$ that fixes H pointwise and whose kernel is N.*
(4) *The restriction of the factor map $\sigma\colon \underline{G} \to \underline{G}/\underline{N}$ to $\underline{H}$ is an isomorphism between $\underline{H}$ and $\underline{G}/\underline{N}$.*
(5) *There exists a short exact sequence*

$$1 \longrightarrow N \xrightarrow{\alpha} G \xrightarrow{\beta} H \longrightarrow 1$$

that splits, *i.e., there is a homomorphism $\rho\colon H \to G$ such that $\beta \circ \rho = \mathrm{id}_H$.*
(6) *$\underline{G}$ is isomorphic to the semidirect product $\underline{N} \rtimes_\theta \underline{H}$ where θ is the action of $\underline{H}$ on N by conjugation in $\underline{G}$.*

PROOF. (1) implies (2): Suppose that $n_1, n_2 \in N$ and h_1, h_2 are such that $n_1h_1 = n_2h_2$. Then in particular, $n_1H = n_2H$, which is the case if and only if $n_2^{-1}n_1 \in N \cap H = \{1\}$, and hence $n_1 = n_2$. Similarly we deduce that $h_1 = h_2$.

(2) implies (3). Let $\mu\colon \underline{G} \to \underline{H}$ be the function that maps $g \in G$ to the unique $h \in H$ such that $g = nh$ for some $n \in N$. Then μ is a homomorphism: if $g_1 = n_1h_1$

and $g_2 = n_2 h_2$

$$\begin{aligned}
\mu(g_1 g_2) &= \mu(n_1 h_1 n_2 h_2) \\
&= \mu(n_1 h_1 n_2 h_1^{-1} h_1 h_2) \\
&= h_1 h_2 && \text{(since } h_1 n_2 h_1^{-1} \in N\text{)} \\
&= \mu(n_1 h_1)\mu(n_2 h_2) \\
&= \mu(g_1)\mu(g_2).
\end{aligned}$$

Then $\mu^{-1}(1) = N$ and for any $h \in H$ we have $\mu(h) = \mu(1h) = h$.

(3) implies (4): The restriction of σ to H is a homomorphism from $\underline{H}$ to $\underline{G}/\underline{N}$. It is injective since for $u \in H$ we have $\sigma(u) = 1_{G/N}$ if and only if $u \in N$ if and only if $\mu(u) = 1_H$ if and only if $u = 1_H$. It is surjective since for every $[g]_N \in G/N$ we have that $\mu(g) = \mu(\mu(g))$, so $[g]_N = [\mu(g)]_N = \sigma(\mu(g))$.

(4) implies (5): let $\tau \colon \underline{H} \to \underline{G}/\underline{N}$ be the restriction of the factor map σ which is an isomorphism by (4). Then $\beta := \tau^{-1}\sigma \colon G \to H$ is a surjective homomorphism whose kernel is N. Choosing $\rho \colon \underline{H} \to \underline{G}$ to be the inclusion map we obtain $\beta \circ \rho = \tau^{-1}\sigma\rho = \mathrm{id}_H$.

(5) implies (6): we may assume that α and ρ are inclusion maps. Define $\theta \colon \underline{H} \to \mathrm{Aut}(\underline{N})$ as $n \mapsto hnh^{-1}$. We claim that $\underline{N} \rtimes_\theta \underline{H}$ is isomorphic to $\underline{G}$, the isomorphism ξ being $(n, h) \mapsto nh$. We verify that ξ is a homomorphism:

$$\begin{aligned}
\xi((n_1, h_1)(n_2, h_2)) &= \xi(n_1 h_1 n_2 h_1^{-1}, h_1 h_2) \\
&= n_1 h_1 n_2 h_1^{-1} h_1 h_2 \\
&= n_1 h_1 n_2 h_2 = \xi(n_1, h_1)\xi(n_2, h_2).
\end{aligned}$$

The homomorphism ξ is injective: if $\xi(n_1, h_1) = n_1 h_1 = 1$ then

$$1 = \beta(n_1 h_1) = \beta(n_1)\beta(h_1) = 1 \cdot \beta(h_1),$$

and hence $h_1 = 1$ since β is injective. Since $n_1 h_1 = 1$ this implies that $n_1 = 1$, too.

To show that ξ is surjective let $g \in G$. We claim that $Ng = N\beta(g)$. It suffices to show that $g\beta(g)^{-1} \in N$, i.e., lies in the kernel of β. And indeed,

$$\beta(g\beta(g)^{-1}) = \beta(g)\beta(\beta(g))^{-1}) = \beta(g)\beta(g)^{-1} = 1.$$

Hence, there exists $n \in N$ such that $g = n\beta(g)$. Then $\xi(n, \beta(g)) = n\beta(g) = g$ which shows that ξ is surjective.

(6) implies (1): this is Proposition 1.5.3. $\square$

If the equivalent conditions in Proposition 1.5.4 apply, then $\underline{G}$ is called a *split extension* of $\underline{N}$ (by $\underline{H}$). We also say that $\underline{G}$ *splits over* $\underline{N}$.

EXAMPLE 20. Revisiting Example 19, we note that the short exact sequence $1 \to A_n \to S_n \overset{\mathrm{sgn}}{\to} \mathbb{Z}_2 \to 0$ splits: any homomorphism ρ from $\mathbb{Z}_2$ to S_n that maps 1 to an element $t \in S_n \setminus A_n$ such that $t^2 = 1$ satisfies $\mathrm{sgn} \circ \rho = \mathrm{id}_{\mathbb{Z}_2}$. Proposition 1.5.4 then implies that $\underline{S}_n$ is isomorphic to $\underline{A}_n \rtimes \mathbb{Z}_2$. $\triangle$

Here is an example of a short exact sequence that does not split.

EXAMPLE 21. Let $\underline{G} := (\mathbb{Z}_6; +)$. Then $h \colon \underline{G} \to (\mathbb{Z}_2; +)$ given by $h(g) := g \bmod 2$ is a surjective homomorphism, and there is an isomorphism i between $\mathbb{Z}_3$ and the kernel N of h. We then have the short exact sequence

$$1 \longrightarrow N \xrightarrow{i} G \xrightarrow{h} \mathbb{Z}_2 \longrightarrow 1.$$

However, there is no homomorphism $r \colon \mathbb{Z}_2 \to \mathbb{Z}_6$ such that $h \circ r = \mathrm{id}_H$ since any non-constant homomorphism $s \colon \mathbb{Z}_2 \to \mathbb{Z}_6$ would have to map 1 to 3 since $s(1) + s(1) = 0$

implies that $s(1) = 3$, but then $h \circ s(1) = h(3) = 0 \neq 1$. So the sequence does not split, and the equivalent conditions from Proposition 1.5.4 do not apply. $\triangle$

Exercises.

(31) Show that in a semidirect product $\underline{N} \rtimes_\theta \underline{H}$, the subgroup $\underline{H}$ is normal if and only if $\theta\colon \underline{H} \to \mathrm{Aut}(\underline{N})$ is trivial (in the sense that it maps every $h \in H$ to id_N), and in this case $\underline{N} \rtimes_\theta \underline{H} = \underline{N} \times \underline{H}$.

1.5.5. Application: the wreath product. We will now describe a natural operation to construct new structures from known structures, and then describe how the semidirect product helps to explain the automorphism groups of the new structures. We start with a simple example.

Example 22. Let $\underline{A}$ the disjoint union of two copies of the 5-element clique $\underline{K}_5 = (\{1, 2, \ldots, 5\}; \neq)$. Note that $\underline{G} = \mathrm{Aut}(\underline{A})$ has a normal subgroup $\underline{N}$ which is isomorphic to $\underline{S}_5 \times \underline{S}_5$. Also note that $\underline{G}/\underline{N}$ is isomorphic to $\mathbb{Z}_2$, and that $\underline{G}$ is in fact isomorphic to $(\underline{S}_5)^2 \rtimes \mathbb{Z}_2$. $\triangle$

Generalising Example 22, we may start from any two structures $\underline{A}$ and $\underline{B}$ with disjoint relational signatures σ and τ. We will define a new structure $\underline{A}[\underline{B}]$; the idea is that we replace the elements of $\underline{A}$ by copies of $\underline{B}$.

Formally, we create a copy $\underline{B}_a$ of $\underline{B}$ for every element a of A such that all the $\underline{B}_a$ have pairwise disjoint domains. Let E be a binary relation symbol that is not already in $\sigma \cup \tau$. Then $\underline{A}[\underline{B}]$ is the $\sigma \cup \tau \cup \{E\}$-structure $\underline{C}$ defined as follows. The τ-reduct of $\underline{C}$ equals the disjoint union of the $\underline{B}_a$. The relation $E^{\underline{C}}$ is the equivalence relation such that $E(x, y)$ holds for $x, y \in C$ if and only if x and y lie in the same copy of $\underline{B}$ in $\underline{C}$. For every relation symbol $R \in \sigma$ of arity k we set

$$R^{\underline{C}} := \{(c_1, \ldots, c_k) \mid \text{there is } (a_1, \ldots, a_k) \in R^{\underline{A}} \text{ and } c_i \in \underline{B}_{a_i} \text{ for } i \in \{1, \ldots, k\}\}.$$

In order to describe $\mathrm{Aut}(\underline{C})$ we need the following definition.

Definition 1.5.5 (Wreath product). *Let $\underline{G}$ be a group and let $\underline{H}$ be a group acting on a set A. Let $\underline{N} := \underline{G}^A$. Note that for every $h \in H$ and $n \in N$ the map $(n_a)_{a \in A} \mapsto (n_{h^{-1}(a)})_{a \in A}$ is an automorphism of $\underline{N}$, and that the map θ that sends $h \in H$ to this automorphism is a homomorphism from $\underline{H}$ to $\mathrm{Aut}(\underline{N})$. Define*

$$\underline{G} \,\mathrm{Wr}\, \underline{H} := \underline{N} \rtimes_\theta \underline{H}.$$

Proposition 1.5.6. *Let $\underline{G}$ be a group acting on a set B and $\underline{H}$ be a group acting on a set A. Then $\underline{G} \,\mathrm{Wr}\, \underline{H}$ has the following action on $B \times A$:*

$$(n, h) \cdot (b, a) := (n_{h(a)}(b), h(a)).$$

Proof. We verify the two conditions from Proposition 1.3.2. Let $(b, a) \in B \times A$. First note that

$$1^{\underline{G} \,\mathrm{Wr}\, \underline{H}}(b, a) = (1^{\underline{G}^A}, 1^{\underline{H}}) \cdot (b, a) = (1^{\underline{G}}(b), 1^{\underline{H}}(a)) = (b, a).$$

If $(n, h), (n', h') \in \underline{G} \,\mathrm{Wr}\, \underline{H}$, then

$$\begin{aligned}
(n', h') \cdot ((n, h) \cdot (b, a)) &= (n', h') \cdot (n_{h(a)}(b), h(a)) \\
&= (n'_{h'(h(a))}(n_{h(a)}(b)), h'h(a)) \\
&= (n'_{h'(h(a))} n_{h(a)}(b), h'h(a)) \\
&= (n'_{h'(h(a))} \theta(h')(n)_{h'h(a)}(b), h'h(a)) \\
&= ((n'\theta(h')(n))_{h'h(a)}(b), h'h(a)) \\
&= (n'\theta(h')(n), h'h) \cdot (b, a) = (n', h')(n, h) \cdot (b, a).
\end{aligned}$$

$\square$

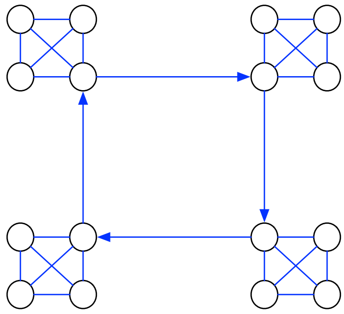


FIGURE 1.2. A digraph; the task of Exercise 33 is to determine its automorphism group. Undirected edges represent directed edges in both directions.

If $\underline{G}$ and $\underline{H}$ are permutation groups, then we also use $\underline{G} \operatorname{Wr} \underline{H}$ for the permutation group on $B \times A$ induced by this action. Note that if $\underline{H}$ acts on B with $|B| > 1$ and $|A| > 1$, then the permutation group $\underline{G} \operatorname{Wr} \underline{H}$ is imprimitive, with block $\{(b, a) \mid b \in B\}$ for every $a \in A$.

PROPOSITION 1.5.7. *For any two structures $\underline{A}$ and $\underline{B}$ we have*

$$\operatorname{Aut}(\underline{A}[\underline{B}]) = \operatorname{Wr}(\operatorname{Aut}(\underline{A}), \operatorname{Aut}(\underline{B})).$$

Exercises.

(32) Give an example of two structures $\underline{A}$ and $\underline{B}$ that illustrates why we need the extra equivalence relation in the definition of the structure $\underline{A}[\underline{B}]$.

(33) Describe the automorphism group of the digraph depicted in Figure 1.2.

(34) Show that there is no tree whose automorphism group is isomorphic to $(\mathbb{Z}_3; +)$. **Hints.**
   - Show that every tree has a *center*, i.e., a vertex or an edge that is fixed by every automorphism.
   - Find an explicit description of point stabiliser of the automorphism group of a tree.

CHAPTER 2

Counting Orbits

It is natural to explore the theory of infinite permutation groups by starting with *large* permutation groups. We first introduce several properties of permutation groups that express certain aspects of *'being large'*. A permutation group G on a set A is

- *k-transitive* if for $s, t \in A^k$ with pairwise distinct entries there is an $g \in G$ such that $g(s) = t$; (recall: the action of G on tuples is componentwise, i.e.,
$$g(s_1, \ldots, s_k) := (g(s_1), \ldots, g(s_k))$$
- *transitive* if it is 1-transitive;
- *k-set transitive* if for all $S, T \subseteq A$ of cardinality k there is a $g \in G$ such that $g(S) = \{g(s) \mid s \in S\} = T$.
- *highly set-transitive* if it is k-set transitive for all $k \geq 1$.
- *highly transitive* if it is k-transitive for all $k \geq 1$.

An example of a highly transitive permutation group is $\mathrm{Sym}(\mathbb{N})$. Clearly, if G is highly transitive, then it is also highly set-transitive. An example of a highly set-transitive but not highly transitive permutation group is $\mathrm{Aut}(\mathbb{Q}; <)$; see Section 3.2.

It is easy to see that a 2-set transitive permutation group G on an infinite set is also transitive. We prove the contraposition: assume that G has more than one orbit. There must be an orbit O with two distinct elements c_1, c_2. Let c_3 be an element not from O. Then there is no automorphism that maps $\{c_1, c_2\}$ to $\{c_1, c_3\}$, and hence G is not 2-set transitive. This fact will be generalised by Proposition 2.1.1 below.

2.1. Two Integer Sequences

For $B \subseteq A$, the *orbit of B under G* is the orbit of B under the action of G on subsets of A of cardinality $|B|$ from Example 11, i.e., the set $\{g(a) \mid g \in G, a \in B\}$.

Proposition 2.1.1 (Cameron [**39**]). *Let G be a permutation group on an infinite set. The number $f_G(n)$ of orbits of n-subsets forms a non-decreasing sequence.*

We will show this proposition in Section 2.3. Being highly set-transitive is equivalent to $f_G(n) = 1$ for all $n \in \mathbb{N}$. There is another important sequence attached to a permutation group. If A is a set and $n \in \mathbb{N}$, we write $A^n_{\neq}$ for the set of all n-tuples with pairwise distinct entries from A.

Definition 2.1.2. *Let G be a permutation group. Then $f^*_G(n)$ denotes the number of orbits of the componentwise action on $A^n_{\neq}$.*

So, G is highly transitive if $f^*_G(n) = 1$ for all $n \in \mathbb{N}$. Note that
$$f_G(n) \leq f^*_G(n) \leq n! f_G(n) \tag{2}$$
since there are $n!$ different orderings of n elements. These two sequences correspond to two different counting paradigms in combinatorics: *labelled* (in the case of f^*_G) and *unlabelled enumeration* (in the case of f_G).

Exercises.

(35) Show that Proposition 2.1.1 is false if G is a permutation group on a finite set.

(36) Prove that $(k+1)$-transitivity implies k-transitivity, for all $k \geq 1$.

(37) Show that if G is a permutation group on an infinite set, then $f^*_G(k) \leq f^*_G(k+1)$, for all $k \geq 1$.

(38) Show that if there exists a k such that $f^*_G(k) = f^*_G(k+1)$, then $f^*_G(k+1) = 1$.

(39) Show that a permutation group G on a set A is highly transitive if and only if $\overline{G} = \mathrm{Sym}(A) = \mathrm{Aut}(A; =)$.

(40) Let $(A; E)$ be a countably infinite structure where E denotes an equivalence relation with infinitely many infinite classes. Describe the automorphism group $\mathrm{Aut}(A; E)$. How many orbits of n-subsets are there?

(41) (Exercise 3 on page 57 in [**39**]) Let $(A; E_2)$ be a countably infinite structure where E_2 denotes an equivalence relation with infinitely many classes of size two, and let $(A; E^2)$ be a structure where E^2 denotes an equivalence relation with two infinite classes. Show that $\mathrm{Aut}(A; E_2)$ and $\mathrm{Aut}(A; E^2)$ have the same number of orbits of n-subsets, for all n.

2.2. Combinatorial Tools

In order to prove Proposition 2.1.1, we need a couple of combinatorial tools.

2.2.1. The Pigeon-hole Principle. If n pigeons fly to less than n holes, there must be one hole that got more than one pigeon. There is an important infinite version of the statement: if infinitely many pigeons fly to finitely many holes, one hole must have gotten infinitely many pigeons. This will be used in the next tools that we present.

2.2.2. König's Tree Lemma. A *walk* in a graph (V, E) (see Example 2) is a sequence $x_0, x_1, \ldots, x_n \in V$ with the property that $\{x_i, x_{i+1}\} \in E$ for all $i \in \{1, \ldots, n-1\}$. A walk is a *path* if all its vertices are distinct. A *cycle* is a walk of length at least three of the form $x_0, x_1, \ldots, x_n = x_0$ such that $x_1, \ldots, x_n$ are pairwise distinct. A *tree* is a connected graph (V, E) (see Section 1.1.6) without cycles. The *degree* of a vertex $u \in V$ is the number of vertices $v \in V$ such that $\{u, v\} \in E$.

Lemma 2.2.1 (König's Tree Lemma). *Let (V, E) be a tree such that every vertex in V has finite degree, and let $v_0 \in V$. If there are arbitrarily long paths that start in v_0, then there is an infinitely long path that starts in v_0.*

Proof. Since the degree of v_0 is finite, there exists a neighbour v_1 of v_0 such that arbitrarily long paths start in v_0 and continue in v_1 (by the infinite pigeon-hole principle). We now construct the infinitely long path by induction. Suppose we have already found a sequence $v_0, v_1, \ldots, v_i$ that can be continued to arbitrarily long paths in (V, E). Since the degree of v_i is finite, v_i must have a neighbour v_{i+1} in $V \setminus \{v_0, v_1, \ldots, v_i\}$ such that $v_0, v_1, \ldots, v_{i+1}$ can be continued to arbitrarily long paths in (V, E). In this way, we define an infinitely long path $v_0, v_1, v_2, \ldots$ in (V, E). □

The degree assumption in Lemma 2.2.1 is necessary, as can be seen from Figure 2.1.

Proofs using König's tree Lemma are often referred to as *compactness arguments* – the link with topology will become clear in Section 4.1. The following proposition illustrates one of the many uses of König's tree Lemma.

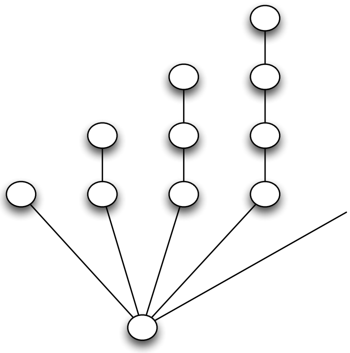


FIGURE 2.1. A tree with arbitrarily long paths, but no infinite paths.

PROPOSITION 2.2.2. *A countably infinite graph G is 3-colourable if and only if every finite subgraph of G is 3-colourable.*

2.2.3. Ramsey's Theorem. To prove Proposition 2.1.1, we also use an important tool from combinatorics: Ramsey theory. We denote the set $\{0, \dots, n-1\}$ also by $[n]$. Subsets of a set of cardinality s will be called s-subsets in the following. Let $\binom{M}{s}$ denote the set of all s-subsets of M. We also refer to mappings $\chi\colon \binom{M}{s} \to [c]$ as a *coloring* of M (with the *colors* $[c]$). In Ramsey theory, one writes

$$L \to (m)^s_c$$

if for every $\chi\colon \binom{L}{s} \to [c]$ there exists an $M \subseteq L$ with $|M| = m$ such that χ is constant on $\binom{M}{s}$. In the following, ω denotes the cardinality of $\mathbb{N}$. Note the following.

- For all $n \in \mathbb{N}$ we have $[n+1] \to (2)^1_n$: this is the pigeon-hole principle.
- For all $c \in \mathbb{N}$ we have $\mathbb{N} \to (\omega)^1_c$: this is the infinite pigeon-hole principle.

We first state and prove a special case of Ramsey's theorem.

THEOREM 2.2.3. $\mathbb{N} \to (\omega)^2_2$.

This statement has the following interpretation in terms of undirected graphs: every countably infinite undirected graph either contains an infinite *clique* (a complete subgraph) or an infinite *independent set* (a subgraph without edges).

PROOF. Let $\chi\colon \binom{\mathbb{N}}{2} \to [2]$ be a 2-colouring of the edges of $\binom{\mathbb{N}}{2}$. We define an infinite sequence $x_0, x_1, \dots$ of numbers from $\mathbb{N}$ and an infinite sequence $V_0 \supseteq V_1 \supseteq \cdots$ of infinite subsets of $\mathbb{N}$. Start with $V_0 := \mathbb{N}$ and $x_0 = 0$. By the infinite pigeon-hole principle, there is a $c_0 \in [2]$ such that $\{v \in V_0 \mid \chi(x_0, v) = c_0\} =: V_1$ is infinite. We now repeat this procedure with any $x_1 \in V_1$ and V_1 instead of V_0. Continuing like this, we obtain sequences $(c_i)_{i \in \mathbb{N}}$, $(x_i)_{i \in \mathbb{N}}$, $(V_i)_{i \in \mathbb{N}}$.

Again by the infinite pigeon-hole principle, there exists $c \in [2]$ such that $c_i = c$ for infinitely many $i \in \mathbb{N}$. Then $P := \{x_i \mid c_i = c\}$ has the desired property. To see this, let $i < j$ be such that $x_i, x_j \in P$. Then $x_j \in V_j \subseteq V_i$ and hence $\chi(\{x_i, x_j\}) = c_i = c$. $\square$

We now state Ramsey's theorem in it's full strength; the proof is similar to the proof of Theorem 2.2.3 shown above.

THEOREM 2.2.4 (Ramsey's theorem). *Let $s, c \in \mathbb{N}$. Then* $\mathbb{N} \to (\omega)^s_c$.

A proof of Theorem 2.2.4 can be found in [**73**] (Theorem 5.6.1); for a broader introduction to Ramsey theory see [**63**]. It is easy to derive the following finite version of Ramsey's theorem from Theorem 2.2.4 via König's tree lemma.

THEOREM 2.2.5 (Finite version of Ramsey's theorem). *For all $c, m, s \in \mathbb{N}$ there is an $l \in \mathbb{N}$ such that $[l] \to (m)^s_c$.*

PROOF. A proof by contradiction: suppose that there are positive integers c, m, s such that for all $l \in \mathbb{N}$ there is a $\chi\colon \binom{[l]}{s} \to [c]$ such that $(*)_{[l]}$ for all m-subsets M of $[l]$ the mapping χ is not constant on $\binom{M}{s}$. We construct a tree as follows. The vertices are the maps $\chi\colon \binom{[l]}{s} \to [c]$ that satisfy $(*)_{[l]}$. We make the vertex $\chi\colon \binom{[l]}{s} \to [c]$ adjacent to $\chi\colon \binom{[l+1]}{s} \to [c]$ if χ is a restriction of χ'. Clearly, every vertex in the tree has finite degree. By assumption, there are arbitrarily long paths that start in the vertex χ_0 where χ_0 is the map with the empty domain. By Lemma 2.2.1, the tree contains an infinite path $\chi_0, \chi_1, \dots$ We use this to define a map $\chi_{\mathbb{N}}\colon \binom{\mathbb{N}}{s} \to [c]$ as follows. For every $n \in \mathbb{N}$, there exists a $c_0 \in [c]$ and an $i_0 \in \mathbb{N}$ such that $\chi_i(S) = c_0$ for all $S \in \binom{[n]}{s}$ and $i \geq i_0$. Define $\chi_{\mathbb{N}}(S) := c_0$ for all $S \in \binom{[n]}{s}$. Then $\chi_{\mathbb{N}}$ satisfies $(*)_{\mathbb{N}}$, a contradiction to Theorem 2.2.4. $\square$

Here comes a variant of Ramsey's theorem.

LEMMA 2.2.6. *Let X be an infinite set. Suppose that $\chi\colon \binom{X}{s} \to [c]$ is surjective. Then there exist infinite sets $X_1, \dots, X_c \subseteq X$ and $k_1, \dots, k_c \in [c]$ such that $k_i \in \chi\binom{X_i}{s}$ for all $i \leq c$ and $k_j \notin \chi\binom{X_i}{s}$ for all $i < j \leq c$.*

PROOF. Ramsey's theorem states that there exists an infinite set X_1 such that $\chi\binom{X_1}{s}$ is constant; we define k_1 to be this constant. Our proof proceeds by induction. Suppose we have already found $X_1, \dots, X_r$ and $k_1, \dots, k_r$ such that $k_i \in \chi\binom{X_i}{s}$ for all $i \leq r$ and $k_j \notin \chi\binom{X_i}{s}$ for all $i < j \leq c$. Let $S \in \binom{X}{s}$ such that $\chi(S) \notin \{k_1, \dots, k_r\}$, and let $Y \subseteq X_r \setminus S$ be infinite. Let $S_0, S_1, \dots$ be an enumeration of all the subsets of S such that $S_i \subseteq S_i \Rightarrow i \leq j$ (the enumeration extends the inclusion order). For $i = 0, 1, \dots,$ define $\chi_i\colon \binom{Y}{s-|S_i|} \to [c]$ as follows: for $B \in \binom{Y}{s-|S_i|}$, set

$$\chi_i(B) = \chi(B \cup S_i)\,.$$

Now by Ramsey's theorem, there exists an infinite set Z_i and $\ell \in [c]$ such that $\chi_i\binom{Z_i}{s} = \{\ell_i\}$. Note that for $i = 0$ we have $S_i = \emptyset$ and $\ell_i \in \{k_1, \dots, k_r\}$ since $Z_i \subseteq Y \subseteq X_r$. On the other hand, for $i = 2^s$ we have $S_i = S$ and $\binom{Y}{s-|S_i|} = \binom{Y}{0} = \{\emptyset\}$, and $\ell_i = \chi_i(\emptyset) = \chi(\emptyset \cup S_i) = \chi(S) \notin \{k_1, \dots, k_r\}$. Let i_0 be smallest such that $\ell_{i_0} \notin \{k_1, \dots, k_r\}$. Then $X_{r+1} := Z_{i_0}$ and $k_{r+1} := \ell_{i_0}$ satisfy the desired properties: Clearly, $k_{r+1} \in \chi\binom{X_{r+1}}{n}$, and $k_j \notin \chi\binom{X_{r+1}}{s}$ for $r+1 < j \leq c$ by the minimal choice of i_0. $\square$

2.3. On the Number of Orbits of n-Subsets

Let G be a permutation group on a countably infinite set D. We want to prove that the number of orbits of n-subsets in a permutation group G forms a non-decreasing sequence (Proposition 2.1.1).

PROOF OF PROPOSITION 2.1.1. Let $O_1, \dots, O_c$ be distinct orbits of n-subsets (we do not assume that these are all orbits of n-subsets, that is, our proof also covers the situation that the group is not oligomorphic). We show that there are at least c orbits of $(n+1)$-subsets, using Ramsey's theorem in the form of Lemma 2.2.6. Let $\chi\colon \binom{D}{n} \to [c]$ be the map that assigns to a subset of D from O_i the number $i \in [c]$, and $i = 1$ if the subset lies in none of $O_1, \dots, O_c$; note that χ is surjective. By Lemma 2.2.6, there exist infinite sets $X_1, \dots, X_c \subseteq D$ and $k_1, \dots, k_c \in [c]$ such that $k_i \in \chi\binom{X_i}{n}$ for all $i \leq c$ and $k_j \notin \chi\binom{X_i}{n}$ for all $i < j \leq c$. For each $i \leq c$, let $B_i \in \binom{X_i}{n+1}$ be such that there exists an $S_i \subset B_i$ with $\chi(S_i) = k_i$. The sets $B_1, \dots, B_c$ lie in distinct orbits of $n+1$-subsets, because no permutation from G can map B_i to B_j for $i < j \leq c$ since $B_j \subseteq X_j$ does not contain n-subsets of color k_i. This proves that G has at least as many orbits of $n+1$-subsets as orbits of n-subsets. $\square$

Exercises.

(42) Let $(X; <)$ be a partially ordered set on a countably infinite set X. Show that $(X; <)$ contains an infinite chain, or an infinite antichain.

(43) Show that an infinite sequence of elements of a totally ordered set contains one of the following:
- a constant subsequence;
- a strictly increasing subsequence;
- a strictly decreasing subsequence.

Derive the Bolzano-Weierstrass theorem (every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence), using the completeness property of $\mathbb{R}^n$.

(44) Show that for every permutation group on an infinite set A with finitely many orbitals there are pairwise distinct $x, y, z \in A$ such that (x, y), (y, z), (x, z) lie in the same orbital.

2.4. Highly Set-transitive Permutation Groups

In this section, we present a classification of highly set-transitive closed subgroups of $\mathrm{Sym}(X)$ for countably infinite X. For $x_1, \dots, x_n \in \mathbb{Q}$ we write $\overrightarrow{x_1 \cdots x_n}$ if $x_1 < \cdots < x_n$.

THEOREM 2.4.1 (of [**37**]). *Let G be a highly set-transitive permutation group that is closed in* $\mathrm{Sym}(X)$ *for a countably infinite set X. Then G is isomorphic (as a permutation group) to one of the following:*

(1) $\mathrm{Aut}(\mathbb{Q}; <)$;
(2) $\mathrm{Aut}(\mathbb{Q}; \mathrm{Betw})$ *where* Betw *is the ternary relation*
$$\{(x, y, z) \in \mathbb{Q}^3 \mid \overrightarrow{xyz} \vee \overrightarrow{zyx}\}\,;$$
(3) $\mathrm{Aut}(\mathbb{Q}; \mathrm{Cycl})$ *where* Cycl *is the ternary relation*
$$\{(x, y, z) \mid \overrightarrow{xyz} \vee \overrightarrow{yzx} \vee \overrightarrow{zxy}\}\,;$$
(4) $\mathrm{Aut}(\mathbb{Q}; \mathrm{Sep})$ *where* Sep *is the 4-ary relation*
$$\begin{aligned}\{(x_1, y_1, x_2, y_2) \mid\ & \overrightarrow{x_1x_2y_1y_2} \vee \overrightarrow{x_1y_2y_1x_2} \vee \overrightarrow{y_1x_2x_1y_2} \vee \overrightarrow{y_1y_2x_1x_2} \\ & \vee \overrightarrow{x_2x_1y_2y_1} \vee \overrightarrow{x_2y_1y_2x_1} \vee \overrightarrow{y_2x_1x_2y_1} \vee \overrightarrow{y_2y_1x_2x_1}\}\,;\end{aligned}$$
(5) $\mathrm{Aut}(\mathbb{Q}; =)$.

The relation Sep is the so-called *separation relation*; note that $\mathrm{Sep}(x_1, y_1, x_2, y_2)$ holds for elements $x_1, y_1, x_2, y_2 \in \mathbb{Q}$ iff all four points x_1, y_1, x_2, y_2 are distinct and the smallest interval over $\mathbb{Q}$ containing x_1, y_1 properly overlaps with the smallest interval containing x_2, y_2 (where properly overlaps means that the two intervals have a non-empty intersection, but none of the intervals contains the other). Illustrations of the four proper closed subgroups of $\mathrm{Sym}(\mathbb{Q}; <)$ that contain $\mathrm{Aut}(\mathbb{Q}; <)$ can be found in Figure 2.2.

To give you some ideas of the proof of Theorem 2.4.1, we give a proof of the following.

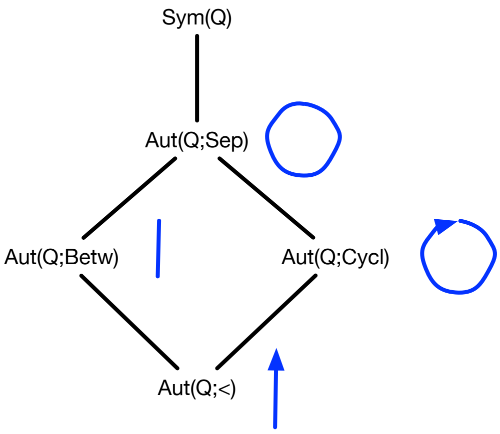


Figure 2.2. Illustrations of $\mathrm{Aut}(\mathbb{Q};<)$, $\mathrm{Aut}(\mathbb{Q};\mathrm{Betw})$, $\mathrm{Aut}(\mathbb{Q};\mathrm{Cycl})$, and $\mathrm{Aut}(\mathbb{Q};\mathrm{Sep})$.

Proposition 2.4.2. *Let G be a permutation group on a countably infinite set D such that G is 3-set transitive but not 2-transitive. Then G is isomorphic to a permutation group that is contained in* $\mathrm{Aut}(\mathbb{Q};<)$.

Proof. By Proposition 2.1.1, G is 2-set transitive, and hence there are at most three orbitals (Definition 1.4.4): to see this, fix distinct $u, v \in D$. Let $O_1 := \{(x,y) \in D^2 \mid \exists \alpha \in G : \alpha(u,v) = (x,y)\}$, $O_2 := \{(x,y) \in D^2 \mid \exists \alpha \in G : \alpha(v,u) = (x,y)\}$, and $O_3 := \Delta_D$. Then the orbits of pairs are either O_1, O_2, O_3, or $O_1 \cup O_2, O_3$. Since G is not 2-transitive, there are exactly the three orbitals O_1, O_2, O_3. The structure $(D; O_1)$ is a special directed graph called *tournament*: for any two distinct $x, y \in D$ we have that either $(x,y) \in O_1$ or $(y,x) \in O_1$. By 3-set transitivity, all of the three-element substructures of $(D; O_1)$ are isomorphic. There are only two possibilities: either these substructures are transitive, or they are directed 3-cycles. An easy case analysis shows that there is no 4-element tournament such that all 3-element substructures are directed 3-cycles. Hence, the first possibility must hold. Hence, $(D; O_1)$ is transitive, and therefore a linear order. The linear order is dense: for all $(a,c) \in O_1$ there exists a $b \in D$ such that $O_1(a,b)$ and $O_1(b,c)$. To see this, let $u, v, w \in D$ be such that $O_1(u,v)$ and $O_1(v,w)$ (and hence $O_1(u,w)$). There exists a $g \in G$ such that $g(u,w) = (a,c)$. Then $b := g(v)$ has the desired properties since $O_1(a, g(v))$ and $O_1(g(v), c)$. Similarly, one can show that $(D; O_1)$ is *unbounded*, that is, for every $b \in D$ there exist $a, c \in D$ such that $O_1(a,b)$ and $O_1(b,c)$. We will see later (Proposition 3.2.1) that every countable dense unbounded linear order is isomorphic to $(\mathbb{Q};<)$, which implies the statement. $\square$

Exercises.

(45) Show that $\mathrm{Aut}(\mathbb{Q};\mathrm{Cycl})$ *strictly* contains $\mathrm{Aut}(\mathbb{Q};<)$. 

(46) Label each edge in Figure 2.2 by its index (that is, the index of the group at the bottom in the group at the top of the edge). 

CHAPTER 3

Oligomorphic Permutation Groups

A permutation group G over a countable set X is *oligomorphic* if G has only finitely many orbits of n-tuples for each $n \geq 1$.

Examples and counterexamples:

- all permutation groups on finite sets;
- $\mathrm{Aut}(\mathbb{Q}; <)$, and all its supergroups;
- $\mathrm{Aut}(\mathbb{Z}; <)$ is a non-example: it has only one orbit, but infinitely many orbitals (orbits of pairs). To see this, note that (u, v) and (x, y) are in the same orbit if and only if $u - v = x - y$.
- $\mathrm{Aut}(\mathbb{Q}; +)$ is another non-example: it has two orbits and infinitely many orbits of pairs.

LEMMA 3.0.1. *Let $G \leq \mathrm{Sym}(\mathbb{N})$ be oligomorphic and $a \in \mathbb{N}^n$ for $n \in \mathbb{N}$. Then G_a is oligomorphic as well.*

If follows from this observation and Theorem 1.2.6 that every closed oligomorphic permutation group must have continuum cardinality.

3.1. sInv-Aut

There is a surprising link between oligomorphicity of permutation groups and first-order logic. Let $\underline{A}$ be a τ-structure and let $\phi(x_1, \dots, x_k)$ be a *first-order formula with free variables from* $x_1, \dots, x_k$, i.e., a formula built in the usual way with existential and universal quantifiers, conjunction, disjunction, negation, equality, relation symbols from τ, terms build from function symbols in τ, variables $x_1, \dots, x_k$, and the quantified variables. If $a_1, \dots, a_k \in A$ then we write $\underline{A} \models \phi(a_1, \dots, a_k)$ if the formula ϕ evaluates in $\underline{A}$ to true when instantiating x_i with a_i. For a rigorous definition, we refer to course notes (e.g., [**18**]) or text books (e.g., [**71**]) in mathematical logic, or text books in model theory (e.g., [**72, 116, 152**]). We say that $\phi(x_1, \dots, x_k)$ *defines* over $\underline{A}$ the relation

$$\{(a_1, \dots, a_k) \in A^k \mid \underline{A} \models \phi(a_1, \dots, a_k)\}.$$

EXAMPLE 23. The formula $x_1 < x_2 < x_3 \vee x_3 < x_2 < x_1$ defines the relation Betw over $(\mathbb{Q}; <)$, and the formula $\forall y (x_1 < y \vee x_1 = y)$ defines the relation $\{0\}$ over $(\mathbb{Q}_{\geq 0}; <)$. △

THEOREM 3.1.1. *Let $\underline{A}$ be a structure such that $\mathrm{Aut}(\underline{A})$ is oligomorphic. Then $R \in \mathrm{sInv}(\mathrm{Aut}(\underline{A}))$ if and only if R is first-order definable over $\underline{A}$.*

The proof of Theorem 3.1.1 can be found below. One direction of the equivalence is easy to show, and holds for general relational structures.

PROPOSITION 3.1.2. *Let $\underline{A}$ be a structure. If R is first-order definable in $\underline{A}$, then $R \in \mathrm{sInv}(\mathrm{Aut}(\underline{A}))$.*

PROOF. Straightforward induction over the syntactic structure of first-order formulas and their semantics. □

Corollary 3.1.3. *There is no linear order that is first-order definable over* $(\mathbb{C}; +, *)$.

Proof. Let $<$ be a linear order on $\mathbb{C}$, and suppose without loss of generality that $-i < i$. The map $a \mapsto \overline{a}$ (complex conjugation) is an automorphism of $\mathbb{C}$, and exchanges $-i$ to i. We thus found an automorphism that violates $<$, and Proposition 3.1.2 implies that $<$ is not first-order definable. $\square$

Corollary 3.1.4. *Let* $\underline{A}$ *be a structure such that* $\mathrm{Aut}(\underline{A})$ *is oligomorphic. Let* $\underline{A}'$ *be a structure with domain* $A' = A$ *such that all relations of* $\underline{A}'$ *are first-order definable in* $\underline{A}$. *Then* $\mathrm{Aut}(\underline{A}')$ *is oligomorphic as well.*

It follows for example that $\mathrm{Aut}(\mathbb{Q}; \mathrm{Betw})$ is oligomorphic.

Exercises.

(47) Show that the relation $\{(x, y) \in \mathbb{R}^2 \mid x = y^2\}$ is *not* first-order definable over the structure $(\mathbb{R}; +)$.

(48) Show that the assumption of Theorem 3.1.1 that $\mathrm{Aut}(\underline{A})$ is oligomorphic is necessary, i.e., find a structure $\underline{A}$ and a relation R such that R is preserved by $\mathrm{Aut}(\underline{A})$ but not definable in $\underline{A}$.

In order to show Theorem 3.1.1, we first show some useful lemmata. Let $x_1, \dots, x_n$ be variables. If $\psi(x_1, \dots, x_n)$ is a τ-formula and Ψ is a set of τ-formulas $\phi(x_1, \dots, x_n)$, and $\underline{A}$ is a τ-structure, then we say that ψ and Ψ are equivalent over $\underline{A}$ if for all $a \in A^n$ we have $\underline{A} \models \psi(a)$ if and only if $\underline{A} \models \phi(a)$ for all $\phi \in \Psi$.

Lemma 3.1.5. *Let* $\underline{A}$ *be a* τ*-structure such that* $\mathrm{Aut}(\underline{A})$ *is oligomorphic, let* $a \in A^n$, *and let* Ψ *be the set of all formulas* $\phi(x_1, \dots, x_n)$ *such that* $\underline{A} \models \phi(a)$. *Then there exists a formula* $\psi(x_1, \dots, x_n)$ *which is equivalent to* Ψ *over* $\underline{A}$.

Proof. Proposition 3.1.2 implies that two n-tuples that lie in the same orbit of $\mathrm{Aut}(\underline{A})$ satisfy the same formulas $\phi(x_1, \dots, x_n)$. Hence, by the oligomorphicity of $\mathrm{Aut}(\underline{A})$ there are only finitely many tuples $b_1, \dots, b_m$ such that any two tuples from $b_1, \dots, b_m$ do not satisfy the same formulas; we may suppose that $b_1 = a$. For every $i \in \{2, \dots, m\}$ there exists a formula ϕ_i that holds in a but not in b_i; then $\phi_2 \wedge \dots \wedge \phi_m$ is equivalent to Ψ. $\square$

Lemma 3.1.6. *Let* $\underline{A}$ *be a structure with an oligomorphic automorphism group, and let* $\bar{s}, \bar{t} \in A^k$ *be tuples that satisfy the same first-order formulas in* $\underline{A}$. *Then for every* $a \in A$ *there exists* $b \in A$ *such that* $(\bar{s}, a) \in A^{k+1}$ *and* $(\bar{t}, b) \in A^{k+1}$ *satisfy the same first-order formulas in* $\underline{A}$.

Proof. Let Ψ be the set of all first-order formulas satisfied by $(\bar{s}, a)$ in $\underline{A}$. By Lemma 3.1.5, there exists a τ-formula ψ which is equivalent to Ψ over $\underline{A}$. Then $\underline{A} \models \exists y. \psi(\bar{s}, y)$, and $\underline{A} \models \exists y. \psi(\bar{t}, y)$ by assumption. So there exists $b \in A$ such that $\underline{A} \models \psi(\bar{t}, b)$. This shows that $(\bar{t}, b)$ satisfies all first-order formulas satisfied by $(\bar{s}, a)$ in $\underline{A}$. The converse holds as well since negation is part of first-order logic. $\square$

Proof of Theorem 3.1.1. One implication of the statement has been shown in Proposition 3.1.2. Conversely, suppose that $R \in \mathrm{sInv}(\mathrm{Aut}(\underline{\mathrm{A}}))$. Let τ be the signature of $\underline{\mathrm{A}}$. Then R is a union of orbits of $\mathrm{Aut}(\underline{\mathrm{A}})$ on A^k; since $\mathrm{Aut}(\underline{\mathrm{A}})$ is oligomorphic, there exists an $m \in \mathbb{N}$ such that $R = O_1 \cup \dots \cup O_m$. We show that orbits of k-tuples are first-order definable in $\underline{A}$; this is sufficient because if ψ_i defines O_i over $\underline{\mathrm{A}}$, then $\psi_1 \vee \dots \vee \psi_m$ defines R. So let O be such an orbit and let $a = (a_1, \dots, a_k) \in O$, and let Ψ be the set of all first-order τ-formulas $\psi(x_1, \dots, x_k)$ in the language of $\underline{\mathrm{A}}$ such that $\underline{\mathrm{A}} \models \psi(a_1, \dots, a_k)$. We prove that if a tuple $\bar{b} = (b_1, \dots, b_k)$ satisfies every formula in

Ψ then $\bar{b} \in O$ by constructing an automorphism of $\underline{\mathrm{A}}$ that maps a to b. This is done by a back-and-forth argument, using Lemma 3.1.6 for going forth, and again using Lemma 3.1.6 for going back (see the proof of Proposition 3.2.1). By Lemma 3.1.5, there is a τ-formula ψ which is equivalent to Ψ over $\underline{A}$, and ψ defines O in $\underline{A}$. $\square$

COROLLARY 3.1.7. *Let $\underline{B}$ and $\underline{C}$ be structures on the same domain such that* $\mathrm{Aut}(\underline{B})$ *and* $\mathrm{Aut}(\underline{C})$ *are oligomorphic. Then* $\mathrm{Aut}(\underline{B}) = \mathrm{Aut}(\underline{C})$ *if and only if $\underline{B}$ and $\underline{C}$ are* (first-order) interdefinable *in the sense that all relations of $\underline{B}$ have a first-order definition in $\underline{C}$ and vice-versa.*

If two countable structures have oligomorphic automorphism groups that are *isomorphic as permutation groups* (Definition 1.0.1), then this corresponds to a model-theoretic relation between the structures.

DEFINITION 3.1.8. *Two structures $\underline{A}$ and $\underline{B}$ are called* bi-definable *if there exists a bijection $f\colon A \to B$ such that every $R \subseteq A^n$ is definable in $\underline{A}$ if and only if $f(A)$ is definable in $\underline{B}$.*

COROLLARY 3.1.9. *Let $\underline{A}$ and $\underline{B}$ be countable structures such that* $\mathrm{Aut}(\underline{A})$ *and* $\mathrm{Aut}(\underline{B})$ *are oligomorphic. Then* $\mathrm{Aut}(\underline{A})$ *and* $\mathrm{Aut}(\underline{B})$ *are isomorphic as permutation groups if and only if $\underline{A}$ and $\underline{B}$ are bi-definable.*

PROOF. Exercise 49. $\square$

COROLLARY 3.1.10. *Let $\underline{A}$ be such that* $\mathrm{Aut}(\underline{A})$ *is oligomorphic and let* $P \subseteq \mathrm{Sym}(A)$. *Then* $\overline{\langle P \rangle} = \mathrm{Aut}(\underline{A})$ *if and only if the set of relations that are definable in $\underline{A}$ equals* $\mathrm{sInv}(P)$.

PROOF. Suppose that $\overline{\langle P \rangle} = \mathrm{Aut}(\underline{A})$. Every relation that is first-order definable in $\underline{A}$ is strongly preserved by $\mathrm{Aut}(\underline{A})$, and hence in particular by $P \subseteq \overline{\langle P \rangle} = \mathrm{Aut}(\underline{A})$. Conversely, if R is strongly preserved by P, then it is also strongly preserved by $\overline{\langle P \rangle}$, and hence first-order definable by Theorem 3.1.1.

Conversely, suppose that the set of relations that are definable in $\underline{A}$ equals $\mathrm{sInv}(P)$. We then have

$$\begin{aligned} \overline{\langle P \rangle} &= \mathrm{Aut}(\mathrm{sInv}(P)) && \text{(by Proposition 1.2.8)} \\ &= \mathrm{Aut}(\underline{A}) && \text{(by assumption).} \end{aligned}$$ $\square$

COROLLARY 3.1.11. *Let $\underline{A}$ and $\underline{B}$ be structures with the same domain and oligomorphic automorphism groups. Then* $\mathrm{Aut}(\underline{A}) \leq \mathrm{Aut}(\underline{B})$ *if and only if every relation that is first-order definable in $\underline{B}$ is first-order definable in $\underline{A}$.*

PROOF. Exercise 50. $\square$

It follows from Theorem 3.1.1 that if $\underline{B}$ is a structure with an oligomorphic automorphism group G, then the congruences of G are exactly the first-order definable equivalence relations in $\underline{B}$. Another application of Theorem 3.1.1 can be found in the following example.

EXAMPLE 24. The *center* of G is the set

$$C(G) := \{\alpha \mid \alpha\beta = \beta\alpha \text{ for all } \beta \in G\}.$$

If $G = \mathrm{Aut}(\underline{A})$ is oligomorphic, then the center contains precisely those automorphisms of $\underline{A}$ that are preserved by all automorphisms of $\underline{A}$. Hence, by Theorem 3.1.1, $C(G)$ consists precisely the automorphisms of $\underline{A}$ that are first-order definable in $\underline{A}$. $\triangle$

The following example demonstrates that Theorem 3.1.1 fails if we keep the assumption that $\mathrm{Aut}(\underline{A})$ has only finitely many orbits of n-tuples, for every $n \in \mathbb{N}$, but do not require that the domain of $\underline{A}$ is countable (which is part of the definition of oligomorphicity).

Example 25. Consider the structure $(\mathbb{R}; \prec)$ where $\prec$ is the binary relation

$$\mathbb{Q} \times (\mathbb{R} \setminus \mathbb{Q}) \cup \{(x, y) \mid x < y, x, y \in \mathbb{Q} \text{ or } x, y \in \mathbb{R} \setminus \mathbb{Q}\}.$$

Let $a \in \mathbb{Q}$ and $b \in \mathbb{R} \setminus \mathbb{Q}$. First note that every formula $\phi(x)$ that holds on a also holds on b; this can be shown by induction over the shape of formulas, using that we can extend isomorphisms between finite substructures $(\mathbb{R}; \prec)$ by one more point in the image or pre-image. However, there is no automorphism of $(\mathbb{R}; \prec)$ that maps a to b, because the set $\{x \mid x \prec a\}$ is countable, but the set $\{x \mid x \prec b\}$ is uncountable. Hence, the unary relation consisting of all elements of $\mathbb{Q}$ is preserved by all automorphisms of $(\mathbb{R}; \prec)$ but not first-order definable in $(\mathbb{R}; \prec)$. It is also easy to see that the automorphism group of $(\mathbb{R}; \prec)$ has finitely many orbits of n-tuples, for all $n \in \mathbb{N}$ (the expansion with the unary relation $\mathbb{Q}$ is homogeneous). △

Remark 3.1.12. *Note that if $\underline{G_1}$ and $\underline{G_2}$ act oligomorphically on A and B, respectively, then the natural intransitive action of $\underline{G_1} \times \underline{G_2}$ is also oligomorphic: when $a(n)$ is the number of orbits of the componentwise action of $\underline{G_1}$ on A^n, and $b(n)$ is the number of orbits of the componentwise action of $\underline{G_2}$ on B, then the number of orbits of the componentwise of $\underline{G_1} \times \underline{G_2}$ on $A \cup B$ is $\sum_{0 \leq i \leq n} a(i)b(n-i)$, and hence finite for all n.*

If $\underline{A}$ and $\underline{B}$ have the same signature τ, then the automorphism group of the τ-structure $\underline{A} \times \underline{B}$ (see Definition 1.1.6) contains the image of the product action of $\mathrm{Aut}(\underline{A}) \times \mathrm{Aut}(\underline{B})$ on $A \times B$. The number of orbits of n-tuples of this action can be bounded by $a_n b_n$ where a_n is the number of orbits of n-tuples in $\underline{A}$ and b_n is the number of orbits on n-tuples in $\underline{B}$. Hence $\mathrm{Aut}(\underline{A} \times \underline{B})$ is oligomorphic.

Exercises.

(49) Prove Corollary 3.1.9.
(50) Prove Corollary 3.1.11. **Hint:** Exercise 11 and Theorem 3.1.1.
(51) Show that the assumption of Example 24 that $\mathrm{Aut}(\underline{A})$ oligomorphic is necessary (there is even a directed graph $\underline{A}$ such that $C(\mathrm{Aut}(\underline{A}))$ contains automorphisms that are not definable in $\underline{A}$).

(52) Show that $(\mathbb{Z}_2)^{\mathbb{N}}$ is isomorphic to the automorphism group of a countable structure.

(53) Show that $(\mathbb{Z}_2)^{\mathbb{N}}$ is not isomorphic to the automorphism group of an ω-categorical structure.

3.2. Countably Categorical Structures

Let τ be a countable signature. A set of (first-order) τ-sentences is called a τ-*theory*. A *model* of a τ-theory T is a τ-structure $\underline{A}$ such that $\underline{A}$ satisfies all sentences in T. Theories that have a model are called *satisfiable*. For every τ-structure $\underline{A}$, we denote by $\mathrm{Th}(\underline{A})$ the *theory of* $\underline{A}$, that is, the set of all τ-sentences that are satisfied by $\underline{A}$.

A satisfiable τ-theory T is called *ω-categorical* (or *$\aleph_0$-categorical*, which we use interchangeably) if all countable models of T are isomorphic. A structure is called *ω-categorical* if its first-order theory is ω-categorical. Note that the theory of a finite structure does not have countable models, and hence is ω-categorical.

Cantor [**43**] proved that the linear order of the rational numbers $(\mathbb{Q};<)$, which we will use as a running example in this section. We will see many more examples of ω-categorical structures later. One of the standard approaches to verify that a structure is ω-categorical is via a so-called *back-and-forth argument*. To illustrate, we give the back-and-forth argument that shows that $(\mathbb{Q};<)$ is ω-categorical; much more about this important concept in model theory can be found in [**73**, **132**].

PROPOSITION 3.2.1. *The structure* $(\mathbb{Q};<)$ *is* ω*-categorical.*

PROOF. Let $\underline{A}$ be a countable model of the first-order theory T of $(\mathbb{Q};<)$. It is easy to verify that T contains (and, as this argument will show, is uniquely given by)

- $\exists x.\ x = x$ (no empty model)
- $\forall x, y, z\ ((x < y \wedge y < z) \Rightarrow x < z)$ (transitivity)
- $\forall x.\ \neg(x < x)$ (irreflexivity)
- $\forall x, y\ (x < y \vee y < x \vee x = y)$ (totality)
- $\forall x\, \exists y.\ x < y$ (no largest element)
- $\forall x\, \exists y.\ y < x$ (no smallest element)
- $\forall x, z\, \exists y\ (x < y \wedge y < z)$ (density).

An isomorphism between $\underline{A}$ and $(\mathbb{Q};<)$ can be defined inductively as follows. Suppose that we have already defined f on a finite subset S of $\mathbb{Q}$ and that f is an embedding of the structure induced by S in $(\mathbb{Q};<)$ into $\underline{A}$. Since $<^{\underline{A}}$ is dense and unbounded, we can extend f to any other element of $\mathbb{Q}$ such that the extension is still an embedding from a substructure of $\mathbb{Q}$ into $\underline{A}$ (*going forth*). Symmetrically, for every element v of $\underline{A}$ we can find an element $u \in \mathbb{Q}$ such that the extension of f that maps u to v is also an embedding (*going back*). We now alternate between going forth and going back; when going forth, we extend the domain of f by the *next* element of $\mathbb{Q}$, according to some fixed enumeration of the elements in $\mathbb{Q}$. When going back, we extend f such that the image of A contains the *next* element of $\underline{A}$, according to some fixed enumeration of the elements of $\underline{A}$. If we continue in this way, we have defined the value of f on all elements of $\mathbb{Q}$. Moreover, f will be surjective, and an embedding, and hence an isomorphism between $\underline{A}$ and $(\mathbb{Q};<)$. $\square$

A second important running example of this section is the *countable random graph* $(\mathbb{V};E)$. This (simple and undirected) graph with a countably infinite number of vertices has the following *extension property*: for all finite disjoint subsets U, U' of $\mathbb{V}$ there exists a vertex $v \in \mathbb{V} \setminus (U \cup U')$ such that v is adjacent to all vertices in U and to no vertex in U'. The existence of such a graph will be show in Example 29.

PROPOSITION 3.2.2. *The random graph* $(\mathbb{V};E)$ *is* ω*-categorical.*

PROOF. Note that the extension property of $(\mathbb{V};E)$ given above is a first-order property; a back-and-forth argument similar to the one given in the proof of Proposition 3.2.1 shows that every countably infinite graph with this property is isomorphic to $(\mathbb{V};E)$. $\square$

The reason why we treat ω-categoricity in this course is the following theorem. An accessible proof can be found in Hodges' book (Theorem 6.3.1 in [**73**]).

THEOREM 3.2.3 (Engeler, Ryll-Nardzewski, Svenonius). *A countable structure* $\underline{B}$ *is* ω*-categorical if and only if* $\mathrm{Aut}(\underline{B})$ *is oligomorphic.*

If the signature of $\underline{B}$ is countable, there is another characterisation of ω-categoricity of $\underline{B}$ via the property of Theorem 3.1.1.

COROLLARY 3.2.4. *Let* $\underline{B}$ *be a structure with a countable signature and a countable domain. Then* $\underline{B}$ *is* ω*-categorical if and only if* $\mathrm{sInv}(\mathrm{Aut}(\underline{B}))$ *equals the set of relations with a first-order definition over* $\underline{B}$*.*

PROOF. The forwards implication is the content of Theorem 3.1.1. Conversely, suppose that $\mathrm{Aut}(\underline{B})$ are infinitely many orbits of n-tuples, for some n. Then the union of any subset of the set of all orbits of n-tuples is preserved by all automorphisms of $\underline{B}$; but there are only countably many first-order formulas over a countable language, so not all the invariant sets of n-tuples can be first-order definable in $\underline{B}$. $\square$

Exercises.

(54) Show that in every ω-categorical structure $\underline{B}$ there exists a unique finest equivalence relation which is definable in $\underline{B}$ and has finitely many classes, and a unique coarsest equivalence which is definable in $\underline{B}$ and has finite classes.

The following exercises are taken from Peter Cameron's book "Oligomorphic permutation groups".

(55) Write down sentences ϕ_n, ψ_n (over the signature $\{=\}$) such that
  (a) any model of ϕ_n has at least n elements;
  (b) any model of ψ_n has exactly n elements.

(56) Write down a sentence, using equality and one binary relation symbol, all of whose models are infinite.
Is this possible with equality alone?

3.3. Homogeneous Structures and Amalgamation Classes

A relational[1] structure $\underline{A}$ is called *homogeneous* (sometimes also called *ultra-homogeneous* [**73**]) if every isomorphism between finite substructures of $\underline{A}$ can be extended to an automorphism of $\underline{A}$.

PROPOSITION 3.3.1. *Let $\underline{A}$ be homogeneous with a finite signature. Then* $\mathrm{Aut}(\underline{A})$ *is oligomorphic (and hence $\underline{A}$ is ω-categorical).*

PROOF. By the homogeneity of $\underline{A}$, the orbit of an n-element subset B of $\mathrm{Aut}(\underline{A})$ is given by the substructure induced by $\underline{A}$ on B. But there are finitely many non-isomorphic substructures of $\underline{A}$ of size n. As we have seen earlier (2), this also bounds the number of orbits of n-tuples of $\mathrm{Aut}(\underline{A})$. $\square$

EXAMPLE 26. The following structures are homogeneous.

- $(\mathbb{Q}; <)$.
- the canonical structure of a permutation group (Definition 1.2.3).
- every expansion of a countable structure $\underline{A}$ with an oligomorphic automorphism group by all *first-order* definable relations is homogeneous. $\triangle$

A versatile tool to construct countable homogeneous structures from classes of finite structures is the *amalgamation technique* à la Fraïssé. We present it here for the special case of *relational structures*; this is all that is needed in the examples we are going to present. For a stronger version of Fraïssé-amalgamation for classes of structures that might involve function symbols, see [**73**].

In the following, let τ be a countable relational signature. The *age* of a τ-structure $\underline{A}$ is the class of all finite τ-structures that embed into $\underline{A}$. A class $\mathcal{C}$ has the *joint embedding property (JEP)* if for any two structures $\underline{B}_1, \underline{B}_2 \in \mathcal{C}$ there exists a structure $\underline{C} \in \mathcal{C}$ that embeds both $\underline{B}_1$ and $\underline{B}_2$.

[1]The entire theory can be adapted to general signatures that might also contain function symbols; to keep the exposition simple, we restrict our focus to relational signatures in this section.

PROPOSITION 3.3.2. *Let $\mathcal{C}$ be a class of finite τ-structures. Then $\mathcal{C}$ is the age of a (countable) relational structure if and only if $\mathcal{C}$*

- *is closed under isomorphisms and substructures,*
- *contains only countably many structures up to isomorphism,*
- *has the JEP.*

REMARK 3.3.3. *The Proposition is false if we drop the second item: take e.g. $\tau := \{R_1, R_2, \dots\}$, for R_i unary, and put all finite τ-structures into $\mathcal{C}$. Then $\mathcal{C}$ satisfies the first and the third, but not the second item.*

The *union* of two relational τ-structures $\underline{B}_1, \underline{B}_2$ is the τ-structure $\underline{C}$ with domain $B_1 \cup B_2$ and relations $R^{\underline{C}} := R^{\underline{B}_1} \cup R^{\underline{B}_2}$ for all $R \in \tau$. The intersection of $\underline{B}_1$ and $\underline{B}_2$ is defined analogously. Let $\underline{B}_1, \underline{B}_2$ be τ-structures such that $\underline{B}_1[B_1 \cap B_2] = \underline{B}_2[B_1 \cap B_2]$; the pair $(\underline{B}_1, \underline{B}_2)$ is then called an *amalgamation diagram.* Then $\underline{B}_1 \cup \underline{B}_2$ is also called the *free amalgam* of $\underline{B}_1, \underline{B}_2$. More generally, a τ-structure $\underline{C}$ is an *amalgam of* $\underline{B}_1$ *and* $\underline{B}_2$ if for $i \in \{1, 2\}$ there are embeddings f_i of $\underline{B}_i$ to $\underline{C}$ such that $f_1(a) = f_2(a)$ for all $a \in B_1 \cap B_2$.

DEFINITION 3.3.4. *An isomorphism-closed class $\mathcal{C}$ of finite τ-structures*

- *has the* free amalgamation property *if for all $\underline{B}_1, \underline{B}_2 \in \mathcal{C}$ the free amalgam of $\underline{B}_1$ and $\underline{B}_2$ is contained in $\mathcal{C}$;*
- *has the* amalgamation property *if every amalgamation diagram $(\underline{B}_1, \underline{B}_2)$ of structures $\underline{B}_1, \underline{B}_2 \in \mathcal{C}$ has an amalgam $\underline{C} \in \mathcal{C}$;*
- *is an* amalgamation class *if it contains at most countably many non-isomorphic structures, has the amalgamation property, and is closed under isomorphisms and taking induced substructures.*

Note that since we only look at relational structures here (and since we allow structures to have an empty domain), the amalgamation property of $\mathcal{C}$ implies the joint embedding property.

EXAMPLE 27. Let $\mathcal{C}$ be the class of all finite linear orders. Then $\mathcal{C}$ is clearly closed under isomorphisms and induced substructures, and has countably many isomorphism types. To show that it also has the amalgamation property, let $\underline{B}_1, \underline{B}_2 \in \mathcal{C}$ and let $\underline{C}$ be the free amalgam of $\underline{B}_1$ and $\underline{B}_2$. Then $\underline{C}$ is an acyclic finite graph; therefore, any depth-first traversal of $\underline{C}$ leads to a linear ordering of the elements that is an amalgam (even a strong amalgam, but not a free amalgam) in $\mathcal{C}$ of $\underline{B}_1$ and $\underline{B}_2$. It follows that $\mathcal{C}$ is an amalgamation class. $\triangle$

THEOREM 3.3.5 (Fraïssé [**58**,**59**]; see [**73**]). *Let τ be a countable relational signature and let $\mathcal{C}$ be an amalgamation class of τ-structures. Then there is a homogeneous and at most countable τ-structure $\underline{C}$ whose age equals $\mathcal{C}$. The structure $\underline{C}$ is unique up to isomorphism, and called the* Fraïssé-limit *of $\mathcal{C}$ (often denoted by* $\mathrm{Flim}(\mathcal{C})$*).*

PROOF. We first prove uniqueness. Let $\underline{C}$ and $\underline{D}$ be countable homogeneous of the same age. We have to show that $\underline{C}$ and $\underline{D}$ are isomorphic, and construct the isomorphism $f \colon \underline{C} \to \underline{D}$ by a back-and-forth argument, similarly to the proof that $(\mathbb{Q}; <)$ is ω-categorical in Proposition 3.2.1. Let $C = \{c_1, c_2, \dots\}$ and $D = \{d_1, d_2, \dots\}$. Suppose f is already defined on a finite subset F of C.

- Going forth: Let $i \in \mathbb{N}$ be smallest so that $c_i \notin F$. Then there is $e \colon \underline{C}[F \cup \{c_i\}] \hookrightarrow \underline{D}$. Note that the finite structures $\underline{D}[e(F)]$ and $\underline{D}[f(F)]$ are isomorphic via $f \circ e^{-1}$. By the homogeneity of $\underline{D}$, this isomorphism can be extended to an automorphism a of $\underline{D}$. Then $a \circ e|_F = f$ and we extend f by setting $f(c_i) := a(e(c_i))$. Clearly, the extension of f thus defined is

an embedding of $\underline{C}[F \cup \{c_i\}]$ into $\underline{D}$ because it is the composition of the embedding e with the automorphism a.
- Going back: let $i \in \mathbb{N}$ be smallest so that $d_i \notin f[F]$. Analogously, find $c \in C$ so that the extension $f(c) := d_i$ is an isomorphism.

A structure $\underline{C}$ is called *weakly homogenous* if for all $\underline{B} \in \mathrm{Age}(\underline{C})$, substructure $\underline{A}$ of $\underline{B}$, and $e\colon \underline{A} \hookrightarrow \underline{C}$ there is $g\colon \underline{B} \hookrightarrow \underline{C}$ which extends e. Note that in the proof above, we only needed weak homogeneity of $\underline{C}$ and $\underline{D}$ to construct f. It follows that weak homogeneity implies homogeneity.

We now prove the existence of the homogeneous structure $\underline{C}$ from the statement of the theorem. We will construct a sequence $(\underline{C}_i)_{i\in\mathbb{N}}$ of τ-structures $\underline{C}_i \in \mathcal{C}$ such that

- $\underline{C}_0$ is the τ-structure with the empty domain,
- $(\underline{C}_i)_{i\in\mathbb{N}}$ is a *chain*, i.e., $\underline{C}_i$ is a substructure of $\underline{C}_j$ if $i \leq j$, and
- if $\underline{A}, \underline{B} \in \mathcal{C}$, with $\underline{A}$ substructure of $\underline{B}$ and $e\colon \underline{A} \hookrightarrow \underline{C}_i$ for some $i \in \mathbb{N}$, then there are $j \in \mathbb{N}$ and $g\colon \underline{B} \hookrightarrow \underline{C}_j$ which extends e.

Then $\underline{C} := \bigcup_{i\in\mathbb{N}} \underline{C}_i$ is weakly homogeneous, and hence homogeneous, by the comments above.

Also note that $\mathrm{Age}(\underline{C}) = \mathcal{C}$. Here, the inclusion $\subseteq$ is clear. For the converse inclusion, first note that for every $\underline{A} \in \mathcal{C}$ there is $\underline{B} \in \mathcal{C}$ such that $\underline{A} \hookrightarrow \underline{B}$ and $\underline{C}_0 \hookrightarrow \underline{B}$ by the JEP. So $\underline{B} \hookrightarrow \underline{C}_j$ for some $j \in \mathbb{N}$, and hence $\underline{A} \hookrightarrow \underline{B} \hookrightarrow \underline{C}$.

Let P be a countable set of representatives for all $(\underline{A}, \underline{B}) \in \mathcal{C}^2$ such that $\underline{A}$ is a substructure of $\underline{B}$. Let $\alpha\colon \mathbb{N}^2 \to \mathbb{N}$ be a bijection such that $\alpha(i,j) \geq i$ for all i, j. Suppose $\underline{C}_k$ already constructed. Let $(\underline{A}_{k,i}, \underline{B}_{k,i}, f_{k,i})_{i\in\mathbb{N}}$ be a list of all triples $(\underline{A}, \underline{B}, f)$ where

- $(\underline{A}, \underline{B}) \in P$ and
- $f\colon \underline{A} \hookrightarrow \underline{C}_k$.

Let i, j be such that $k = \alpha(i,j)$. Construct $\underline{C}_{k+1}$ as amalgam of $\underline{C}_k$ and $\underline{B}_{i,j}$ so that $f_{i,j}$ extends to $\underline{B}_{i,j} \hookrightarrow \underline{C}_{k+1}$. □

Example 28. Let $\mathcal{C}$ be the class of all finite partially ordered sets. Amalgamation can be shown by computing the transitive closure: if $\underline{C}$ is the free amalgam of $\underline{B}_1$ and $\underline{B}_2$ over $\underline{A}$, then the transitive closure of $\underline{C}$ gives an amalgam in $\mathcal{C}$. The Fraïssé-limit of $\mathcal{C}$ is called the *homogeneous universal partial order.* △

Exercises.

(57) Recall that a *tournament* is a directed graph without self-loops such that for all pairs x, y of distinct vertices exactly one of the pairs (x, y), (y, x) is an arc in the graph. Show that the class of all finite tournaments has the amalgamation property.

Example 29. Let $\mathcal{C}$ be the class of all finite graphs. It is even easier than in the previous examples to verify that $\mathcal{C}$ is an amalgamation class, since here the free amalgam itself shows the amalgamation property. The Fraïssé-limit of $\mathcal{C}$ is the *countable random graph* $(\mathbb{V}; E)$ (also called the *Rado graph*) which we already encountered in Section 3.2. △

Proposition 3.3.6. *For every k, there is a permutation group which is k-transitive but not $(k+1)$-transitive.*

Proof. Let R be a relation symbol of arity $k+1$. Let $\mathcal{C}$ be the class of all finite $\{R\}$-structures where R denotes a relation that only contains tuples with pairwise distinct entries. The class $\mathcal{C}$ is clearly closed under isomorphism, substructures, and

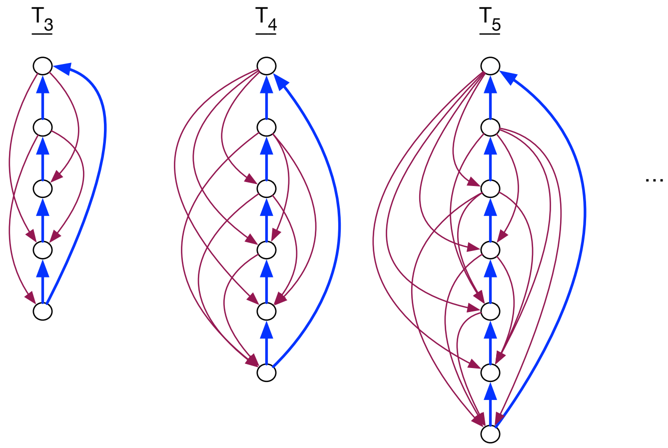


FIGURE 3.1. Henson's family of finite tournaments that are pairwise homomorphically incomparable.

has only countably many isomorphism classes of structures. It also has the free amalgamation property. Note that any two structures with at most k elements are isomorphic, since R denotes the empty relation in those structures. Since the Fraïssé-limit is homogeneous, its automorphism group is therefore k-transitive. On the other hand, the class contains non-isomorphic structures of size $k+1$ (e.g., a structure where R denotes the empty relation and a structure where R is non-empty), and hence the automorphism of $\mathrm{Flim}(\mathcal{C})$ is not $k+1$-transitive. $\square$

EXAMPLE 30. Let $\mathcal{C}$ be the class of all finite *triangle-free graphs*, that is, all graphs that do not contain K_3 as a subgraph. Again, we have the free amalgamation property. The Fraïssé-limit is up to isomorphism uniquely described as the triangle-free graph $\underline{A}$ such that for any finite $S, T \subset A$ such that S *is stable* (i.e., induces a graph with no edges; such a vertex subset is sometimes also called an *independent set*) there exists $v \in A \setminus (S \cup T)$ which is connected to all points in S, but to no point in T. $\triangle$

We now introduce a convenient tool to describe classes of finite τ-structures. If $\mathcal{N}$ is a class of τ-structures, we say that a structure $\underline{A}$ is $\mathcal{N}$*-free* if no $\underline{B} \in \mathcal{N}$ embeds into $\underline{A}$. The class of all finite $\mathcal{N}$-free structures we denote by $\mathrm{Forb}(\mathcal{N})$.

EXAMPLE 31. Henson [**65**] used Fraïssé limits to construct 2^ω many homogeneous directed graphs. Note that for all classes $\mathcal{N}$ of finite tournaments, $\mathrm{Forb}(\mathcal{N})$ is an amalgamation class, because if $\underline{A}_1$ and $\underline{A}_2$ are directed graphs in $\mathrm{Forb}(\mathcal{N})$ such that $\underline{A} = \underline{A}_1 \cap \underline{A}_2$ is an induced substructure of both $\underline{A}_1$ and $\underline{A}_2$, then the free amalgam $\underline{A}_1 \cup \underline{A}_2$ is also in $\mathrm{Forb}(\mathcal{N})$.

Henson in his proof specified an infinite set $\mathcal{T}$ of tournaments $\underline{T}_3, \underline{T}_4, \dots$ with the property that $\underline{T}_i$ does not embed into $\underline{T}_j$ if $i \neq j$. The tournament $\underline{T}_n$, for $n \geq 3$, in Henson's set $\mathcal{T}$ has vertices $0, \dots, n+1$, and the following edges (see Figure 3.1):

- $(i, i+1)$ for $0 \leq i \leq n$;
- $(0, n+1)$;
- (j, i) for $j > i+1$ and $(i, j) \neq (0, n+1)$.

Note that this property implies that for two distinct subsets $\mathcal{N}_1$ and $\mathcal{N}_2$ of $\mathcal{T}$ the two sets $\mathrm{Forb}(\mathcal{N}_1)$ and $\mathrm{Forb}(\mathcal{N}_2)$ are distinct as well. Since there are 2^ω many subsets of the infinite set $\mathcal{T}$, there are also that many distinct homogeneous directed graphs; they are often referred to as *Henson digraphs*. △

The structures from Example 31 can be used to prove various negative results about homogeneous structures with finite signature. A better behaved class of homogeneous structures are those whose age is *finitely bounded* (this is the same terminology as in [**110**]).

DEFINITION 3.3.7. *A class $\mathcal{C}$ of finite relational τ-structures (or a structure with age $\mathcal{C}$) is called* finitely bounded *if τ is finite and there exists a finite set of finite τ-structures $\mathcal{N}$ such that $\mathcal{C} = \mathrm{Forb}(\mathcal{N})$.*

Fraïssé's theorem (Theorem 3.3.5) has a converse.

PROPOSITION 3.3.8. *The age of every homogeneous relational structure has the amalgamation property.*

PROOF. Let $\underline{A}$ be a homogeneous structure and let $(\underline{B}_1, \underline{B}_2)$ be an amalgamation diagram with $\underline{B}_1, \underline{B}_2 \in \mathrm{Age}(\underline{A})$. Then there are embeddings $e_i \underline{B}_1 \hookrightarrow \underline{A}$, for $i \in \{1, 2\}$, and by the homogeneity of $\underline{A}$ there exists an automorphism $\alpha \in \mathrm{Aut}(\underline{A})$ such that $\alpha(e_1(x)) = e_2(x)$ for every $x \in B_1 \cap B_2$. Let $\underline{C}$ be the substructure of $\underline{A}$ with domain $\alpha(e_1(B_1)) \cup e_2(B_2)$. Then the embedding $f_1 \colon B_1 \to \underline{C}$ given by $\alpha \circ e_1$ and the embedding e_2 show that $\underline{C}$ is an amalgam of $(\underline{B}_1, \underline{B}_2)$. □

Exercises.

(58) Is the class of finite forests (i.e., simple acyclic graphs) an amalgamation class?

(59) Let $\mathcal{C}$ be the class of all finite graphs $\underline{G}$ such that there is no embedding from the 5-cycle $\underline{C}_5$ into $\underline{G}$. Is $\mathcal{C}$ an amalgamation class?

(60) Show that the assumption that the domain of $\underline{A}$ is countable is necessary for the third item in Example 26, that is, show that there exists an uncountable structure whose automorphism group has finitely many orbits of n-tuples, for all n, but whose expansion by all first-order definable relations is *not* homogeneous.

(61) Show the age $\mathcal{C}$ of a structure has the amalgamation property if and only if it has the *1-point amalgamation property*, i.e., if for all $\underline{A}, \underline{B}_1, \underline{B}_2 \in \mathcal{C}$ and embeddings $e_1 \colon \underline{A} \hookrightarrow \underline{B}_1$ and $e_2 \colon \underline{A} \hookrightarrow \underline{B}_2$ such that $|B_1| = |B_2| = |A| + 1$ there are a $\underline{C} \in \mathcal{C}$ and embeddings $f_i \colon \underline{B}_i \hookrightarrow \underline{C}$ for $i \in \{1, 2\}$ such that $f_1 \circ e_1 = f_2 \circ e_2$.

(62) Let $\underline{D}$ be the tournament obtained from the directed cycle $\vec{C}_3$ of length three by adding a new vertex u, and adding the edges (u, v) for every vertex v of $\vec{C}_3$. Let $\underline{D}'$ be the tournament obtained from $\underline{D}$ by flipping the orientation of each edge. Show that the class of all finite tournaments that embeds neither $\underline{D}$ nor $\underline{D}'$ is an amalgamation class.

(63) Let P be a unary relation symbol. Let $\mathcal{D}$ be the class of all finite $\{P, <\}$-structures $\underline{A}$ such that $<^{\underline{A}}$ is a linear order.
 (a) Show that $\mathcal{D}$ is an amalgamation class.

(b) Let $\underline{B}$ be the Fraïssé-limit of the class $\mathcal{D}$, and define $E \subseteq B^2$ by $(u, v) \in E$ if
- $u < v$ and $(u \in P \Leftrightarrow v \in P)$, or
- $u > v$ and not $(u \in P \Leftrightarrow v \in P)$.

Show that $(B; E)$ is a tournament.

(c) Show that the class $\mathrm{Age}(B; E)$ equals the class of all tournaments that can be obtained from tournaments $\underline{T}$ in $\mathrm{Age}(\mathbb{Q}; <)$ by performing the following operation: pick $u \in T$ and reverse all edges between u and other elements of T (we 'switch edges at u').

(d) Show that $(B; E)$ is homogeneous.

(e) Show that $\mathrm{Age}(B; E)$ equals the class $\mathcal{C}$ from Exercise 62.

Solution suggestion. First note that every tournament obtained from the tournament $\underline{D}$ or the tournament $\underline{D}'$ from Exercise 62 by switching edges at a vertex (see Part (c)) is isomorphic to $\underline{D}$ or to $\underline{D}'$, and hence not in the age of $(B; E)$; it follows that $\mathrm{Age}(B; E) \subseteq \mathcal{C}$. For the converse inclusion, let $(V; E) \in \mathcal{C}$. Note that for any fixed $u \in V$, the set

$$V_1 := \{u\} \cup \{v \in V \mid (u, v) \in E\}$$

induces a linear order in $(V; E)$, because $(V; E)$ is a tournament that does not embed $\underline{D}$. Choose u such that $|V_1|$ is maximal after having applied a finite number of switches at vertices from V. From now on, $(V; E)$ denotes the graph after having performed this finite number of switches. If $V_1 = V$, then $(V; E) \in \mathrm{Age}(B; E)$ by Part (c) of the exercise and we are done. Also the set $V_2 := \{v \in V \mid (v, u) \in E\}$ induces a linear order in $(V; E)$, because $(V; E)$ is a tournament that does not embed $\underline{D}'$. Let $w \in V_2$ be the maximal element with respect to E. Let $w' \in V_1$ be minimal with respect to E such that $(w', w) \in E$; such an element must exist due to the choice of u since otherwise $V_1 \cup \{w\}$ induces a linear order in $(V; E)$. Also note that $w' \neq u$ because $(w, u) \in E$. Let $w'' \in V_1$ be the predecessor of w' with respect to E; such an element must exist because $w' \neq u$. Let $V_1' := \{v \in V_1 \mid (w', v) \in E\}$. Note that $(w, w''), (w'', w'), (w', w) \in E$; hence, for every $v \in V_1'$ we have $(v, w) \in E$ since otherwise $\{v, w, w', w''\}$ induce a copy of $\underline{D}'$ in (V, E). Then after switching edges at each $v \in V_1' \cup \{w'\}$, the set $V_1 \cup \{w\}$ induces a linear order (with least element w, then w', followed by the elements of V_1', then u, followed by all other elements of V_1; the final vertex is w''), and we again reach a contradiction to the choice of u such that $|V_1|$ is maximal.

(f) Show how solutions to the previous sub-exercises provide a solution to Exercise 62.

(g) Show that $(B; E)$ is isomorphic to the tournament whose vertices are a countable dense subset $S \subseteq \mathbb{R}^2$ of the unit circle without antipodal points, and where the edges are oriented in clockwise order, i.e., put $((u_1, u_2), (v_1, v_2)) \in E$ if and only if $u_1 v_2 - u_2 v_1 > 0$.

3.4. Strong Amalgamation and Algebraicity

A *strong amalgam* of $\underline{B}_1, \underline{B}_2$ is an amalgam $\underline{C}$ of $\underline{B}_1, \underline{B}_2$ with embeddings $f_i \colon \underline{B}_i \to \underline{C}$ such that $f_1(B_1) \cap f_2(B_2) = f_1(B_1 \cap B_2) = f_2(B_1 \cap B_2)$. We say that a class $\mathcal{C}$ has the *strong amalgamation property* if all $\underline{B}_1, \underline{B}_2 \in \mathcal{C}$ have a strong amalgam in $\mathcal{C}$. A class $\mathcal{C}$ is called a *strong amalgamation class* if it is an amalgamation class with the strong amalgamation property.

Examples:

- The age of the Random Graph.
- The age of $(\mathbb{Q}; <)$.
- The age of all other structures we have seen so far.
- An example of an amalgamation class that does *not* have strong amalgamation: let P be a unary relation symbol, and let $\mathcal{C}$ be the class of all finite $\{P\}$-structures where P contains at most one element. Then $\mathcal{C}$ is an amalgamation class, and the Fraïssé-limit is a countably infinite structure where P contains only one element. But $\mathcal{C}$ does not have strong amalgamation. Picture with an amalgamation diagram that fails to have a strong amalgam.
- Another non-example is the $(\mathbb{N}; E_2)$ (where E_2 is an equivalence relation with infinitely many classes of size two; see Exercise 41).

3.4.1. Algebraicity. If $\mathcal{C}$ is a strong amalgamation class, then the automorphism group of the Fraïssé-limit of $\mathcal{C}$ has a remarkable property. If $\underline{A}$ is a structure and $B \subseteq A$, then we say that $R \subseteq A^n$ is *(first-order) definable in $\underline{A}$ with parameters from B* (also: *definable in $\underline{A}$ over B*) if R is definable in the expansion of $\underline{A}$ by a constant for each element of B. We say that R is *parameter definable* if R is definable in $\underline{A}$ with parameters from A.

Definition 3.4.1 (Model-theoretic algebraic closure). *Let $\underline{A}$ be a structure and $B \subseteq A$. Then $\mathrm{acl}_{\underline{A}}(B)$ denotes the elements of $\underline{A}$ that lie in finite sets that are first-order definable over $\underline{A}$ with parameters from B.*

Example 32. Let $\underline{A} = (\mathbb{N}; E_2)$ where E_2 is an equivalence relation with infinitely many classes of size two. Then we have $\mathrm{acl}_{\underline{A}}(\emptyset) = \emptyset$, and for every $a \in \mathbb{A}$ we have $\mathrm{acl}_{\underline{A}}(\{a\}) = \{a, b\}$ where b is the unique element that is distinct from a and lies in the same E_2 class as a. $\triangle$

Definition 3.4.1 and Theorem 3.1.1 motivates the following definition.

Definition 3.4.2 (Group-theoretic algebraic closure). *Let $G \leq \mathrm{Sym}(A)$ be a permutation group and $B \subseteq A$ finite. Then $\mathrm{acl}_G(B)$ denotes the set of elements of A that lie in finite orbits in $G_{(B)}$.*

Lemma 3.4.3. *Both the model theoretic and the group theoretic algebraic closure are closure operators. If G is oligomorphic and the automorphism group of an (ω-categorical) structure $\underline{A}$, then the algebraic closure of B in G coincides precisely with the model-theoretic algebraic closure of B in $\underline{A}$, i.e.,*

$$\mathrm{acl}_{\underline{A}}(B) = \mathrm{acl}_{\mathrm{Aut}(\underline{A})}(B). \tag{3}$$

In this case, the algebraic closure of a finite set is finite.

Proof. Both for the group theoretic and the model theoretic version, the verification that $B \subseteq \mathrm{acl}(B)$, $B \subseteq C \Rightarrow \mathrm{acl}(B) \subseteq \mathrm{acl}(C)$, and $\mathrm{acl}(\mathrm{acl}(B)) = \mathrm{acl}(B)$ is easy and left to the reader. The statements about acl in an ω-categorical structure $\underline{A}$ follow from Theorem 3.1.1. $\square$

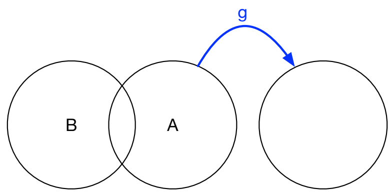


FIGURE 3.2. An illustration of Lemma 3.4.3.

We say that $B \subseteq A$ is *algebraically closed* in $\underline{A}$ if $\operatorname{acl}_{\underline{A}}(B) = B$. We say that $\underline{A}$ has *no algebraicity* if all finite $B \subseteq A$ are algebraically closed in $\underline{A}$. We use the analogous terminology for permutation groups G instead of structures, i.e., $B \subseteq A$ is *algebraically closed* in G if $\operatorname{acl}_G(B) = B$, and G has *no algebraicity* if all finite $B \subseteq A$ are algebraically closed in G.

THEOREM 3.4.4 (See (2.15) in [**39**]). *Let $\underline{A}$ be a homogeneous structure. Then the age of $\underline{A}$ has strong amalgamation if and only if $G := \operatorname{Aut}(\underline{A})$ has no algebraicity.*

PROOF. We first show that strong amalgamation of the age implies no algebraicity of G. Let $t \in A^n$, $F = \{t_1, \dots, t_n\}$, and $u \in A \setminus F$. We want to show that the orbit of u in G_t is infinite. Let $m \in \mathbb{N}$ and let $\underline{B}$ be the structure induced by $F \cup \{u\}$ in $\underline{A}$. Then there exists a strong amalgam $\underline{B}' \in \operatorname{Age}(\underline{A})$ of $\underline{B}$ with $\underline{B}$ over $\underline{A}[F]$. We iterate this, taking a strong amalgam of $\underline{B}$ with $\underline{B}'$ over $\underline{A}[F]$, showing that, because of homogeneity, there are m distinct elements in $A \setminus F$ that lie in the same orbit as u in G_t. Since $m \in \mathbb{N}$ and $u \in A \setminus F$ were chosen arbitrarily, the group G_t has no finite orbits outside F.

For the other direction, we rely on the following lemma of Peter Neumann (see Figure 3.2).

LEMMA 3.4.5. *Let G be a permutation group on D without finite orbits, and let $A, B \subset D$ be finite. Then there exists $g \in G$ with $g(A) \cap B = \emptyset$.*

PROOF. The proof is by induction on $|A|$. The induction base $A = \emptyset$ is trivial. We assume the result for any set A' with $|A'| < |A|$. We may also assume without loss of generality that there exists $a \in A \setminus B$, because for any $a' \in A$ we choose $k \in G$ such that $k(a') \notin B$ since G has no finite orbits. If $g' \in G$ shows that the statement holds for $k(A)$ instead of A, then $g := g'k$ is the map we are looking for.

Let $h_1, \dots, h_k \in G$ be such that $\big\{h_1(a), \dots, h_k(a)\big\} = (G \circ a) \cap B$. By the inductive assumption, there exists $h \in G$ such that

$$h(A \setminus \{a\}) \cap \big\{B \cup h_1(B) \cup \cdots \cup h_k(B)\big\} = \emptyset.$$

If $h(a) \notin B$, then $h(A) \cap B = \emptyset$ and we are done. Otherwise, $h(a) = h_i(a)$ for some $i \in \{1, \dots, k\}$. Define $g := h_i^{-1} h$. Then

- $g(a) = a \notin B$ since $a \in A \setminus B$.
- $g(A \setminus \{a\}) \cap B = h_i^{-1} h(A \setminus \{a\}) \cap B = \emptyset$, since $h(A \setminus \{a\}) \cap h_i(B) = \emptyset$.

Therefore, $g(A) \cap B = \emptyset$. □

We now continue with the reverse implication of Theorem 3.4.4. Suppose G has no algebraicity, and let $\underline{B}_1, \underline{B}_2 \in \operatorname{Age}(\underline{A})$. By the homogeneity of $\underline{A}$ we can furthermore assume that $\underline{B}_1, \underline{B}_2$ are substructures of $\underline{A}$; that is, the structure induced by $B_1 \cup B_2$ in $\underline{A}$ is an amalgam, but possibly not a strong one. Let $t \in A^n$ be such that $\{t_1, \dots, t_n\} = B_1 \cap B_2$. Since G has no algebraicity, G_t has no finite orbits outside

$B_1 \cap B_2$. By the lemma, there exists $g \in G_t$ such that $(B_1 \setminus B_2) \cap g(B_2 \setminus B_1) = \emptyset$. Then the substructure of $\underline{A}$ induced on $B_1 \cup g(B_2 \setminus B_1)$ is a strong amalgam of $\underline{B}_1$ and $\underline{B}_2$. $\square$

Exercises.

(64) Give an example of a homogeneous structure with a transitive automorphism group whose age does not have strong amalgamation.

(65) Let G be a permutation group on a set A. Show that the following are equivalent.
  (a) There exists a structure $\underline{A}$ with finite relational signature such that $G = \mathrm{Aut}(\underline{A})$;
  (b) There exists a relation $R \subseteq A^n$ such that $G = \mathrm{Aut}(A, R)$;
  (c) There exists a structure $\underline{A}$ with finite relational signature such that $G = \mathrm{Aut}(\underline{A})$, and all relations in $\underline{A}$ have pairwise distinct entries.

(66) If in the previous exercise we additionally assume that the domain A is *finite*, we can combine the two conditions, and the above items are equivalent to
  (d) There exists a relation $R \subseteq A^n$ such that $G = \mathrm{Aut}(A, R)$ and R has pairwise distinct entries.

(67) Prove the equivalence of (a) and (d) in the previous exercises for oligomorphic G.

(68) Prove the equivalence of (a) and (d) in the previous exercises for an arbitrary permutation group G on an infinite set.

(69) For Equation 3, can we replace the assumption that $\mathrm{Aut}(\underline{A})$ is oligomorphic by the assumption that $\underline{A}$ is homogeneous?

3.4.2. Generic superpositions. For strong amalgamation classes there is a powerful construction to obtain new strong amalgamation classes from known ones.

DEFINITION 3.4.6. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be classes of finite structures with disjoint relational signatures τ_1 and τ_2, respectively. Then the* generic superposition of $\mathcal{C}_1$ and $\mathcal{C}_2$*, denoted by $\mathcal{C}_1 * \mathcal{C}_2$, is the class of $(\tau_1 \cup \tau_2)$-structures $\underline{A}$ such that the τ_i-reduct of $\underline{A}$ is in $\mathcal{C}_i$, for $i \in \{1, 2\}$.*

The following lemma has a straightforward proof by combining amalgamation in $\mathcal{C}_1$ with amalgamation in $\mathcal{C}_2$.

LEMMA 3.4.7. *If $\mathcal{C}_1$ and $\mathcal{C}_2$ are strong amalgamation classes, then $\mathcal{C}_1 * \mathcal{C}_2$ is also a strong amalgamation class.*

If $\underline{A}_1$ and $\underline{A}_2$ are countable homogeneous structures with countable signatures and without algebraicity, then $\underline{A}_1 * \underline{A}_2$ denotes the (up to isomorphism unique) Fraïssé limit of the generic superposition of the age of $\underline{A}_1$ and the age of $\underline{A}_2$.

EXAMPLE 33. For $i \in \{1, 2\}$, let $\tau_i = \{<_i\}$, let $\mathcal{C}_i$ be the class of all finite τ_i-structures where $<_i$ denotes a linear order, and let $\underline{A}_i$ be the Fraïssé limit of $\mathcal{C}_i$. Then $\underline{A}_1 * \underline{A}_2$ is known as the *random permutation* (see e.g. [**34**, **107**, **147**]). $\triangle$

PROPOSITION 3.4.8. *Let $\underline{A}_1, \underline{A}_2$ be countable homogeneous structures with countable signatures and without algebraicity. Then for $i \in \{1, 2\}$ the τ_i-reduct of $\underline{A}_1 * \underline{A}_2$ is isomorphic to $\underline{A}_i$.*

Proof. Let $a_0, a_1, a_2, \dots$ be an enumeration of the elements of $\underline{A}_1 * \underline{A}_2$, and let $b_0, b_1, b_2, \dots$ be an enumeration of the elements of $\underline{A}_1$. Suppose that f is an isomorphism between the τ_1-reduct $\underline{B}$ of a finite substructure of $\underline{A}_1 * \underline{A}_2$ and a finite substructure of $\underline{A}_1$.

- Going forth: let $j \in \mathbb{N}$ be smallest so that $a_j \notin B$. Then the τ_1-reduct of $\underline{A}_1 * \underline{A}_2[B \cup \{a_j\}]$ has an embedding e into $\underline{A}_1$. By the homogeneity of $\underline{A}_1$ we may assume that e extends f.
- Going back: let $j \in \mathbb{N}$ be smallest so that $b_j \notin f(B)$, and let $k \in \mathbb{N}$ be such that $a_k \notin B$. By definition, the age of $\underline{A}_1 * \underline{A}_2$ contains a structure $\underline{C}$
  - whose τ_2-reduct equals the τ_2-reduct of $\underline{A}_1 * \underline{A}_2[B \cup \{a_k\}]$.
  - the extension of f that maps a_k to b_j is an isomorphism between the τ_1-reduct of $\underline{C}$ and $\underline{A}_1[f(B) \cup \{b_j\}]$.

  Let e be an embedding of $\underline{C}$ into $\underline{A}_1 * \underline{A}_2$; by the homogeneity of $\underline{A}_1 * \underline{A}_2$ we may assume that e is the identity on B. Then $f(e(a_k)) := b_j$ is an extension of f whose image contains b_j, as desired.

If we repeat going forth and going back for countably many steps, each time extending the isomorphism f as described above, the result is an embedding that is defined on all of $\underline{A}_1 * \underline{A}_2$ (by the way we defined going forth) and which is surjective onto $\underline{A}_1$ (by the way we defined going back). An isomorphism between $\underline{A}_1 * \underline{A}_2$ and $\underline{A}_2$ can be constructed analogously. $\square$

Exercises.

(70) Construct an permutation group G on a set X with precisely $n!$ orbits of n-element subsets. Extra question (I don't know the answer to it): does this property characterise G uniquely up to isomorphism?

(71) Show that the random graph can be partitioned into two subsets so that both parts are isomorphic to the random graph.

(72) Show that the previous statement is not true for all partitions of the random graph into two infinite subsets.

(73) Show that for every partition of the random graph, one of the parts induces a subgraph which is isomorphic to the random graph.

(74) Show that an amalgamation class $\mathcal{C}$ has the free amalgamation property if and only if $\mathcal{C} = \mathrm{Forb}(\mathcal{N})$ for some set of finite structures $\mathcal{N}$ whose *Gaifman graphs* are cliques. The *Gaifman graph* of a structure $\underline{A}$ is a graph with vertex set A where two distinct $u, v \in A$ are adjacent if and only if $\underline{A}$ contains a relation with a tuple where both u and v appear in some entry.

(75) Let $\underline{A}$ be a homogeneous relational structure whose age has the free amalgamation property. Show that if $\underline{B}, \underline{C} \in \mathrm{Age}(\underline{A})$, then there is $g \in \mathrm{Aut}(\underline{A})_{(B \cap C)}$ such that $\underline{A}[g(B) \cup C]$ is isomorphic to $\underline{B} \cup \underline{C}$.

(76) Show that if $\mathcal{C}_1 * \mathcal{C}_2$ is a strong amalgamation class, then $\mathcal{C}_1$ and $\mathcal{C}_2$ are strong amalgamation classes.

(77) Let $\mathcal{C}_1, \mathcal{C}_2$ be amalgamation classes with Fraïssé-limits $\underline{A}_1, \underline{A}_2$ such that $\mathrm{Aut}(\underline{A}_1)$ and $\mathrm{Aut}(\underline{A}_2)$ are oligomorphic. Show that if $\mathcal{C}_1$ does not have the strong amalgamation property, then $\mathcal{C}_1 * \mathcal{C}_2$ is not an amalgamation class, unless $\mathrm{Aut}(\mathrm{Flim}(\mathcal{C}_2)) = \mathrm{Sym}(A_2)$.

(78) Show that there are permutation groups G_1, G_2 on a countably infinite set such that both G_1 and G_2 are isomorphic (as permutation groups) to $\mathrm{Aut}(\mathbb{Q}; <)$, but $G_1 \cap G_2 = \{\mathrm{id}\}$.

3.5. The Weak Amalgamation Property

Definition 3.5.1. *A class $\mathcal{K}$ of finite structures satisfies the* weak amalgamation property (WAP) *if every $\underline{A} \in \mathcal{K}$ has an extension $\underline{A}' \in \mathcal{K}$ which is determined on $\underline{A}$, that is, for all $\underline{B}_1, \underline{B}_2 \in \mathcal{K}$ and embeddings $e_i \colon \underline{A}' \to \underline{B}_i$ for $i \in \{1,2\}$ there exists $\underline{C} \in \mathcal{K}$ and embeddings $f_i \colon \underline{B}_i \to \underline{C}$ such that $f_1 \circ e_1|_A = f_2 \circ e_2|_A$.*

Example 34. A class of relational structures with JEP and WAP, but not the AP is the age of $(\mathbb{Z}; \{(x,y) \mid x = y+1\})$ (Exercise 79). An ω-categorical structure whose age has the WAP but not the AP is the structure $(\mathbb{Q}_0^+; <, \{0\})$ (Exercise 80). $\triangle$

Exercises.

(79) Prove that the age of $(\mathbb{Z}; \{(x,y) \mid x = y+1\})$ does not have the AP, but the WAP.

(80) Prove that the age of $(\mathbb{Q}; <, \{0\})$ does not have the AP, but the WAP.

(81) Prove that the age of $(\mathbb{Q}; \{(x,y,z) \mid x < y, z \neq x, z \neq y\})$ does not have the AP, but the WAP.

(82) A countable ω-categorical τ-structure $\underline{B}$ is called *model-complete* if every first-order τ-formula is over $\underline{B}$ equivalent to an existential τ-formula. Show that $\underline{B}$ is model-complete if and only if $\mathrm{Aut}(\underline{B})$ is dense in $\mathrm{Emb}(\underline{B}) \subseteq B^B$, where $\mathrm{Emb}(\underline{B})$ is the set of all self-embeddings of $\underline{B}$.

(83) Let $\underline{A}$ be an ω-categorical structure. The a *model companion* of $\underline{A}$ is a structure $\underline{B}$ with the same age as $\underline{A}$ which is model-complete (see the previous exercise). Show that every ω-categorical structure has a model companion, and that the model companion is unique up to isomorphism. We refer to $\underline{B}$ as *the* model companion of $\underline{A}$.

(84) (Todor Tsankov, personal communication 2012) Prove that the age of *every* ω-categorical structure $\underline{A}$ has the WAP.

(85) A structure $\underline{A}$ with a finite relational signature is called *homogenizable* if there exists a definable expansion $\underline{B}$ of $\underline{A}$ by finitely many new relations such that $\underline{B}$ is homogeneous. We say that $\underline{A}$ is *boundedly homogenizable* if it is homogenizable and for every $a \in A^n$, $n \in \mathbb{N}$, there exists $b \in A^m$, $m \in \mathbb{N}$, such that the type of (a,b) in $\underline{A}$ is equivalent to a quantifier-free formula. Find a structure which is model-complete, homogenizable, but not boundedly homogenizable.

More on the WAP can be found in Example **??**.

3.6. First-order Interpretations

Many ω-categorical structures can be derived from other ω-categorical structures via interpretations (our definition follows [**73**]).

DEFINITION 3.6.1. *A relational σ-structure $\underline{B}$ has a* (first-order[2]) interpretation *I in a τ-structure $\underline{A}$ if there exists a natural number d, called the* dimension *of I, and*

- *a τ-formula $\delta_I(x_1, \dots, x_d)$ – called the* domain formula*,*
- *for each atomic σ-formula $\phi(y_1, \dots, y_k)$ a τ-formula $\phi_I(\overline{x}_1, \dots, \overline{x}_k)$ where the $\overline{x}_i$ denote disjoint d-tuples of distinct variables – called the* defining formulas*,*
- *a surjective map $h \colon D \to B$, where*

$$D := \{(a_1, \dots, a_d) \in A^d \mid \underline{A} \models \delta_I(a_1, \dots, a_d)\}$$

  – called the coordinate map*,*

such that for all atomic σ-formulas ϕ and all tuples $\overline{a}_i \in D$

$$\underline{B} \models \phi(h(\overline{a}_1), \dots, h(\overline{a}_k)) \;\Leftrightarrow\; \underline{A} \models \phi_I(\overline{a}_1, \dots, \overline{a}_k) \, .$$

Note that the coordinate map h determines the defining formulas up to logical equivalence; hence, we sometimes identify I with h. Note that the kernel of h coincides with the relation defined by $(x = y)_I$, for which we also write $=_I$, the defining formula for equality.

We say that $\underline{B}$ is *interpretable in $\underline{A}$ with finitely many parameters* if there are $a_1, \dots, a_n \in A$ such that $\underline{B}$ is interpretable in the expansion of $\underline{A}$ by the singleton relations $\{a_i\}$ for all $1 \leq i \leq n$.

LEMMA 3.6.2 (see Theorem 7.3.8 in [**72**]). *Let $\underline{A}$ be an ω-categorical structure. Then every structure $\underline{B}$ that is interpretable in $\underline{A}$ with finitely many parameters is ω-categorical or finite.*

PROOF. An easy consequence of Theorem 3.2.3. $\square$

Note that in particular all reducts of an ω-categorical structure and all expansions of an ω-categorical structure by finitely many constants are again ω-categorical.

EXAMPLE 35. Allen's interval algebra: studied in *temporal reasoning* in computer science. Domain: closed bounded intervals over the rational numbers. Relations: containment, disjointness, precedence, etc. Formally a structure $\underline{A}$ that is best described by a first-order interpretation I in $(\mathbb{Q}; <)$:

- the dimension of the interpretation is two;
- the domain formula $\delta_I(x, y)$ is $x < y$;
- for each inequivalent $\{<\}$-formula ϕ with four variables a binary relation R such that (a_1, a_2, a_3, a_4) satisfies ϕ if and only if $((a_1, a_2), (a_3, a_4)) \in R$.

By Lemma 3.6.2, $\underline{A}$ is ω-categorical. $\triangle$

3.6.1. Composing interpretations. Interpretations can be composed. In order to conveniently treat these compositions, we first describe how an interpretation of a σ-structure $\underline{B}$ gives rise to interpreting formulas for arbitrary σ-formulas $\psi(x_1, \dots, x_n)$. Replace each atomic σ-formula $\phi(y_1, \dots, y_n)$ in ψ by the formula $\phi_I(y_{1,1}, \dots, y_{1,d}, \dots, y_{n,1}, \dots, y_{n,d})$; we write $\psi_I(x_{1,1}, \dots, x_{1,d}, \dots, x_{n,1}, \dots, x_{n,d})$ for the resulting τ-formula, and call it the *interpreting formula* for ψ. Note that if ψ defines the relation R in $\underline{B}$, then ϕ_I defines $I^{-1}(R)$ in $\underline{A}$. For all d-tuples $a_1, \dots, a_k \in I^{-1}(B)$

$$\underline{B} \models \phi(I(a_1), \dots, I(a_k)) \;\Leftrightarrow\; \underline{A} \models \phi_I(a_1, \dots, a_k) \, .$$

[2]'First-order' refers to the usage of first-order logic in the definition; one may replace first-order logic by fragments or extensions of first-order logic, or by entirely different logics, and in this way one obtains different forms of interpretations that have been studied in the literature. We often omit the specification of 'first-order' in this text since all interpretations studied here are first-order.

Definition 3.6.3. *Let $\underline{C}$, $\underline{B}$, $\underline{A}$ be structures with the relational signatures ρ, σ, and τ. Suppose that*

- *$\underline{C}$ has a d-dimensional interpretation I in $\underline{B}$, and*
- *$\underline{B}$ has an e-dimensional interpretation J in $\underline{A}$.*

Then $\underline{C}$ has a natural $(d \cdot e)$-dimensional interpretation $I \circ J$ in $\underline{A}$: the domain of $I \circ J$ is the set of all de-tuples in A that satisfy the τ-formula $(\top_I)_J$, and we define

$$I \circ J(a_{1,1}, \ldots, a_{1,e}, \ldots, a_{d,1}, \ldots, e_{d,e}) := I(J(a_{1,1}, \ldots, a_{1,e}), \ldots, J(a_{d,1}, \ldots, e_{d,e}))$$

(which is a partial function, composed from partial functions, and which is defined on all tuples where all partial functions that appear in the composition are defined).

Let ϕ be a τ-formula which defines a relation R over $\underline{A}$. Then the formula $(\phi_I)_J$ defines in $\underline{A}$ the preimage of R under $I \circ J$.

3.6.2. Bi-interpretations. Two interpretations I_1 and I_2 of $\underline{B}$ in $\underline{A}$ are called *homotopic*[3] if the relation $\{(\bar{x}, \bar{y}) \mid I_1(\bar{x}) = I_2(\bar{y})\}$ is first-order definable in $\underline{D}$. Note that id_C is an interpretation of $\underline{C}$ in $\underline{C}$, called the *identity interpretation* of $\underline{C}$ (in $\underline{C}$).

Definition 3.6.4. *Two structures $\underline{A}$ and $\underline{B}$ with an interpretation I of $\underline{B}$ in $\underline{A}$ and an interpretation J of $\underline{A}$ in $\underline{B}$ are called* mutually interpretable*. If both $I \circ J$ and $J \circ I$ are homotopic to the identity interpretation (of $\underline{A}$ and of $\underline{B}$, respectively), then we say that $\underline{A}$ and $\underline{B}$ are* bi-interpretable (via I and J).

Example 36. The directed graph $\underline{C} := (\mathbb{N}^2; E)$ where

$$E := \big\{((u_1, u_2), (v_1, v_2)) \mid u_2 = v_1\big\}$$

and the structure $\underline{D} := (\mathbb{N}; =)$ are bi-interpretable. The interpretation I_1 of $\underline{C}$ in $\underline{D}$ is 2-dimensional, the domain formula is true, and the coordinate map is the identity. The interpretation I_2 of $\underline{D}$ in $\underline{C}$ is 1-dimensional, the domain formula is true, and the coordinate map sends (x, y) to x.

Then $I_2(I_1(x, y)) = z$ is definable by the formula $x = z$, and hence $I_1 \circ I_2$ is homotopic to the identity interpretation of $\underline{D}$. Moreover, $I_1(I_2(u), i_2(v)) = w$ is first-order definable by

$$E(w, v) \wedge \exists p \, (E(p, u) \wedge E(p, w)) \; ,$$

so $I_2 \circ I_1$ is also homotopic to the identity interpretation of $\underline{C}$. △

Example 37. It is easy to see that Allen's interval algebra (Example 35) is bi-interpretable with $(\mathbb{Q}; <)$. △

We show that the property to have essentially infinite language is preserved by bi-interpretability.

Proposition 3.6.5. *Let $\underline{B}$ and $\underline{C}$ be ω-categorical structures that are first-order bi-interpretable. Then $\underline{B}$ has essentially infinite signature if and only if $\underline{C}$ has.*

Proof. Let τ be the signature of $\underline{B}$. We have to show that if $\underline{C}$ has finite signature, then $\underline{B}$ is interdefinable with a structure $\underline{B}'$ with a finite signature. Let $\sigma \subseteq \tau$ be the set of all relation symbols that appear in all the formulas of the interpretation of $\underline{C}$ in $\underline{B}$. Since the signature of $\underline{C}$ is finite, the cardinality of σ is finite as well.

We will show that there is a first-order definition of $\underline{B}$ in the σ-reduct $\underline{B}'$ of $\underline{B}$. Suppose that the interpretation I_1 of $\underline{C}$ in $\underline{B}$ is d_1-dimensional, and that the interpretation I_2 of $\underline{B}$ in $\underline{C}$ is d_2-dimensional. Let $\theta(x, y_{1,1}, \ldots, y_{d_1,d_2})$ be the formula

[3] We follow the terminology from [**3**].

that shows that $I_2 \circ I_1$ is homotopic to the identity interpretation. That is, θ defines in $\underline{B}$ the $(d_1 d_2 + 1)$-ary relation that contains a tuple $(a, b_{1,1}, \ldots, b_{d_1,d_2})$ iff

$$a = h_2(h_1(b_{1,1}, \ldots, b_{1,d_2}), \ldots, h_1(b_{d_1,1}, \ldots, b_{d_1,d_2})) \; .$$

Let ϕ be an atomic τ-formula with k free variables $x_1, \ldots, x_k$. We will specify a σ-formula that is equivalent to ϕ over $\underline{B}'$.

$$\begin{aligned} \exists y^1_{1,1}, \ldots, y^k_{d_1,d_2} \; \Big(& \bigwedge_{i \leq k} \theta(x_i, y^i_{1,1}, \ldots, y^i_{d_1,d_2}) \\ & \wedge \; \phi_{I_1 I_2}(y^1_{1,1}, \ldots, y^k_{1,d_2}, y^1_{2,d_2}, \ldots, y^k_{2,d_2}, \ldots, y^k_{d_1,d_2})\Big) \end{aligned}$$

is equivalent to $\phi(x_1, \ldots, x_k)$ over $\underline{B}$. Indeed, by surjectivity of h_2, for every element a_i of $\underline{B}$ there are elements $c^i_1, \ldots, c^i_{d_2}$ of $\underline{C}$ such that $h_2(c^i_1, \ldots, c^i_{d_2}) = a_i$, and by surjectivity of h_1, for every element c^i_j of $\underline{C}$ there are elements $b^i_{1,j}, \ldots, b^i_{d_1,j}$ of $\underline{B}$ such that $h_1(b^i_{1,j}, \ldots, b^i_{d_1,j}) = c^i_j$. Then

$$\begin{aligned} \underline{B} \models R(a_1, \ldots, a_k) \; & \Leftrightarrow \underline{C} \models \phi_{I_2}(c^1_1, \ldots, c^1_{d_2}, \ldots, c^k_1, \ldots, c^k_{d_2}) \\ & \Leftrightarrow \underline{B}' \models \phi_{I_1 I_2}(b^1_{1,1}, \ldots, b^k_{1,d_2}, b^1_{2,d_2}, \ldots, b^k_{2,d_2}, \ldots, b^k_{d_1,d_2}) \qquad \Box \end{aligned}$$

We will come back to bi-interpretability in Section 5.3.

CHAPTER 4

Topological Groups

We start with a very brief introduction to concepts from topology that will be relevant in what follows.

4.1. Topological Spaces

Topological spaces are mathematical structures that are used to capture the notions of "closeness" and "continuity" on a very general level. A *topological space* is a set S together with a collection of subsets of S, called the *open* sets of S, such that

(1) the empty set and S are open;
(2) arbitrary unions of open sets are open;
(3) the intersection of two open sets is open.

Complements of open sets are called *closed*. Note that S and the empty set are *both* open and closed.

Example 38. On every set S, there is the *trivial topology* where the only open sets are S and the empty set. △

Example 39. Every set S can be equipped with the *discrete topology*: in this topology, every subset of S is open (and hence also closed). △

Example 40. The *standard topology on* $\mathbb{R}$: a set $U \subseteq \mathbb{R}$ is open exactly if for every $x \in U$ there exists an $\epsilon > 0$ such that the every $y \in \mathbb{R}$ with $x - \epsilon < y < x + \epsilon$ is contained in U. The standard topology on $\mathbb{Q}$ is defined analogously. △

Example 41. The *standard topology on* $\mathbb{R}^d$, $d \in \mathbb{N}$: a set $U \subseteq \mathbb{R}^d$ is open exactly if for every $x \in U$ there exists an $\epsilon > 0$ such that the ϵ-ball around x is contained in U. △

Example 42. Every set S can be equipped with the *cofinite topology*: in this topology, the open sets are the empty set and every cofinite subset of S. The only closed subsets in this topology are the finite sets and the entire set S. △

Example 43. The topology of pointwise convergence on $\mathrm{Sym}(\mathbb{N})$; see Proposition 1.2.4. △

For $E \subseteq S$, the *closure* of E, denoted by $\bar{E}$, is the intersection over all closed sets over S that contain E. A subset E of S is called *dense (in S)* if its closure is the full space S. The *interiour* of E is the largest open set contained in E; and is denoted by $\mathrm{Int}(E)$. Equivalently, $\mathrm{Int}(E) := S \setminus \overline{S \setminus E}$.

The *subspace of S induced on E* is the topological space on E where the open sets are exactly the intersections of E with the open sets of S. When we work with permutation groups $G \subseteq \mathrm{Sym}(X)$ then we will always work with the topology on G inherited from $\mathrm{Sym}(X)$ in this way.

4.1.1. Countability, separation, and connection. A *basis* (or *base*) of S is a collection of open subsets of S such that every open set in S is the union of sets from the collection. Clearly, a topology is uniquely given by any of its bases. For $p \in S$, a collection of open subsets of S is called a *basis at* p if each set from the collection contains p, and every open set containing p also contains an open set from the collection. A topological space S is called

- *first-countable* if for all $s \in S$ there exists a countable basis at s;
- *second-countable* if it has a countable basis;
- *separable* if it contains a countable dense set.

Note that being second-countable implies first-countable (to obtain a countable basis at $s \in S$, select all members of the countable basis of S that contain s.) Every second-countable space is also separable: if $\{U_1, U_2, \dots\}$ is a countable basis of nonempty sets, choosing any $x_n \in U_n$ gives a countable dense set $\{x_1, x_2, \dots\}$. (Suppose for contradiction that there exists $p \in S \setminus \overline{\{x_1, x_2, \dots\}}$. Then there is a closed set V containing $\{x_1, x_2, \dots\}$ but not p. Hence, there is an open set U that contains p but no point in $\{x_1, x_2, \dots\}$. Since $\{U_1, U_2, \dots\}$ is a basis, there exists an $i \in \mathbb{N}$ such that $U_i \subseteq U$. But then $x_i \in U_i \subseteq U$, a contradiction.)

Example 44. We revisit the examples that we have seen above.

- The standard topology on $\mathbb{R}$ and, more generally, on $\mathbb{R}^d$ for $d \in \mathbb{N}$ are second-countable (and in particular first-countable), a countable basis being the set of all open balls with rational center and rational radius.
- The topology of pointwise convergence on $\mathrm{Sym}(\mathbb{N})$ is second-countable: there are countably many basic open sets $S(a, b)$ with $a, b \in \mathbb{N}^n$ for $n \in \mathbb{N}$.
- The discrete topology on any set S is first-countable (for every $p \in S$, the set $\{\{p\}\}$ is a basis at p), but if S is uncountable, S is not separable (we have $\bar{U} = U$ for all $U \subseteq X$), and therefore also not second-countable.
- The cofinite topology on an uncountably infinite set S is not first-countable, but separable: the closure of any countably infinite subset of S is S. $\triangle$

A topological space S is called *Hausdorff* if for any two distinct points u, v of S there are disjoint open sets U and V that contain u and v, respectively.

Example 45. The standard topology on $\mathbb{R}$, $\mathbb{R}^d$, $\mathbb{Q}$, and the topology of pointwise convergence on $\mathrm{Sym}(\mathbb{N})$ are Hausdorff. The trivial topology on a set with at least two elements is not Hausdorff. The cofinite topology on an infinite set is a more interesting example that is not Hausdorff: the intersection of any two non-empty open sets is infinite, so in particular not empty. So we cannot separate two distinct points with open disjoint sets. $\triangle$

We have seen that $\mathbb{R}$ and $\mathrm{Sym}(\mathbb{N})$ share some countability and separation properties. But in some other respects, these two spaces are very different. A topological space S is called *disconnected* if it is the union of two or more disjoint nonempty open subsets, and *connected* otherwise. A subset of S is said to be *connected* if it is connected under its subspace topology. The inclusion-wise maximal connected subsets of a non-empty topological space are called the *connected components* of that space. Picture in $\mathbb{R}^2$!

Example 46. The standard topology on $\mathbb{Q}$ is disconnected: the two sets

$$\{x \in \mathbb{Q} \mid x < \pi\} \text{ and } \{x \in \mathbb{Q} \mid x > \pi\}$$

are open, and for any irrational π they partition $\mathbb{Q}$. On the other hand, $\mathbb{R}$ and $\mathbb{R}^d$ are connected[1]. △

A topological space S is *totally disconnected* if all connected subsets of X are one-element sets.

EXAMPLE 47. The topology of pointwise convergence on $\mathrm{Sym}(\mathbb{N})$ is totally disconnected: if $f, g \in \mathrm{Sym}(\mathbb{N})$ are distinct, there exists an $i \in \mathbb{N}$ such that $f(i) \neq g(i)$. Then $S(i, f(i))$ is an open set that contains f, and $\mathrm{Sym}(\mathbb{N}) \setminus S(i, f(i)) = \bigcup_{j \neq f(i)} S(i, j)$ is a disjoint set that contains g, and it is open as a union of basic open sets. Hence, no set that contains more than one element is connected. △

Exercises.

(86) Show the claim from Example 46 that the standard topology on $\mathbb{R}$ is connected.

(87) On which sets X is the cofinite topology separable?

(88) Show that $\mathbb{R}$ with the standard topology is not homeomorphic to a closed subgroup of $\mathrm{Sym}(\mathbb{N})$.

(89) Show that the cofinite topology on an uncountably infinite set is not first-countable.

4.1.2. Continuity and convergence. A map between two topological spaces is called *continuous* if the pre-images of open sets are open, and *open* if images of open sets are open. A bijective open and continuous map is called a *homeomorphism*.

There are equivalent characterisations of continuity of maps from a first-countable space S to a topological space T that are often easier to work with. For a sequence $(s_n)_{n \geq 1}$ of elements of S, we say that s_n *converges against* s if for every open set U of S that contains s there exists an n_0 such that $s_n \in U$ for all $n \geq n_0$. Note that if T is Hausdorff, then s is unique, and called the *limit* of $(s_n)_{n \geq 1}$, and we write

$$\lim_{n \to \infty} s_n = s.$$

For $x \in S$, we say that f is *continuous at* x if for every open $V \subseteq T$ containing $f(x)$ there is an open $U \subseteq S$ containing x whose image $f(U)$ is contained in V.

PROPOSITION 4.1.1. *Let S be a first-countable and T an arbitrary topological space. Then for every $f \colon S \to T$ the following are equivalent.*

(1) *f is continuous.*
(2) *For all sequences s_n, if s_n converges against s, then $f(s_n)$ converges against $f(s)$.*
(3) *f is continuous at every $x \in S$.*

PROOF. The implication from (1) to (2) is true even without the assumption that S is first-countable. Let $(s_n)_{n \geq 1}$ be such that $\lim_{n \to \infty} s_n = s$, and let V be open so that $f(s) \in V$. Then $U := f^{-1}(V)$ is open, and $s \in U$. So there exists an n with $s_n \in U$. For this n we have $f(s_n) \in V$. So $\lim_{n \to \infty} f(s_n) = f(s)$.

For the implication from (2) to (3), we show the contraposition. Suppose that f is not continuous at some $s \in S$. That is, there exists an open set V containing $f(s)$ such that all open sets U that contain s have an image that is not contained in V. Since S is first-countable, there exists a countable collection U_n of open sets containing s so that any open set that contains s also contains some U_n. Replacing U_n by $\cap_{k=1}^{n} U_k$ where

[1]This relies on the fact that the real numbers are (by definition!) *Dedekind-complete*: every non-empty subset S of $\mathbb{R}$ with an upper bound in $\mathbb{R}$ has a least upper bound.

necessary, we may assume that $U_1 \supset U_2 \supset \cdots$. If $U_n \subseteq f^{-1}(V)$, then $f(U_n) \subseteq V$, in contradiction to our assumption; so we can pick an $s_n \in U_n \setminus f^{-1}(V)$ for all n, and obtain a sequence that converges to s. But $s_n \notin f^{-1}(V)$ for all n, and so $f(s_n)$ does not converge to $f(s) \in V$.

Finally, the implication from (3) to (1) again holds in arbitrary topological spaces. Let $V \subseteq T$ be open. We want to show that $U := f^{-1}(V)$ is open. When s is a point from U, then because f is continuous at s, and V contains $f(s)$ and is open, there is an open set $U_s \subseteq S$ containing s whose image $f(U_s)$ is contained in V. Then $\bigcup_{s \in U} U_s = U$ is open as a union of open sets. □

4.1.3. Product spaces. The *product* $\prod_{i \in I} S_i$ of a family of topological spaces $(S_i)_{i \in I}$ is the topological space on the cartesian product $\prod_{i \in I} S_i$ where the open sets are unions of sets of the form $\prod_{i \in I} U_i$ where U_i is open in S_i for all $i \in I$, and $U_i = S_i$ for all but finitely many $i \in I$. When I has just two elements, say 1 and 2, we also write $S_1 \times S_2$ for the product. We denote by S^k for the k-th power $S \times \cdots \times S$ of S, equipped with the product topology as described above.

We also write S^I to a $|I|$-th power of S, where the factors are indexed by the elements of I. In this case, we can view each element of $T := S^I$ as a function from I to S in the obvious way. The product topology on T is also called the *topology of pointwise convergence*, due to the following.

PROPOSITION 4.1.2. *Let S be a topological space, and I be a set. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of elements of the product space $T := S^I$. Then $\lim_{n \to \infty} f_n = f$ if and only if $\lim_{n \to \infty} f_n(j) = f(j)$ in S for all $j \in I$.*

PROOF. Suppose first that $\lim_{n \to \infty} f_n = f$ in T. Let $j \in I$ be arbitrary and let V be an open set that contains $f(j)$. Then the set $U := \prod_{i \in I} T_i$ where $T_i = V$ if $i = j$, and $T_i = S$ otherwise, is open in T and contains f. So there is an n_0 such that $f_n \in U$ for all $n \geq n_0$. But then $f_n(j) \in V$ for all $n \geq n_0$, and so $\lim_{n \to \infty} f_n(j) = f(j)$.

Now suppose that $\lim_{n \to \infty} f_n(j) = f(j)$ in S for all $j \in I$, and let V be an open set of T that contains f. Then there exists a finite $J \subseteq I$ and open subsets $(V_j)_{j \in J}$ of S such that $f \in \prod_{i \in I} T_i$ where $T_i = V_i$ if $i \in J$ and $T_i = S$ otherwise. For each $j \in J$ there exists an n_j so that $f_n(j) \in V_j$ for all $n \geq n_j$. Then $f_n \in V$ for all $n \geq max_{j \in J} n_j$, and hence $\lim_{n \to \infty} f_n = f$. □

EXAMPLE 48. When we equip the natural numbers $\mathbb{N}$ with the discrete topology, then $\mathbb{N}^{\mathbb{N}}$ with the topology of pointwise convergence is called the *Baire space*. Note that the open sets are exactly the unions of sets of the form

$$S(\bar{a}, \bar{b}) := \{g \in \mathbb{N} \to \mathbb{N} \mid g(\bar{a}) = \bar{b}\}$$

for some $\bar{a}, \bar{b} \in \mathbb{N}^k$, $k \in \mathbb{N}$. The closed subspace $2^{\mathbb{N}}$ is called the *Cantor space*. The subspace on $\mathrm{Sym}(\mathbb{N})$ is precisely the topology that we introduced in Proposition 1.2.4. △

THEOREM 4.1.3 (Baire). *$\mathbb{N}^{\mathbb{N}}$ is homeomorphic to the irrational numbers $\mathbb{P}$.*

PROOF. It suffices to construct a mapping from $\mathbb{Z}^{\mathbb{N}}$ to $\mathbb{P}$. Note that the sets of the form $\{x \in \mathbb{Z}^{\mathbb{N}} \mid x_1 \dots x_k = s\}$ for $s \in \mathbb{Z}^k$, $k \in \mathbb{N}$ form a basis of open sets for $\mathbb{Z}^{\mathbb{N}}$. Let $(q_n)_{n \in \mathbb{N}}$ be an enumeration of the rational numbers. Inductively construct a sequence of open intervals $(I_s)_{s \in \mathbb{Z}^k, k \in \mathbb{N}}$ satisfying the following:

(1) $I_s = \mathbb{R}$ if $s \in \mathbb{Z}^0$;
(2) if $s \in \mathbb{Z}^k$, for $k > 0$, then I_s is an open interval in $\mathbb{R}$ with rational endpoints of length less than $1/k$;
(3) for every $n \in \mathbb{Z}$ we have $I_{(s,n)} \subseteq I_s$,

(4) the right end point of $I_{(s,n)}$ is the left end point of $I_{(s,n+1)}$,
(5) $\{I_{(s,n)} \mid n \in \mathbb{Z}\}$ covers $I_s \cap \mathbb{P}$,
(6) the n-th rational q_n is an endpoint of I_s for some $s \in \mathbb{Z}^k$ with $k \leq n+1$.

Define the function $f\colon \mathbb{Z}^{\mathbb{N}} \to \mathbb{P}$ as follows. Given $x \in \mathbb{Z}^{\mathbb{N}}$ the set $\bigcap_{n\in\mathbb{N}} I_{x_1\ldots x_n}$ must consist of a singleton irrational:

- it is nonempty because $\overline{I_{x_1\ldots x_n x_{n+1}}} \subseteq I_{x_1\ldots x_n}$;
- it is a singleton because the length of $I_{x_1\ldots x_n}$ tends to zero for $n \to \infty$.

So we can define f by

$$\{f(x)\} := \bigcap_{n\in\mathbb{N}} I_{x_1\ldots x_n}.$$

The function f is injective because if s and t are not prefixes of each other then I_s and I_t are disjoint, and f is surjective because for every $u \in \mathbb{P}$ and $k \in \mathbb{N}$ there is a unique $s \in \mathbb{Z}^k$ with $u \in I_s$. Finally, f is a homeomorphism because

$$f(\{x \in \mathbb{Z}^{\mathbb{N}} \mid x_1 \ldots x_k = s\}) = I_s \cap \mathbb{P}$$

and the sets of the form $I_s \cap \mathbb{P}$ form a basis for $\mathbb{P}$. □

4.1.4. Metric spaces. Important examples of topologies come from metric spaces. A *pseudometric space* is a pair (M, d) where M is a set and d is a *pseudometric* on M, i.e., a function

$$d\colon M \times M \to \mathbb{R}$$

such that for any $x, y, z \in M$, the following holds:

(1) $d(x, y) \geq 0$ (*non-negativity*)
(2) $d(x, y) = d(y, x)$ (*symmetry*)
(3) $d(x, z) \leq d(x, y) + d(y, z)$ (*subadditivity* or *triangle inequality*)

If d additionally satisfies

(4) $d(x, y) = 0 \Leftrightarrow x = y$ (*indiscernibility*)

then d is called a *metric*, and (M, d) is called a *metric space*. If $M' \subseteq M$ then the restriction of d to M' is clearly a metric, too. Every metric on M gives rise to a topology on M, namely the topology with the basis

$$\big\{\{y \in M \mid d(x, y) < \epsilon\} \mid 0 < \epsilon \in \mathbb{R}, x \in M\big\}.$$

A topological space S is *metrisable* if there exists a metric d on S which is *compatible* with the topology, i.e., the topology equals the topology that arises from the metric as described above.

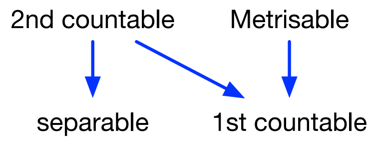
Example 49. The discrete metric ρ on X is defined by

$$\rho(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y \end{cases}$$

for any $x, y \in X$. In this case (X, ρ) is called a *discrete metric space* or a *space of isolated points*. △

Example 50. The distance function $d(x, y) = |x - y|$ (absolute difference) defines a metric on $\mathbb{R}$, $\mathbb{R}^d$, and on $\mathbb{Q}$. The topology that arises from this metric is precisely the standard topology on $\mathbb{R}$, $\mathbb{R}^d$, and on $\mathbb{Q}$. △

Proposition 4.1.4. *A metric space is second countable if and only if it is separable.*



Proof. Suppose that X is separable, and let A be a countable set which is dense in X. Then open balls with rational radii and centres from A form a countable basis. Why? Conversely, when X is second-countable, we choose for every U from a countable base of X one element; this will give a countable dense subset of X. $\square$

4.1.5. Complete Metrics. A sequence $(s_n)_{n\in\mathbb{N}}$ of elements of a metric space (M,d) is called a *Cauchy sequence* if

$$\forall \epsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n, m > n_0 : d(s_n, s_m) < \epsilon\,;$$

in this case, we write

$$\lim_{n,m\to\infty} d(s_n, s_m) = 0\,.$$

A metric space (M,d) is called *complete* if every Cauchy sequence converges against an element of M. A topological space S is called *completely metrizable* if it has a compatible complete metric. It is called *Polish* if S is separable and completely metrizable.

Example 51. The standard distance metric on $\mathbb{R}$ is complete. The same metric on $\mathbb{Q}$ is not complete. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ and countable, it follows that $\mathbb{R}$ is separable and hence Polish. $\triangle$

Example 52. The symmetric group $\mathrm{Sym}(D)$ on a countably infinite set D has the following compatible metric d. Let $b_1, b_2, \ldots$ be an enumeration of D. Then for elements $f, g \in \mathscr{G}$ we define $d(f,g) := 0$ if $f = g$, and otherwise $d(f,g) := 1/2^n$ where n is the least natural number such that $f(b_n) \neq g(b_n)$. In fact, d is an *ultrametric*, that is, it satisfies

$$d(x,z) \leq max(d(x,y), d(y,z))$$

for all x, y, z. This metric is not complete: to see this, let f be an arbitrary injective non-surjective mapping from $D \to D$. For each n, there exists a permutation h_n of D such that $h_n(b_i) = f(b_i)$ for all $i \leq n$. Hence, the sequence $(h_n)_{n\geq 1}$ is Cauchy, but it does not converge to a permutation. $\triangle$

Example 53. Similarly to the previous example, the Baire space (Example 48) can be equipped with a compatible metric: define $d(f,g) := 0$ if $f = g$, and otherwise $d(f,g) := 1/2^n$ where n is the least natural number such that $f(n) \neq g(n)$. For the Baire space, this is a complete metric. The restriction of this metric to the Cantor space shows that the Cantor space is completely metrisable as well. $\triangle$

Example 54. The topology on $\mathrm{Sym}(D)$ on a countably infinite set D is also completely metrizable. Again, let $b_1, b_2, \ldots$ be an enumeration of D. We define a compatible complete metric d' on $\mathrm{Sym}(D)$ by setting $d'(f,g) = 0$ if $f = g$, and otherwise $d'(f,g) = 1/2^n$ where n is the least natural number such that $f(b_n) \neq g(b_n)$ or $f^{-1}(b_n) \neq g^{-1}(b_n)$. To prove that this metric is complete, let $(g_m)_{m\in\mathbb{N}}$ be a Cauchy sequence with respect to d'. For $b \in D$, define $g(b) := g_i(b)$ if $i \in \mathbb{N}$ is such that $g_r(b) = g_s(b)$ for all $r, s \geq i$. Note that this is well-defined. Also note that such an i exists for every $b \in D$: if $b = b_j$ for $j \in \mathbb{N}$, then (g_m) being Cauchy with respect to d' implies that there exists i such that for all $r, s \geq i$ we have $d'(g_r, g_s) < 1/2^j$ which means that $g_r(b_j) = g_s(b_j)$. Note that the map $g\colon D \to D$ is injective, and also surjective. To show that $b \in D$ lies in the image of g, let j be such that $b = b_j$. Then there exists $i' \in \mathbb{N}$ such that $c := g_r^{-1}(b) = g_s^{-1}(b)$ for all $r, s \geq i$, again by the assumption that (g_m) is Cauchy with respect to d'. Then $g(c) = b$ by the definition of g. Since $\mathrm{Sym}(D)$ is also separable, we have that $\mathrm{Sym}(D)$ is Polish. △

EXAMPLE 55. The Hilbert cube $[0,1]^{\mathbb{N}}$. Here, the interval $[0,1]$ is equipped with the usual topology inherited from $\mathbb{R}$, and $[0,1]^{\mathbb{N}}$ carries the product topology. △

Exercises.

(90) Every metrisable space X has a compatible metric d which satisfies $d(x,y) \leq 1$ for all $x, y \in X$.

(91) Let (X,d) be a complete metric space and $S \subseteq X$. Then S is closed in X if and only if $(S, d|_{S^2})$ is complete.

(92) Show that every metrisable space is Hausdorff.

(93) Show that every metrisable space X is *regular*: for any $x \in X$ and any open set U that contains x there is an open set O that contains x such that $\bar{O} \subseteq U$.

(94) Show that every metrisable space X is *normal*: for any disjoint closed $C, F \subseteq X$ there are disjoint open $O, U \subseteq X$ such that $C \subseteq O$ and $F \subseteq U$.

(95) Let X be a metrisable space and $Y \subseteq X$ be closed. Then Y is a countable intersection of open sets.

(96) True or false: the closure of an open ball of radius r in a metric space is the closed ball of radius r in that metric space.

(97) Show that a countable product of metrisable spaces is metrisable.

(98) Every separable metrisable space is homeomorphic to a subspace of the Hilbert cube $[0,1]^{\mathbb{N}}$.

For a metric space (X,d) and $A \subseteq X$, define $\mathrm{diam}_d(A) := \sup_{b,c \in A} d(b,c)$. Note that for every $\epsilon > 0$ and every open subset U of X we find an open $V \subseteq U$ such that $\mathrm{diam}_d(V) < \epsilon$.

DEFINITION 4.1.5. *Countable intersections of open sets are called G_δ.*[2]

In the proof of the following lemma we use the axiom of dependent choice (Appendix A.2).

LEMMA 4.1.6. *Every Polish subspace Y of a Polish space X is G_δ.*

PROOF. Let d_X be a complete compatible metric on X and let d_Y be a complete compatible metric on Y. Define $V_n \subseteq X$ as the union of all open sets $U \subseteq X$ that satisfy

(1) $U \cap Y \neq \emptyset$;
(2) $\mathrm{diam}_{d_X}(U) < \frac{1}{n}$;
(3) $\mathrm{diam}_{d_Y}(U \cap Y) < \frac{1}{n}$;

It suffices to show that $Y = \bigcap_{n \in \mathbb{N}} V_n$. Let $x \in Y$ and $n \in \mathbb{N}$; we want to show that $x \in V_n$. Let $U_1 \subseteq Y$ be an open set that contains x such that $\mathrm{diam}_{d_Y}(U_1) < \frac{1}{n}$. By the definition of the relative topology, there is an open set U_2 in X such that $U_2 \cap Y = U_1$. Let U_3 be an open subset of X that contains x such that $\mathrm{diam}_{d_X}(U_3) < \frac{1}{n}$. Then $U := U_2 \cap U_3$ satisfies all of the three conditions above. Hence, $x \in V_n$.

[2] The origin of the notation is G for the German word *Gebiet* and δ for intersection.

Conversely, suppose that $x \in \bigcap_{n\in\mathbb{N}} V_n$. Then for every $n \in \mathbb{N}$, there exists an open subset U_n of X that contains x and satisfies the three conditions given above. By the first condition, $U_n \cap Y \neq \emptyset$. Choose $x_n \in U_n \cap Y$. Then the second condition implies that $\lim_{n\to\infty} x_n = x$. The third condition implies that $(x_n)_{n\in\mathbb{N}}$ is Cauchy with respect to d_Y. Since d_Y is complete, $\lim_{n\to\infty} x_n = x \in Y$. □

REMARK 4.1.7. *The converse of Lemma 4.1.6 is true as well: every G_δ set is Polish. We do not need this statement in the following and therefore omit the proof.*

We finally state an important property of completely metrizable spaces.

THEOREM 4.1.8 (The Baire Category Theorem). *Every Polish[3] space S is* Baire*, i.e., has the property that countable intersections of dense open sets are dense.*

PROOF. Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of open dense sets. We want to show that $U := \bigcap U_n$ is dense. It is sufficient to show that any non-empty open set W in S contains an element of U. Since U_1 is dense, there is $x_1 \in U_1 \cap W$. Hence, there is an r_1 with $0 < r_1 < 1$ such that $\{z \in S \mid d(x_1, z) \leq r_1\} \subseteq U_1 \cap W$ where d is the compatible complete metric. We can continue recursively to find a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of S and a sequence $(r_n)_{n\in\mathbb{N}}$ of elements of $\mathbb{R}$ such that

$$\{z \in S \mid d(z, x_n) \leq r_n\} \subseteq U_n \cap B_{r_{n-1}}(x_{n-1}) \tag{4}$$

as follows: if we have defined $x_1, \dots, x_n$ and $r_1, \dots, r_n$ satisfying (4), then density of U_{n+1} guarantees that $B_{r_n}(x_n) \cap U_{n+1}$ is non-empty. Using the axiom of dependent choices, we may choose an element x_{n+1} from this set such that

$$\text{there is an } r_{n+1} \in \mathbb{R} \text{ with } \{z \in S \mid d(z, x_{n+1}) \leq r_{n+1}\} \subseteq U_{n+1} \cap B_{r_n}(x_n). \tag{5}$$

Instead of using this axiom we can instead use the assumption that there exists a countable set $\{u_1, u_2, \dots\}$ which is dense in S: we may then choose $x_{n+1} := u_k$ for k smallest possible so that (5) is satisfied. Since $x_m \in B_{r_n}(x_n)$ for all $m > n$, we have that $(x_n)_{n\in\mathbb{N}}$ is Cauchy, and hence converges to some limit x by the completeness of d. For any n, the set $\{z \in S \mid d(z, x_n) \leq r_n\}$ is closed and hence contains x. Therefore, $x \in W$ and $x \in U_n$ for all n. □

4.1.6. Metric Completions. Let (M_1, d_1) and (M_2, d_2) be two metric spaces. An *isometry* between (M_1, d_1) and (M_2, d_2) is a function $i \colon M_1 \to M_2$ such that $d_1(x_1, x_2) = d_2(i(x_1), i(x_2))$ (note that i must be injective, but it is not required to be surjective). Two metrics are called *isometric* if there exists a bijective isometry between them.

Metric spaces have the advantage that we can use Cauchy sequences to talk about points that aren't really there. More formally:

DEFINITION 4.1.9. *A* completion *of a metric space (M, d) is a complete metric space (M^*, d^*) together with an isometry $i \colon M \to M^*$ such that $i(M)$ is dense in M^*.*

PROPOSITION 4.1.10. *Every metric space has a completion.*

PROOF. Let (M, d) be a metric space. Let C be the collection of all Cauchy sequences in M. Define an equivalence relation $\sim$ on C by setting $(x_n) \sim (y_n)$ if $\lim_{n\to\infty} d(x_n, y_n) = 0$ for $(x_n), (y_n) \in C$. Define

- M^* to be the set of all equivalence classes of $\sim$,

$$X^* := \big\{[(x_n)] \mid (x_n) \in C\big\},$$

[3] Using the axiom of dependent choices (DC), this assumption can be weakened from Polish to completely metrisable (in fact, the modified statement is then equivalent to DC; see Appendix A.2).

- $d^*\colon X^* \times X^* \to \mathbb{R}$ by setting for $[(x_n)], [(y_n)] \in M^*$
$$d^*([(x_n)], [(y_n)]) := \lim_{n\to\infty} d(x_n, y_n),$$
- $i\colon M \to M^*$ by setting for $x \in M$
$$\phi(x) := [(x, x, \dots)].$$

Claim 1. d^* is well-defined. Let (x_n') and (y_n') be two Cauchy sequences such that $(x_n') \sim (x_n)$ and $(y_n') \sim (y_n)$. By the triangle inequality
$$d(x_n, y_n) \leq d(x_n, x_n') + d(x_n', y_n') + d(y_n', y_n)$$
and thus
$$|d(x_n, y_n) - d(x_n', y_n')| \leq |d(x_n, x_n') - d(y_n', y_n)|$$
which tends to 0 for $n \to \infty$, and proves that $\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x_n', y_n')$.

Claim 2. d^* is a metric on M^*. This is straightforward.

Claim 3. i is an isometry:
$$d^*(i(x), i(y)) = \lim_{n\to\infty} d(x, y) = d(x, y)\,.$$

Claim 4. $i(M)$ is dense in M^*. Let $[(x_n)] \in M^*$ and $\epsilon > 0$. Since (x_n) is Cauchy, there exists an $n_0 \in \mathbb{N}$ such that $d(x_m, x_n) < \epsilon$ for all $m, n \geq n_0$. For $z := i(x_{n_0})$ we have
$$d^*([(x_n)], i(z)) = \lim_{n\to\infty} d(x_n, x_{n_0}) \leq \epsilon\,.$$

Claim 5. d^* is complete. Let $[(x_n^1)], [(x_n^2)], \dots$ be Cauchy in (M^*, d^*). Diagonal argument: We define a function $k\colon \mathbb{N} \to \mathbb{N}$ as follows. Set $k(1) = 1$, and $k(2)$ such that $d(x^2_{k(2)}, x^2_l) < 1/2$ whenever $l \geq k(2)$. For $s \in \mathbb{N}$, choose $k(s)$ such that

- $N(s) \geq N(s-1)$
- $d(x^k_{k(s)}, x^k_{k(l)}) < 1/k$ whenever $l \geq k(s)$.

Then $(x^s_{k(s)})$ is a Cauchy sequence:
$$\begin{aligned} d(x^s_{k(s)}, x^t_{k(t)}) &\leq \limsup_{j\in\mathbb{N}} (d(x^s_{k(s)}, x^s_j) + d(x^s_j, x^t_j) + d(x^t_j, x^t_{k(t)})) \\ &\leq 1/s + 0 + 1/t \end{aligned}$$
which tends to 0 for $n, m \to \infty$.

Moreover, $\lim_{m\to\infty}[(x_n^m)] = [(x^n_{k(n)})]$: let $\epsilon > 0$ and choose $n_0 \in \mathbb{N}$ such that $1/n_0 < \epsilon/2$ and if $n, m \geq n_0$ then $d(x^n_{k(n)}, x^m_{k(m)}) < \epsilon/2$. Now, for $m \geq n_0$:
$$\begin{aligned} d^*([(x_n^m)], [(x^n_{k(n)})]) &= \lim_{n\to\infty} d(x_n^m, x^n_{k(n)}) \\ &= \limsup_{n\to\infty} d(x_n^m, x^m_{k(m)}) + \limsup_{n\to\infty} d(x^m_{k(m)}, x^n_{k(n)}) \\ &\leq 1/n_0 + \epsilon/2 < \epsilon \end{aligned}$$ □

We will in the following we refer to *the* completion of (M, d) because completions are essentially unique:

Proposition 4.1.11. *Let (M_1^*, d_1^*, i_1) and (M_2^*, d_2^*, i_2) be two completions of (M, d). Then there is a unique bijective isometry f between M_1^* and M_2^* such that $f \circ i_1 = i_2$.*

Proof. Let $x \in M_1^*$. Since $i_1(M)$ is dense in M_1^* for each $n \in \mathbb{N}$ there exists $x_n \in i_1(M)$ with $d_1^*(x_n, x) \leq \frac{1}{n}$. Let $y_n := i_2(i_1^{-1}(x_n))$. Since i_1 and i_2 are isometries, we have $d_2(y_n, y_m) = d_1(x_n, x_m)$ for all $m, n \in \mathbb{N}$. The sequence $(x_n)_{n\in\mathbb{N}}$ converges against x, so it is Cauchy, and it follows that $(y_n)_{n\in\mathbb{N}}$ is Cauchy, too. Since M_2^* is complete the sequence $(y_n)_{n\in\mathbb{N}}$ must converge to some $y \in M_2^*$.

Claim 1. The map $f\colon M_1^* \to M_2^*$ defined by $f(x) := y$ is well-defined. Suppose that $(x'_n)_{n\in\mathbb{N}}$ is another sequence of elements of M_1 that converges to x. For $n \in \mathbb{N}$, let $y'_n := i_2(i_1^{-1}(x'_n)$; we have to show that $\lim_{n\to\infty} y'_n = y$.

Let $\epsilon > 0$. Since $\lim_{n\in\mathbb{N}} y_n = y$ there exists $m \in \mathbb{N}$ such that $d_2^*(y_n, y) < \epsilon/2$ for all $n \geq m$. There is also a $k \in \mathbb{N}$ such that for all $n \geq k$ we have $d_1^*(x_n, x) < \epsilon/4$ and $d_1^*(x'_n, x) < \epsilon/4$. Hence, $d_1(x_n, x'_n) \leq d_1^*(x_n, x) + d_1^*(x, x'_n) < \epsilon/4 + \epsilon/4 = \epsilon/2$. Since i_1 and i_2 are isometries we have $d_2(y_n, y'_n) = d_1(x_n, x'_n)$. Hence, $d_2(y_n, y'_n) < \epsilon/2$ for all $n \geq k$. So for $n \geq max(k, m)$ we have $d_2^*(y'_n, y) \leq d_2(y'_n, y_n) + d_2(y_n, y) < \epsilon/2 + \epsilon/2 = \epsilon$, showing that $\lim_{n\to\infty} y'_n = y$.

Claim 2. f is an isometry. Let $x, x' \in M_1^*$ and let $(x_n)_{n\in\mathbb{N}}$ and $(x'_n)_{n\in\mathbb{N}}$ be sequences of elements of $i_1(M)$ that converge to x and x', respectively. Define $y_n := i_2(i_1^{-1}(x_n))$ and $y'_n := i_2(i_1^{-1}(x'_n))$, and we have seen that $\lim_{n\to\infty} y_n = f(x)$ and $\lim_{n\in\mathbb{N}} y'_n = f(x')$. Then

$$d_2^*(f(x), f(x')) = \lim_{n\in\mathbb{N}} d_2(y_n, y'_n) = \lim_{n\in\mathbb{N}} d_1(x_n, x'_n) = d(x, x').$$

Claim 3. f is surjective. Let $y \in M_2^*$. Since $i_2(M)$ is dense in M_2^* there is a a sequence $(y_n)_{n\in\mathbb{N}}$ of elements of $i_2(M)$ converging to y. Similarly as above it can be shown that $(i_1(i_2^{-1}(y_n)))_{n\in\mathbb{N}}$ is a sequence in $i(M_1)$ that converges to some $x \in M_1^*$, and that $f(x) = y$. □

If the isometry i of a metric completion (M^*, d^*) of (M, d) is not specified, we typically assume that $M \subseteq M^*$ and i is the identity. Clearly, the completion of a separable metric space is separable, too.

Example 56. $(\mathbb{R}; d)$ is the completion of $(\mathbb{Q}; d)$. △

Example 57. Let d be the ultrametric on $\mathrm{Sym}(\mathbb{N})$ from Example 52. Then the metric completion of $(\mathrm{Sym}(\mathbb{N}), d)$ equals the set of all injections from $\mathbb{N}$ to $\mathbb{N}$. △

4.1.7. Uniform continuity. Given metric spaces (X, d_1) and (Y, d_2), a function $f\colon X \to Y$ is called *uniformly continuous* if

$$\forall \epsilon > 0\, \exists \delta > 0\, \forall x, y \in X\, (d_1(x, y) < \delta \Rightarrow d_2(f(x), f(y)) < \epsilon).$$

For comparison: continuity of f with respect to the topologies induced by d_1 and d_2 only requires that

$$\forall \epsilon > 0,\, x \in X\, \exists \delta > 0\, \forall y \in X\, (d_1(x, y) < \delta \Rightarrow d_2(f(x), f(y)) < \epsilon).$$

Example 58. The function $x \mapsto x^2$ from $\mathbb{R} \to \mathbb{R}$ is continuous, but not uniformly continuous: given an arbitrarily small positive real ϵ, uniform continuity requires the existence of a positive number δ such that for all x_1, x_2 with $|x_1 - x_2| < \delta$ we have $|f(x_1) - f(x_2)| < \epsilon$. But $(x + \delta')^2 - x^2 = 2x\delta' + (\delta')^2$ is larger than ϵ for sufficiently large x. △

Example 59. An endomorphism ξ of the Baire space (Example 53) is uniformly continuous if for every finite $F \subseteq \mathbb{N}$ there exists a finite $G \subseteq \mathbb{N}$ such that for all $f, g \in \mathbb{N}^{\mathbb{N}}$ if $f|_G = g|_G$ then $\xi(f)|_F = \xi(g)|_F$.

For comparison: an endomorphism of the Baire space is continuous if and only if for every finite $F \subseteq \mathbb{N}$ and every $f \in \mathbb{N}^{\mathbb{N}}$ there exists a finite $G \subseteq \mathbb{N}$ such that if $g \in \mathbb{N}^{\mathbb{N}}$ is such that $f|_G = g|_F$ then $\xi(f)|_F = \xi(g)|_F$. △

PROPOSITION 4.1.12. *A uniformly continuous map f between metric spaces maps Cauchy sequences to Cauchy sequences.*

PROOF. Let $(s_n)_{n\in\mathbb{N}}$ be a Cauchy sequences, and let $\epsilon > 0$. By uniform continuity of f there exists $\delta > 0$ such that $d(f(x) - f(y)) < \epsilon$ for $d(x - y) < \delta$. Since s_n is Cauchy, there exists an $n_0 > 0$ such that $d(s_n - s_m) < \delta$ for all $n, m > n_0$. Hence, $d(f(s_n) - f(s_m)) < \epsilon$ for all $n, m > n_0$. Therefore, $(f(s_n))_{n\in\mathbb{N}}$ is Cauchy. $\square$

4.1.8. Compactness. A topological space S is called *compact* if for an arbitrary collection $\mathcal{U} = \{U_i\}_{i\in A}$ of open subsets of S with $S = \bigcup_{i\in A} U_i$ (also called an *open cover*) there is a finite subset B of A such that $S = \bigcup_{i\in B} U_i$ (the collection $\{U_i\}_{i\in B}$ is called a *subcover* of $\mathcal{U}$). Clearly, finite spaces are compact. We state some closure properties for compactness.

PROPOSITION 4.1.13. *Closed subsets of compact spaces are compact.*

PROOF. Let T be a compact space and let C be a closed subspace of T. Let $\mathcal{U}$ be an open cover of C. By assumption, $T \setminus C$ is open in T. Hence, $\mathcal{U} \cup \{T \setminus C\}$ is an open cover of T. As T is compact, there is a finite subcover of $\mathcal{U}$, say $\{U_1, U_2, \dots, U_r\}$. This also covers C by the fact that it covers T. If $T \setminus C$ is among $U_1, U_2, \dots, U_r$, then it can be removed and the remaining sets still cover C. Thus we have found a finite subcover of $\mathcal{U}$ which covers C, and hence C is compact. $\square$

The following is more substantial (actually, equivalent to the axiom of choice).

THEOREM 4.1.14 (Tychonoff). *Products of compact spaces are compact.*

PROOF. We only prove the statement for *countable* products of compact spaces; this is all that will be needed in this text anyway. We first show that if X and Y are compact, then so is $X \times Y$. Let $\mathcal{U}$ be a collection of open subsets of $X \times Y$ such that no finite subset of $\mathcal{U}$ covers $X \times Y$; we will show that $\mathcal{U}$ does not cover $X \times Y$.

Claim 1. There exists $x_0 \in X$ such that for every open $U \subseteq X$ that contains x_0 the set $U \times Y$ is not finitely covered by $\mathcal{U}$. Suppose otherwise that for every $x \in X$ there exists an open set $U_x \subseteq X$ that contains x such that $U \times Y$ is covered by finitely many elements of $\mathcal{U}$. By the compactness of X, finitely many of the U_x cover X, so finitely many sets of the form $U_x \times Y$ cover $X \times Y$, contradicting the assumptions.

Claim 2. There exists $y_0 \in Y$ such that for every open $U \subseteq X$ that contains x_0 and every open $V \subseteq Y$ that contains y_0 no finite subset of $\mathcal{U}$ covers $U \times V$. Otherwise, for every $y \in Y$ there is an open $U_y \subseteq X$ containing x_0 and an open $V_y \subseteq Y$ containing y such that $U_y \times V_y$ is covered by finitely many elements of $\mathcal{U}$. By the compactness of Y, there is a finite subset $F \subseteq Y$ such that $Y = \bigcup_{y\in F} V_y$. Set $U := \bigcap_{u\in F} U_y$. Then U is open and contains x_0, and

$$U \times Y = \bigcup_{y\in F} U \times V_y \subseteq \bigcup_{y\in F} U_y \times V_y$$

is covered by finitely many elements of $\mathcal{U}$, contradicting Claim 1.

It follows that no basic open set containing (x_0, y_0) is covered by finitely many elements of $\mathcal{U}$. In particular, no basic open set containing (x_0, y_0) can be contained in an element of $\mathcal{U}$, so (x_0, y_0) is not covered by $\mathcal{U}$. This finishes the proof that $X \times Y$ is compact. To prove the statement for countable products, we first slightly generalise the proof of Claim 2 to get the following.

Claim 3. Suppose that $\mathcal{U}$ is a family of open subsets of $X \times Y \times Z$ where Y is compact, and suppose that there is an $x_0 \in X$ such that for every open $U \subseteq X$ that contains x_0 the set $U \times Y \times Z$ is not covered by finitely many elements of $\mathcal{U}$. Then there exists an $y_0 \in Y$ such that for every open $U \subseteq X$ that contains x_0 and every open $V \subseteq Y$ that contains y_0, the set $U \times V \times Z$ is not finitely covered by $\mathcal{U}$.

We finally prove that if $X_1, X_2, \dots$ are compact, then $X = \prod_{i \in \mathbb{N}} X_i$ is compact. Let $\mathcal{U}$ be a family of open sets that that no finite subset of $\mathcal{U}$ covers X. We will construct an element $x = (x_1, x_2, \dots)$ of X that is not covered by $\mathcal{U}$. Note first that there is an $x_1 \in X_1$ such that for every open $U_1 \subseteq X_1$ that contains x_1 the set $U_1 \times X_2 \times X_2 \times \cdots$ is not finitely covered; the proof is as the proof of Claim 1, with $X_2 \times X_3 \times \cdots$ playing the role of Y. Next, we can find $x_2 \in X_2$ such that such that for every open $U_1 \subseteq X_1$ that contains x_1 and every open $U_2 \subseteq X_2$ that contains x_2 the set $U_1 \times U_2 \times X_3 \times X_4 \times \cdots$ is not covered by finitely many elements of $\mathcal{U}$; this follows from Claim 3 applied to $X_1 \times X_2 \times (X_3 \times X_4 \times \cdots)$. Continuing in this way, we inductively define $x_1, x_2, x_3, \dots$ such that for each n and all open $U_i \subseteq X_i$ for $i \leq n$ such that U_i contains x_i, the set $U_1 \times \cdots \times U_n \times X_{n+1} \times \cdots$ is not covered by finitely many elements of $\mathcal{U}$. The element $(x_1, x_2, \dots) \in X$ is then not covered by $\mathcal{U}$. $\square$

Exercises.

(99) Prove that a finite union of compact sets is compact.

In order to discuss which subsets of $\mathbb{R}$ and of $\mathrm{Sym}(\mathbb{N})$ are compact (with respect to the subspace topology), we need the following definition for metric spaces.

Definition 4.1.15. *A subset S of a metric space (M, d) is* bounded *if it is contained in an open ball of finite radius, i.e., if there exists $x \in M$ and a real $\epsilon > 0$ such that for all $s \in S$, we have $d(x, s) < \epsilon$.*

The open ball of radius ϵ and center x will be denoted by $B_x(\epsilon)$ in the following.

Example 60. A subset of $\mathbb{R}^d$ is compact if and only if it is closed and bounded – this is the theorem of Heine-Borel. $\triangle$

Which subsets of $\mathrm{Sym}(\mathbb{N})$ are compact?

Proposition 4.1.16. *Any compact subset S of a Hausdorff topological space X is closed in X.*

Proof. If S is compact but not closed then there exists $a \in \bar{S} \setminus S$. For each $x \in S$ there exists an open set U_x that contains x but does not intersect an open set V_x that contains a, because X is Hausdorff. Then $\mathcal{U} := \{U_x \mid x \in S\}$ is an open cover of S, and by compactness of S there exists a finite subcover $\{U_{x_1}, \dots, U_{x_n}\}$ of $\mathcal{U}$. But then $V := V_{x_1} \cap \cdots \cap V_{x_n}$ is open and contains a, and hence contains a point b in S since $a \in \bar{S}$. Since V is disjoint from each of $U_{x_1}, \dots, U_{x_n}$, we have $b \notin U_{x_1} \cup \cdots \cup U_{x_n}$, in contradiction to $\{U_{x_1}, \dots, U_{x_n}\}$ being a cover of S. $\square$

With compactness, we ask for much: in general topological spaces, we might in general even have open covers without *countable* subcovers! However, this can't happen if S is second-countable.

Proposition 4.1.17 (Lindelöf). *Let S be second-countable. Then every open cover of S has a* countable *subcover.*

Proof. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of S. Each U_α can be written as a union $\bigcup_{\beta \in J_\alpha} V^\alpha_\beta$ of basic open sets V^α_β. Then $\mathcal{V} := \{V^\alpha_i\}_{\alpha \in A, \beta \in J_\alpha}$ covers S. Since S is second-countable, there are only countably many basic open sets, so $\mathcal{V}$ be we written as $\{V_1, V_2, \dots\}$. By construction, for each $i \in \mathbb{N}$ there exists $\beta(i) \in A$ such that $V_i \subseteq U_{\beta(i)}$. Then $\{U_{\beta(i)}\}_{i \in \mathbb{N}}$ forms a countable subcover of $\mathcal{U}$. $\square$

The set S is *totally bounded* if for every real $\epsilon > 0$ there exists a finite collection of open balls in M of radius ϵ whose union contains S. Clearly, a totally bounded space is bounded, but the converse is not true: the discrete metric is bounded, but not totally bounded: for $\epsilon = 1/2$, we need infinitely many open ϵ-balls (points!) to cover the infinite set.

PROPOSITION 4.1.18. *If a metric space is totally bounded, then it is separable.*

PROOF. If X is totally bounded then for each positive $n \in \mathbb{N}$ there exists a finite $A_n \subseteq X$ such that $X = \bigcup_{x \in A_n} B_x(1/n)$. Let $A := \bigcup_{n \geq 0} A_n$. Clearly A is countable. We claim that $\bar{A} = X$. Let $x \in X$. For any $n \in \mathbb{N}$ there is some $y_n \in A_n$ such that $x \in B_{y_n}(1/n)$. This gives a sequence (y_n) with $d(x, y_n) < 1/n$. Thus $\lim y_n = x$ which proves the claim, and separability of X. $\square$

In the proof of the following theorem, we assume the axiom of countable choice (see Appendix A.2).

THEOREM 4.1.19. *For a metric space (X, d), the following are equivalent.*

(1) *X is compact;*
(2) *Every collection of closed sets in X with the* finite intersection property *(every finite subcollection has a nonempty intersection) has a nonempty intersection;*
(3) *If $F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$ is a decreasing sequence of nonempty closed sets in X, then $\bigcap_{n \geq 1} F_n$ is nonempty;*
(4) *X is* sequentially compact*, that is, every sequence in X has a convergent subsequence;*
(5) *X is totally bounded and complete.*

PROOF. (1) $\Rightarrow$ (2). Suppose that $\mathcal{C}$ is a collection of closed sets with empty intersection. Then $\mathcal{U} := \{X \setminus C \mid C \in \mathcal{C}\}$ is an open cover of X, and hence contains a finite subcover of X. The complements of the members of the subcover in X give the collection with the finite intersection property.

(2) $\Rightarrow$ (3). A decreasing sequence of non-empty closed sets obviously has the finite intersection property.

(3) $\Rightarrow$ (4). Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of points in X, and let F_n be the closure of the set $\{x_n, x_{n+1}, \ldots\}$. Then $F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$ and all the sets F_n are nonempty and closed. Therefore, by (3), the set $\bigcap_{n \geq 1} F_n$ contains at least one point a. Then $(x_n)_{n \in \mathbb{N}}$ contains a subsequence converging to a: to see this, let d be the compatible metric, and set $n_0 = 1$. Now suppose that n_k has already been defined for $k \in \mathbb{N}$. Since a is in the closure of $\{x_{n_k+1}, x_{n_k+2}, \ldots\}$ there exists an $n \in \{n_k + 1, n_k + 2, \ldots\}$ such that $d(x_n, a) < 1/(k+1)$. Let n_{k+1} the the smallest such n. Then $\lim_{k \to \infty} x_{n_k} = a$.

(4) $\Rightarrow$ (5). To prove that X is complete, let (x_n) be any Cauchy sequence in X. By (4), there is a subsequence converging to some point $a \in X$. But then the whole sequence (x_n) converges to a. This shows that X is complete.

Now suppose that X is not totally bounded, i.e., there exists a number $\epsilon > 0$ such that X has no finite covering by open balls of radius ϵ. Then we can define a sequence $(x_n)_{n \geq 1}$ of points in X having $d(x_i, x_j) \geq \epsilon$ for all $i \neq j$, by the following inductive construction: First let x_1 be any point in X. Then, supposing that $x_1, \ldots, x_{n-1}$ have been chosen, we know $B_{x_1}(\epsilon) \cup \cdots \cup B_{x_{n-1}}(\epsilon)$ is not the whole space. Hence we can choose a point x_n satisfying $d(x_i, x_n) \geq \epsilon$ for all $i < n$. On the other hand, the sequence (x_n) cannot have any convergent subsequence; for if it had a subsequence (x_{n_k}) converging to a, then there would exist an integer k_0 such that $d(x_{n_k}, a) < \epsilon/2$ for all $k \geq k_0$, and hence by the triangle inequality $d(x_{n_k}, x_{n_{k'}}) < \epsilon$ for all $k, k' \geq k_0$, contrary to the definition of the sequence (x_n).

(5) $\Rightarrow$ (4). Let $(x_i)_{i\in\mathbb{N}}$ be a sequence of elements from X. Let $S = \{x_n \mid n \in \mathbb{N}\}$. If S is finite then the statement is trivial so assume that S is infinite. Since X is totally bounded, there exists a finite cover of X with open balls of radius $\epsilon_1 := 1$. One of those balls, call it B_1, must contain infinitely many elements from S. Again by total boundedness, there exists a finite cover of X with open balls of radius $\epsilon_2 := 1/2$, and again, one of those balls must contain infinitely many elements from $B_1 \cap S$; this ball we call B_2. We continue this process, producing a sequence of balls (B_k) of radius $1/k$ so that $B_k \cap B_{k-1} \cap \cdots \cap B_1$ contains infinitely many elements of the sequence x_i. Pick now indices $n_1 < n_2 < n_3 < \cdots$ such that $y_k := x_{n_k} \in B_k$. It is easy to see that (y_i) is Cauchy and so by the completeness assumption on X it must have a convergent subsequence.

(4) $\Rightarrow$ (1). Let $\mathcal{U}$ be an open cover of X. From the implication (4) $\Rightarrow$ (5) we have that X is totally bounded, and Proposition 4.1.18 implies that X is second-countable. By Proposition 4.1.17 (Lindelöf) we can therefore assume that $\mathcal{U}$ is countable, $\mathcal{U} = \{U_1, U_2, \dots\}$. Suppose for contradiction that $\mathcal{U}$ does not have a finite subcover. Pick $x_n \in X \setminus (U_1 \cup \cdots \cup U_n)$ arbitrarily. Then by assumption, the sequence (x_n) has a subsequence (y_n) which converges to some $y_0 \in X$. Since $\mathcal{U}$ is a cover of X there is some $m \in \mathbb{N}$ with $y \in U_m$. But then $y_j \notin U_m$ for all $j \geq m$, which is a contradiction. $\square$

We mention that sequential compactness and compactness are not equivalent in general topological spaces; for example, $[0,1]^{\mathbb{R}}$ is compact by Theorem 4.1.14, but it can be shown that it is not sequentially compact.

Local compactness. A topological space S is called *locally compact* if every $p \in S$ is contained in an open set which is itself contained in a compact subset of S. Clearly, every compact space S is also locally compact (take S itself as compact open set that contains p).

Example 61. $\mathbb{R}$ is locally compact, but not compact. $\triangle$

Example 62. The discrete space on S is locally compact, but only compact if S is finite. $\triangle$

4.2. Topological Groups

A *topological group* is an (abstract) group $\underline{G}$ together with a topology on the elements G of $\underline{G}$ such that $(x, y) \mapsto xy$ is continuous from G^2 to G and $x \mapsto x^{-1}$ is continuous from G to G. Two topological groups are said to be *isomorphic* if the groups are isomorphic, and the isomorphism is a homeomorphism between the respective topologies.

Example 63. The groups $(\mathbb{R}, +)$ and $(\mathbb{Q}, +)$ are topological groups with respect to their standard topology. (Why?) $\triangle$

Example 64. The elements of the group $\underline{G} := \mathrm{Sym}(\mathbb{N})$ form a (non-closed!) subset of the Baire space $\mathbb{N}^{\mathbb{N}}$ (Example 48), and the topology induced by the Baire space on $\mathrm{Sym}(\mathbb{N})$ is also called the topology of pointwise convergence. Observe that a set of permutations of a set X is a closed subset of $\mathrm{Sym}(X)$ if and only if it is locally closed as defined in Proposition 1.2.2.

Composition is continuous as a map from $G^2 \to G$. If $U \subseteq G$ is a basic open set $S(\bar{a}, \bar{c})$ for $\bar{a}, \bar{c} \in \mathbb{N}^n$ (we use the terminology from Example 48), then the preimage of

U is

$$\{(f,h) \in G^2 \mid f \circ h \in S(\bar{a},\bar{c})\} = \{(f,h) \in G^2 \mid \exists \bar{b}\,(h \in S(\bar{a},\bar{b}) \text{ and } f \in S(\bar{b},\bar{c}))\}$$
$$= \bigcup_{\bar{b} \in \mathbb{N}^n} \big(S(\bar{b},\bar{c}) \times S(\bar{a},\bar{b})\big)$$

which is open as a union of open sets. The preimage of $S(\bar{a},\bar{b})$ under the inverse map is $S(\bar{b},\bar{a})$, which is open, too. $\triangle$

Proposition 4.2.1. *Let $\underline{G}$ be a topological group, $g \in G$, and $U \subseteq G$ open. Then gU is open, too. If U is an open subgroup, then it is also closed.*

Proof. As a consequence of Proposition 4.1.2, for every $g \in G$ the function $t_g \colon G \to G$ defined by $t_g(x) := gx$ is continuous. The pre-image of U under the function $t_{g^{-1}}$ is gU. Therefore, this set is open as the pre-image of an open set under a continuous function. The second part follows since the complement of U in G equals $\bigcup_{g \in G \setminus U} gU$, a union of open sets, hence open. $\square$

Remark 4.2.2. *Proposition 4.2.1 also implies that the topology on G is given by a basis at $1^{\underline{G}}$: if $\mathcal{B}$ is a basis of open sets at the identity, and $g \in G$, then $\{gU \mid U \in \mathcal{B}\}$ is a basis at g.*

Exercises.

(100) Let $\underline{G}$ be a topological group and let $A, B \subseteq G$. If A is open, then so is $AB := \{ab \mid a \in A, b \in B\}$.

(101) Show that a group $\underline{G}$ with a topology on G is a topological group if and only if the map $(x,y) \mapsto xy^{-1}$ is continuous from G^2 to G.

(102) Show that for all $n \in \mathbb{N}$ the groups $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{GL}(n,\mathbb{C})$ of invertible real or complex matrices are topological groups with respect to the standard topology.

A topological group $\underline{G}$ is *Hausdorff (first-countable, metrizable, Polish)* if the topology of $\underline{G}$ is Hausdorff (first-countable, metrizable, Polish, respectively). Note that G is first-countable if and only if G has a countable basis at the identity (see Remark 4.2.2).

4.2.1. Continuous group actions. Recall from Section 1.3 that an action of a group $\underline{G}$ on a set S is a homomorphism from $\underline{G}$ to $\mathrm{Sym}(S)$.

Definition 4.2.3. *An action ξ of a topological group $\underline{G}$ on a topological space S is called* continuous *if $(g,s) \mapsto \xi(g)(s)$ is continuous as a map from $G \times S \to S$.*

Example 65. Recall the faithful action of $\underline{G}$ on G by left multiplication from the proof of Cayley's theorem, Theorem 1.3.3). This is the special case of Example 12 where $\underline{H} = \{1\}$. This action is continuous since it equals the group composition which is continuous by definition. $\triangle$

If S is a topological space, then $\mathrm{Homeo}(S) \subseteq \mathrm{Sym}(S)$ denotes the set of all homeomorphisms of S. We view $\mathrm{Homeo}(S)$ as a topological space with the subspace topology inherited from S^S which carries the product topology[4].

[4]Note that it is not clear (and depends on S) whether $\mathrm{Homeo}(S)$ with this topology is a topological group.

Proposition 4.2.4. *Every continuous action of a topological group $\underline{G}$ on a topological space S is a continuous homomorphism from $\underline{G}$ into* $\mathrm{Homeo}(S)$.

Proof. Suppose that $\xi\colon \underline{G} \to \mathrm{Sym}(S)$ is a continuous action of $\underline{G}$ on S, so the map $\chi(g,s) := \xi(g)(s)$ is continuous from $G \times S$ to S. For every $g \in G$, the map t_g defined by $s \mapsto \chi(g,s)$ is continuous. The inverse of t_g is $s \mapsto \chi(g^{-1},s)$, which is also continuous. Hence, t_g is a homeomorphism. To show that ξ is continuous, let U be a basic open subset of $\mathrm{Homeo}(S)$, i.e., $U = \prod_{s \in S} U_s$ where U_s is open in S for all $s \in S$, and there exists a finite set F such that $U_s = S$ for all $s \in S \setminus F$. Note that for fixed s, the map $t_s\colon G \to S$ given by $g \mapsto \xi(g)(s)$ is continuous, and hence for all $s \in F$ the set $\{g \in G \mid \xi(g)(s) \in U_s\}$ is open. Therefore,

$$\begin{aligned}\xi^{-1}(U) &= \{g \in G \mid \xi(g)(s) \in U_s \text{ for all } s \in F\}\\ &= \bigcap_{s \in F} \{g \in G \mid \xi(g)(s) \in U_s\}\end{aligned}$$

is a finite intersection of open sets and hence open. □

If S carries the discrete topology (in which case $\mathrm{Homeo}(S) = \mathrm{Sym}(S)$), the statement of Proposition 4.2.4 can be strengthened to obtain an equivalent characterisation of continuity of actions.

Lemma 4.2.5. *Let $\underline{G}$ be a topological group and ξ an action of $\underline{G}$ on a set S equipped with the discrete topology. Then ξ is continuous if and only if ξ is continuous as a map from $\underline{G}$ to* $\mathrm{Sym}(S)$.

Proof. The forward implication follows from Proposition 4.2.4. For the converse implication, we show that the function $\chi\colon G \times S \to S$ given by $(g,s) \mapsto \xi(g)(s)$ is continuous. Since S is discrete, it suffices to show that for every $s' \in S$ there exists an open $U \subseteq G$ and an open $T \subseteq S$ such that $\chi(U,T)$ contains s' and is contained in $\chi^{-1}(\{s'\})$, because then $\chi^{-1}(\{s'\})$ is a union of open sets $U \times T$. Let $g \in G$ and $s \in S$ be such that $\chi(g,s) = s'$. Since S is discrete, in particular $T := \{s\}$ is open. Let $U := \xi^{-1}(S(s,s'))$ which is by assumption an open subset of G and contains g. Then $\chi(U,T) = \{\chi(u,s) \mid u \in S(s,s')\} = \{s'\}$. □

An important example of a continuous action is the action by conjugation from Example 13.

Example 66. Let $\underline{G}$ be a topological group. Then the conjugation action $\xi\colon G \to G$ given by $\xi(g)(h) := ghg^{-1}$ is continuous since composition and inverse in a topological group are continuous. △

Further important examples of continuous actions of a topological group arise from the action by left translation (Example 12). We also view G/H as a topological space, with the *quotient topology.* We first define $p\colon G \to G/H$ by setting $p(g) = gH$ (the projection map). Define $U \subseteq G/H$ to be open if and only if $p^{-1}(U)$ is open in G (in this way, p will necessarily be continuous). In other words, a set of left-cosets is open in G/H if and only if their union is open in G.

Proposition 4.2.6. *Let $\underline{H}$ be an open subgroup of a topological group $\underline{G}$. Then the action of $\underline{G}$ on G/H by left translation is continuous.*

Proof. Let ξ be the action of $\underline{G}$ on G/H by left translation. Let $S \subseteq G/H$ be open, and let $gH \in S$. It suffices to show that there are open subsets $U \subseteq G$ and $T \subseteq G/H$ such that $gH \in \{\xi(u)(t) \mid u \in U, t \in T\} \subseteq S$. By the definition of the quotient topology $p^{-1}(S)$ is open in G. Since composition in G is continuous, the set $\{(g_1,g_2) \in G^2 \mid g_1g_2 \in p^{-1}(S)\}$ is open in G^2. This set contains $(1,g)$, because

$p(1g) = gH \in S$. So there exists an open $U \subseteq G$ containing 1 and an open $V \subseteq G$ containing g such that $\{uv \mid u \in U, v \in V\} \subseteq p^{-1}(S)$. Then $T := p(V) = \{vH \mid v \in V\}$ is open in G/H, and

$$gH \in \{\xi(u)(t) \mid u \in U, t \in T\} = \{uvH \mid u \in U, v \in V\} \subseteq S. \qquad \square$$

Also, if $\underline{G} \leq \mathrm{Sym}(X)$ then for every $n \in \mathbb{N}$ the componentwise action of $\underline{G}$ on X^n and the setwise action of $\underline{G}$ on $\binom{X}{n}$ are continuous. A general result about the continuous actions of a permutation group is Theorem 5.2.1 in Chapter 5. We present an example of a discontinuous group action of an oligomorphic permutation group.

EXAMPLE 67. Let $\mathcal{K}$ be the class of all finite structures $(A; E_0, E_1, \dots)$ where E_i denotes an equivalence relation on $A^i_{\neq}$ with at most two equivalence classes. Clearly, $\mathcal{K}$ is closed under substructures and isomorphism, and countable up to isomorphism. It is easy to verify that it also has the amalgamation property (Section 3.3). Let $\underline{A}$ be the Fraïssé-limit of $\mathcal{K}$. Then $\mathrm{Aut}(\underline{A})$ has a continuous homomorphism ξ_1 to $(\mathbb{Z}_2)^{\mathbb{N}}$ (which is equipped with the product topology for $\mathbb{Z}_2$ discrete): for $\alpha \in \mathrm{Aut}(\underline{A})$ we define $\xi_1(\alpha) = (\alpha_i)_{i \in \mathbb{N}}$ where $\alpha_i = 0$ if α fixes the equivalence classes of E_i and $\alpha_i = 1$ otherwise. This map is clearly a group homomorphism, and it is continuous. Indeed, if U is an open subset of $(\mathbb{Z}_2)^{\mathbb{N}}$, then it is a union of sets of the form $U = \{u \in (\mathbb{Z}_2)^{\mathbb{N}} \mid u_1 = x_1, \dots, u_n = x_n\}$ for some $n \in \mathbb{N}$ and $x \in (\mathbb{Z}_2)^n$. Then

$$\begin{aligned}\xi_1^{-1}(U) = & \bigcap_{i \in \{1,\dots,n\}, x_i = 0} \left(\bigcup_{a,b \in A^i_{\neq}, E_i(a,b)} \mathrm{Aut}(\underline{A}) \cap S(a,b) \right) \\ & \cap \bigcap_{i \in \{1,\dots,n\}, x_i = 1} \left(\bigcup_{a,b \in A^i_{\neq}, \neg E_i(a,b)} \mathrm{Aut}(\underline{A}) \cap S(a,b) \right)\end{aligned}$$

is open as a finite intersection of unions of open sets in $\mathrm{Aut}(\underline{A})$.

To construct a discontinuous group homomorphism, let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ (Appendix A.1), and let $\xi_2 \colon (\mathbb{Z}_2)^{\mathbb{N}} \to \mathbb{Z}_2$ be the function given by

$$(\alpha_i)_{i \in \mathbb{N}} := \begin{cases} 0 & \text{if } \{i \in \mathbb{N} \mid \alpha_i = 0\} \in \mathcal{U} \\ 1 & \text{otherwise.} \end{cases}$$

Claim 1. ξ_2 is a group homomorphism: we have

$\xi_2(\alpha + \beta) = 0$
$\Leftrightarrow \{i \mid \alpha_i + \beta_i = 0\} \in \mathcal{U}$
$\Leftrightarrow \{i \mid \alpha_i = \beta_i = 0\} \cup \{i \mid \alpha_i = \beta_i = 1\} \in \mathcal{U}$
$\Leftrightarrow \{i \mid \alpha_i = \beta_i = 0\} \in \mathcal{U}$ or $\{i \mid \alpha_i = \beta_i = 1\} \in \mathcal{U}$
$\Leftrightarrow (\{i \mid \alpha_i = 0\} \in \mathcal{U} \wedge \{i \mid \beta_i = 0\} \in \mathcal{U}) \vee (\{i \mid \alpha_i = 1\} \in \mathcal{U} \wedge \{i \mid \beta_i = 1\} \in \mathcal{U})$
$\Leftrightarrow \xi_2(\alpha) + \xi_2(\beta) = 0.$

Claim 2. ξ_2 is continuous if and only if $\mathcal{U}$ is principal. If there exists $a \in \mathbb{N}$ such that $\mathcal{U} = \{Y \subseteq \mathbb{N} \mid a \in Y\}$, then

$$\begin{aligned}\xi_2^{-1}(0) &= \{\alpha \in (\mathbb{Z}_2)^{\mathbb{N}} \mid \{i \in \mathbb{N} \mid \alpha_i = 0\} \in \mathcal{U}\} \\ &= \{\alpha \in (\mathbb{Z}_2)^{\mathbb{N}} \mid \alpha_a = 0\}\end{aligned}$$

is a basic open set in $(\mathbb{Z}_2)^{\mathbb{N}}$. We have the analogous statement for $\xi_2^{-1}(1)$, which proves the continuity of ξ_2. Conversely, if $\xi_2^{-1}(0)$ is open, then it is a union of sets of the form $U = \{\alpha \in (\mathbb{Z}_2)^{\mathbb{N}} \mid \alpha_{a_1} = b_1, \dots, \alpha_{a_k} = b_k\}$ for some $k \in \mathbb{N}$, $a_1, \dots, a_k \in \mathbb{N}$ and $b_1, \dots, b_k \in \mathbb{Z}_2$. If $\xi_2^{-1}(0)$ contains U, then $\{a_i \mid i \in \{1, \dots, k\}, b_i = 0\} \in \mathcal{U}$.

Hence, there exists $i \in \{1, \dots, k\}$ such that $\{a_i\} \in \mathcal{U}$ (see Lemma A.1.3) and $\mathcal{U}$ is principal.

In the same way it can be shown that for a non-principal ultrafilter $\mathcal{U}$, the map $\xi_2 \circ \xi_1$ is a discontinuous group homomorphism from an oligomorphic permutation group to $\mathbb{Z}_2$. $\triangle$

Proposition 4.2.7 (Proposition 13 and Proposition 14 in [**35**]). *Let $\underline{G}$ be a topological group, and let $\underline{H}$ be a subgroup of $\underline{G}$. Then*

- *H is open in G if and only if G/H is discrete;*
- *H is closed in G if and only if G/H is Hausdorff.*

Proof. Part 1: G/H is discrete if each left coset gH is open, which is the case if and only if H is open.

Part 2: If G/H is Hausdorff, then every coset of $\underline{H}$ that is distinct from H is contained in an open set O that does not intersect H. The union of all those open sets is open, and defines the complement of H in G/H. Hence, H is closed.

Conversely, suppose that H is closed. Then the equivalence relation

$$R := \{(x, y) \mid x^{-1}y \in H\} = \bigcup_{g \in G} (gH)^2$$

on G is closed in $G \times G$, since it is the inverse image of H under the continuous mapping $(x, y) \mapsto x^{-1}y$. Hence, $\bigcup_{g \in G} \{gH\}^2$ is closed in G/H, and

$$U := (G/H)^2 \setminus \bigcup_{g \in G} \{gH\}^2$$

is open in G/H. Let $g_1H, g_2H \in G/H$ be distinct, that is, $(g_1, g_2) \notin R$. By the definition of the product topology there exist open sets U_1, U_2 of G/H such that $\{g_1H\} \times \{g_2H\} \subseteq U_1 \times U_2 \subseteq U$. The sets U_1 and U_2 are disjoint: otherwise, if $gH \in U_1 \cap U_2$ for some $g \in G$, then $\{gH\} \times \{gH\} \in U$. So $(g, g) \in G^2 \setminus R$, a contradiction to the fact that R contains (g, g) for all $x \in G$. This proves that G/H is Hausdorff. $\square$

Example 68 (The Logic Action). Let τ be a relational signature, and let $\mathcal{C}$ be a class of finite τ-structures which is closed under substructures and isomorphisms, has the JEP, and contains only finitely many non-isomorphic structures with n elements for each $n \in \mathbb{N}$. Let $X_{\mathcal{C}}$ be the space of all structures with domain $\mathbb{N}$ whose age is contained in $\mathcal{C}$. The basic open sets in $X_{\mathcal{C}}$ are given by elements $\underline{A}$ of $\mathcal{C}$ together with a map $\alpha \colon A \to \mathbb{N}$ as follows:

$$\{\underline{B} \in X_{\mathcal{C}} \mid \alpha \colon \underline{A} \hookrightarrow \underline{B} \text{ is an embedding}\}.$$

Note that this topology is compact because it is a closed subset of a product of finite spaces (TODO: explain more, Proposition 4.1.13, Theorem 4.1.14).

We now define the so-called *logic action* of $\mathrm{Sym}(\mathbb{N})$ on $X_{\mathcal{C}}$: for $g \in \mathrm{Sym}(\mathbb{N})$ and $\underline{B} \in X_{\mathcal{C}}$, define $g(\underline{B})$ to be the unique structure $\underline{B}'$ in $X_{\mathcal{C}}$ such that g is an isomorphism between $\underline{B}$ and $\underline{B}'$. Clearly, this action is continuous. $\triangle$

Exercises.

(103) Show that the componentwise action of $\underline{G} \leq \mathrm{Sym}(X)$ on X^n and the setwise action of $\underline{G}$ on $\binom{X}{n}$ are continuous.

4.2.2. Topologically faithful actions. Recall that an action on S is called *faithful* if it is an injective homomorphism to $\mathrm{Sym}(S)$. A faithful action of a closed subgroup of $\mathrm{Sym}(\mathbb{N})$ on a discrete space S is called *topologically faithful* if it is continuous and its image is closed in $\mathrm{Sym}(S)$.

EXAMPLE 69. Let $\underline{H}$ be an open subgroup of $\underline{G}$. By Proposition 4.2.7 the quotient space G/H is discrete, so in particular the image of the action of $\underline{G}$ on G/H by left translation is closed. △

We also give an example of a continuous injective homomorphism from a closed subgroup $\underline{G}$ of $\mathrm{Sym}(\mathbb{N})$ to $\mathrm{Sym}(\mathbb{N})$ whose image is not closed. That is, we have a faithful continuous action of $\underline{G}$ which is not topologically faithful; $\underline{G}$ is even oligomorphic.

EXAMPLE 70. This example is due to Dugald Macpherson and can be found in Hodges' model theory [**72**] (on page 354). Let $\underline{Q}$ be the structure $(\mathbb{Q}; <, P)$ where

- $<$ is the usual strict order of the rational numbers, and
- $P \subseteq \mathbb{Q}$ is such that both P and $O := \mathbb{Q} \setminus P$ are dense in $(\mathbb{Q}; <)$.

Let $\underline{P}$ be the substructure induced by P in $\mathbb{Q}$. It is easy to see (and follows from more general principles that will be presented in Corollary 5.3.3) that the mapping which sends $f \in \mathrm{Aut}(\underline{Q})$ to $f|_P$ induces a continuous homomorphism ξ from $\mathrm{Aut}(\underline{Q})$ to $\mathrm{Aut}(\underline{P})$ whose image is dense in $\mathrm{Aut}(\underline{P})$. We claim that ξ is not surjective. To prove this, we consider *Dedekind cuts* (S,T) *of* $\underline{P}$, that is, partitions of P into subsets S, T with the property that for all $s \in S$ and $t \in T$ we have $s < t$. Note that for each element $o \in O$ we obtain a Dedekind cut (S,T) with $S := \{a \in P \mid a < o\}$ and $T := \{a \in P \mid a > o\}$. But since there are uncountably many Dedekind cuts (they are in bijection with the real numbers, see 56) and only countably many elements of O, there also exists a Dedekind cut (S', T') which is not of this form. By a standard back-and-forth argument, there exists an $\alpha \in \mathrm{Aut}((P,<))$ that maps S to S' and T to T'. Suppose for contradiction that there is $\beta \in \mathrm{Aut}(\underline{Q})$ with $\beta|_P = \alpha$. Then $s < \beta(o) < t$ for all $s \in S'$, $t \in T'$, in contradiction to the assumptions on (S', T'). △

Some other groups have the remarkable property that *every* faithful continuous action is topologically faithful; this is for example known for $\mathrm{Sym}(\mathbb{N})$ (due to [**61**]; see Theorem 1.3 in [**162**] for a more recent and more powerful result in this context).

4.2.3. Metrics on topological groups. The (ultra-) metric d on $\mathrm{Sym}(D)$ from Example 52 is *left-invariant*, i.e., $d(gh_1, gh_2) = d(h_1, h_2)$ for all $g, h_1, h_2 \in G$, because if $n \in \mathbb{N}$ is smallest such that $h_1(n) \neq h_2(n)$, then n is also smallest such that $g(h_1(n)) \neq g(h_2(n))$.

THEOREM 4.2.8 (Birkhoff, Kakutani; see Theorem 9.1 in [**88**]). *A topological group G is metrisable if and only if G is Hausdorff and first-countable. Every metrisable topological group has a compatible left-invariant metric.*

LEMMA 4.2.9. *Let G be a group with a left-invariant metric d. Then for all $g, h \in G$*

$$d(gh, 1^{\underline{G}}) \leq d(g, 1^{\underline{G}}) + d(h, 1^{\underline{G}}).$$

PROOF. $d(gh, 1^{\underline{G}}) = d(h, g^{-1}) \leq d(h, 1^{\underline{G}}) + d(1^{\underline{G}}, g^{-1}) = d(g, 1^{\underline{G}}) + d(h, 1^{\underline{G}})$. □

PROPOSITION 4.2.10. *Let $\xi \colon \underline{G} \to \underline{H}$ be a continuous homomorphism between topological groups with compatible left-invariant metrics d_1 and d_2. Then f is uniformly continuous.*

PROOF. Let $\epsilon > 0$. Since ξ is continuous, there exists a $\delta > 0$ such that for all $g \in G$ with $d_1(1^{\underline{G}}, g) < \delta$ we have $d_2(1^{\underline{H}}, \xi(g)) < \epsilon$. Let $g_1, g_2 \in G$ be such that $d_1(g_1, g_2) < \delta$. Then $d_1(1^{\underline{G}}, g_1^{-1}g_2) < \delta$, and hence

$$d_2(\xi(g_1), \xi(g_2)) = d_2(1^{\underline{H}}, \xi(g_1)^{-1}\xi(g_2)) = d_2(1^{\underline{H}}, \xi(g_1^{-1}g_2)) < \epsilon$$

which shows uniform continuity of ξ. □

We have seen in Example 52 an example of a left-invariant metric d on $\mathrm{Sym}(D)$ which is not complete.

LEMMA 4.2.11. *Let G be a topological group with a compatible left-invariant metric d. Then*

$$d'(g, h) := d(g, h) + d(g^{-1}, h^{-1})$$

is a compatible metric, too.

PROOF. Clearly, d' is non-negative, indiscernible, symmetric, and subadditive. We have to show that d' induces the same topology on G as d. Let $\epsilon > 0$. The set $S := \{g \in G \mid d(1, g) < \epsilon\}$ is open with respect to d and contains the identity $1 \in G$. This set contains the set $S' := \{g \in G \mid d'(1, g) < \epsilon\}$ which is open with respect to d' and also contains 1. Conversely, the set S' contains the set $\{g \in G \mid d(1, g) < \epsilon/2\}$ (which is open with respect to d and contains 1): if g is such that $d(1, g) < \epsilon/2$, the by the left-invariance of d we have that $d(g^{-1}, 1) < \epsilon/2$, and hence

$$d'(1, g) \leq d(1, g) + d(1, g^{-1}) < \epsilon/2 + \epsilon/2 < \epsilon,$$

so $g \in S_2$. Since the topology on G is given by a basis of open sets at 1, the statement follows. □

LEMMA 4.2.12. *Let $\underline{G}$ be a topological group, let d be a compatible left-invariant metric, and let d' be the compatible metric defined in Lemma 4.2.11. Let $(g_i)_{i\in\mathbb{N}}$ and $(h_i)_{i\in\mathbb{N}}$ be Cauchy sequences in (G, d'). Then $(g_i^{-1}h_i)_{i\in\mathbb{N}}$ is Cauchy in (G, d) and in (G, d').*

PROOF. Let $\epsilon > 0$. Then there exists an $n_0 \in \mathbb{N}$ such that

$$d'(h_n, h_m) = d'(h_m^{-1}h_n, 1^{\underline{G}}) < \epsilon/3 \tag{6}$$

for all $n, m \geq n_0$. By the continuity of the multiplication operation and since d is a compatible metric there exists a $\delta > 0$ such that for all $k \in G$ with $d(k, 1^{\underline{G}}) < \delta$

$$d(h_{n_0}^{-1}kh_{n_0}, 1^{\underline{G}}) < \epsilon/3.$$

Let $n_1 \geq n_0$ be such that $d'(g_n, g_m) < \delta$ for all $n, m \geq n_1$. Then for all $n, m \geq n_1$

$$d(g_m g_n^{-1}, 1^{\underline{G}}) = d(g_n^{-1}, g_m^{-1}) \leq d'(g_n, g_m) < \delta$$

and hence

$$d(h_{n_0}^{-1}g_m g_n^{-1}h_{n_0}, 1^{\underline{G}}) < \epsilon/3. \tag{7}$$

Therefore

$$\begin{aligned}
&d(g_n^{-1}h_n, g_m^{-1}h_m) \\
&= d(h_m^{-1}g_m g_n^{-1}h_n, 1^{\underline{G}}) \\
&\leq d(h_m^{-1}h_{n_0}, 1^{\underline{G}}) + d(h_{n_0}^{-1}g_m g_n^{-1}h_{n_0}, 1^{\underline{G}}) + d(h_{n_0}^{-1}h_n, 1^{\underline{G}}) && \text{(Lemma 4.2.9)} \\
&\leq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon && \text{(by (6), (7)), and (6))}
\end{aligned}$$

which proves that $(g_i^{-1}h_i)_{i\in\mathbb{N}}$ is Cauchy in (G, d). Note that by symmetry also the sequence $(h_i^{-1}g_i)_{i\in\mathbb{N}}$ is Cauchy in (G, d), and it follows that both sequences are also Cauchy in (G, d'). □

LEMMA 4.2.13. *Let G be a topological group with a compatible left-invariant metric d, and let d' be the metric defined in Lemma 4.2.11. Let (G^*, d^*) be the metric completion of (G, d'). Then the group multiplication can be extended uniquely to G^* such that G^* becomes a topological group with the compatible complete metric d^*.*

PROOF. To define an extension of the multiplication operation of G to G^*, pick representatives $(g_n)_{n\in\mathbb{N}}$ and $(h_n)_{n\in\mathbb{N}}$ of elements of G^* (recall our construction of G^* in Proposition 4.1.10). Then $(g_n h_n)_{n\in\mathbb{N}}$ is Cauchy with respect to d' by Lemma 4.2.12. Define

$$[(g_n)_{n\in\mathbb{N}}] \cdot [(h_n)_{n\in\mathbb{N}}] := [(g_n h_n)_{n\in\mathbb{N}}].$$

To show that this is well-defined, let $(g'_n)_{n\in\mathbb{N}}$ and $(h'_n)_{n\in\mathbb{N}}$ be Cauchy sequences in (G, d') such that $\lim_{n\in\mathbb{N}} d(g_n, g'_n) = 0$ and $\lim_{n\in\mathbb{N}} d(h_n, h'_n) = 0$.

Let $\epsilon > 0$. We will show that $d(g'_n h'_n, g_n h_n) \leq \epsilon$. Since $(h_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, there exists $n_0 \in \mathbb{N}$ such that $d(h_n, h_m) < \epsilon/3$ for all $n, m \geq n_0$. By the continuity of the multiplication operation there exists a $\delta > 0$ such that for all $k \in G$ with $d(k, 1^{\underline{G}}) < \delta$

$$d(h_{n_0}^{-1} k h_{n_0}, 1^{\underline{G}}) < \epsilon/3.$$

Let $n_1 \geq n_0$ be such that $d'(g'_n, g_n) < \delta$ for all $n \geq n_1$. Then for $n \geq n_1$ and by Lemma 4.2.9

$$\begin{aligned} d(g'_n h'_n, g_n h_n) &= d(h_n^{-1} g_n^{-1} g'_n h'_n, 1^{\underline{G}}) \\ &\leq d(h_n^{-1} h_{n_0}, 1^{\underline{G}}) + d(h_{n_0}^{-1} g_n^{-1} g'_n h_{n_0}, 1^{\underline{G}}) + d(h_{n_0}^{-1} h'_n, 1^{\underline{G}}) \\ &\leq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

This shows that $[(g'_n h'_n)_{n\in\mathbb{N}}] = [(g_n h_n)_{n\in\mathbb{N}}]$ and hence the multiplication on G^* is indeed well-defined.

The multiplication operation defined on G^* is associative and has the neutral element $[(1^{\underline{G}})_{n\in\mathbb{N}}]$. The inverse of $[(g_n)_{n\in\mathbb{N}}]$ is $[(g_n^{-1})_{n\in\mathbb{N}}]$ (Lemma 4.2.12 implies that $(g_n^{-1})_{n\in\mathbb{N}}$ is Cauchy). We use Proposition 4.1.1 to verify that the multiplication and taking inverses in G^* is continuous with respect to the topology induced by d^*: if $\lim_{m\to\infty} \lim_{n\to\infty} g_{n,m} = g$ and $\lim_{m\to\infty} \lim_{n\to\infty} h_{n,m} = h$ then

$$\begin{aligned} \lim_{m\to\infty} \Big([\lim_{n\to\infty} g_{n,m}^{-1}] \cdot [\lim_{n\to\infty} h_{n,m}] \Big) &= \lim_{m\to\infty} \Big[\lim_{n\to\infty} g_{n,m}^{-1} \cdot \lim_{n\to\infty} h_{n,m} \Big] \\ &= \Big[\lim_{m\to\infty} \lim_{n\to\infty} g_{n,m}^{-1} \cdot \lim_{n\to\infty} h_{n,m} \Big] \\ &= \Big[\lim_{n,m\to\infty} g_{n,m}^{-1} h_{n,m} \Big] = [g^{-1} \cdot h] \end{aligned}$$

so that we indeed obtain a topological group $\underline{G}^*$. □

LEMMA 4.2.14 (Proposition 2.2.1 in [**60**]). *Let $\underline{G}$ be a Polish group and $\underline{U}$ a subgroup. Then $\underline{U}$ is Polish if and only if U is closed in G.*

PROOF. Suppose first that U is closed in G. Let $S \subseteq G$ be a countable set that is dense in G and let d be a compatible metric on G. Let $n \in \mathbb{N}$. For every $u \in U$ there exists $s_u \in S$ such that $d(u, s_u) < \frac{1}{n+1}$. From all the elements $v \in U$ such that $s_u = s_v$, select one. Let T be the set of all elements of U selected like this, for all values of $n \in \mathbb{N}$. Then T is countable because S is countable. Also note that T is dense in U, so U is separable. The restriction of a complete compatible metric for G metric to U is certainly also a compatible metric for U, and it is complete since U is closed.

Conversely, we assume that U is Polish. Let $\bar{U}$ be the closure of U in G; note that $\bar{U}$ is Polish. By Lemma 4.1.6 we have that U is a countable intersection of open sets.

Suppose for contradiction that there exists a $g \in \bar{U} \setminus U$. Since U is dense in $\bar{U}$, the coset gU is dense in $\bar{U}$ and also a countable intersection of open sets. By the Baire category theorem (Theorem 4.1.8) applied to the Polish group $\bar{U}$, the intersection of gU and U is dense in $\bar{U}$, and in particular non-empty, which is impossible. $\square$

LEMMA 4.2.15. *Let $\underline{G}$ be a Polish group with a compatible left-invariant metric d. Then $d'(g,h) := d(g,h) + d(g^{-1},h^{-1})$ is a compatible complete metric for G.*

PROOF. Let (G^*, d^*) be the metric completion of (G, d') (see Proposition 4.1.10). By Lemma 4.2.13, G^* can be viewed as a topological group $\underline{G}^*$ with the compatible complete metric d^*, and since G^* is also separable we conclude that $\underline{G}^*$ is Polish. Since $\underline{G}$ is Polish, too, Lemma 4.2.14 implies that G is closed in G^*. Therefore, $G = G^*$. This shows that d' is a compatible complete metric on G. $\square$

Exercises.

(104) Show that $\mathrm{Sym}(\mathbb{N})$ does not admit a compatible complete and left-invariant metric.

(105) Show that a Polish group has a compatible and complete left-invariant metric if and only if every compatible left-invariant metric is complete.

(106) Show that no oligomorphic permutation group G on a countably infinite set has a compatible complete and left-invariant metric.

Hint. We may assume that G is closed, so that we can write $G = \mathrm{Aut}(\underline{A})$. Then apply the compactness theorem of first-order logic to construct an elementary self-embedding of $\underline{A}$ which is not surjective.

Alternative direct solution. Let d be the complete left-invariant metric from Example 52. By the previous exercise, it suffices to show that d is not complete. We define the sequence $(a_i)_{i \in \mathbb{N}}$ of elements of $\mathbb{N}$ inductively as follows:

- Let a_1 be any natural number in an infinite orbit of G.
- $A_0 := \emptyset$.
- $A_N := \{0, 1, 2, 3, \ldots, a_n\}$.
- Let a_{n+1} be any element strictly larger than a_n such that a_n and a_{n+1} are in the same infinite orbit with respect to $G_{A_{n-1}}$ and the orbit of a_{n+1} with respect to G_{A_n} is infinite. This is possible, because the infinite $G_{A_{n-1}}$-orbit containing a_n only splits into finitely many G_{A_n} orbits which need to contain an infinite orbit.
- For every $n \in \mathbb{N}$, let $g_n, h_n \in G$ be such that $h_0 = \mathrm{id}_{\mathbb{N}}, h_{n+1} = h_n \circ g_n$, and g_n with $n \geq 1$ is an element of $G_{A_{n-1}}$ mapping a_{n+1} to a_n.

For $n < m$, we have

$$\begin{aligned} d(h_n, h_m) &= d(g_1 g_2 \ldots g_n, g_1 g_2 \ldots g_m) \\ &= d(1, g_{n+1} \ldots g_m) \leq 2^{-a_n} \leq 2^{-n} \end{aligned}$$

using the left invariance of d and that $g_{n+1} \ldots g_m \in G_{A_n}$. This shows that $(h_n)_{n \in \mathbb{N}}$ is Cauchy with respect to d. But this sequence is not convergent since the preimage of a_1 after $n-1$ steps is a_n which is a non-convergent sequence.

4.3. Closed Subgroups

In this section we give a topological characterisation of those topological groups that appear as automorphism groups of countable structures. Since the automorphism group of a structure $\underline{A}$ is a closed subgroup of $\mathrm{Sym}(A)$ (Proposition 1.2.2), and since $\mathrm{Sym}(A)$ is Polish (Example 54), it follows from Lemma 4.2.14 that $\mathrm{Aut}(\underline{A})$ is Polish. But also $(\mathbb{R}; +)$ is Polish, and it certainly isn't the automorphism group of a countable structure, so we need to identify more properties of closed subgroups of $\mathrm{Sym}(\mathbb{N})$.

A topological group is *non-archimedian* if it has a basis at the identity consisting of open subgroups.

Example 71. The group $(\mathbb{R}; +)$ is *archimedian*: for all $a, b \in \mathbb{R}$ with $0 < a \leq b$ there exists an $n \in \mathbb{N}$ such that $na := a + a + \cdots + a \geq b$. Hence, the open interval $(-b, b)$ does not contain any non-trivial open subgroup, since if the subgroup contains $a \in (0, b)$, then it also contains elements larger than b. This implies that $(\mathbb{R}; +)$ is *not* non-archimedian in the sense above and motivates the terminology. △

Example 72. The group $\mathrm{Sym}(D)$ is non-archimedian: the point stabilisers G_a for $a \in D^n$, $n \in \mathbb{N}$, are open subgroups of G and they form a basis at the identity. △

Remark 4.3.1. *Note that if $\underline{G}$ and $\underline{H}$ are non-archimedian, and if we want to verify that a given group homomorphism $h \colon \underline{G} \to \underline{H}$ is continuous, it suffices to verify that the preimage of every open subgroup of $\underline{H}$ is open in G (see Remark 4.2.2). Likewise, h is open if the image of every open subgroup of $\underline{G}$ is open in H.*

Automorphism groups of countable structures can be characterised in topological terms.

Theorem 4.3.2 (Section 1.5 in [**9**]; also see Theorems 2.4.1 and 2.4.4 in [**60**]). *Let $\underline{G}$ be a topological group. Then the following are equivalent.*

(1) *$\underline{G}$ is topologically isomorphic to the automorphism group of a countable relational structure.*
(2) *$\underline{G}$ is topologically isomorphic to a closed subgroup of* $\mathrm{Sym}(\mathbb{N})$.
(3) *$\underline{G}$ is Polish and admits a compatible left-invariant ultrametric.*
(4) *$\underline{G}$ is Polish and non-archimedian.*

Proof. The equivalence of (1) and (2) has been shown in Proposition 1.2.2. The implication from (2) to (3) has been explained in the paragraphs preceding the statement of the proposition. So it suffices to show (3) $\Rightarrow$ (4) $\Rightarrow$ (2).

For the implication from (3) to (4), let d be a left-invariant ultrametric on G. Let $U_n = \{g \in G \mid d(g, 1) < 2^{-n}\}$, for $n \in \mathbb{N}$. We claim that the set of all those U_n forms a basis at the identity consisting of open subgroups. Since d is a left-invariant ultra-metric, for $g, h \in U_n$ we have

$$d(g^{-1}h, 1) = d(h, g) \leq max(d(h, 1), d(1, g)) < 2^{-n} .$$

Hence, $g^{-1}h \in U_n$ and U_n is indeed a subgroup.

For the implication from (4) to (2), let $\{B_1, B_2, \dots\}$ be an at most countable basis at the identity (which exists since G is metrisable). Each B_i has an open subset V_i which is a subgroup, since G has a basis at the identity consisting of open subgroups. Then $\{V_1, V_2, \dots\}$ is a countable basis of the identity consisting of open subgroups. Each V_i has at most countably many cosets since G is separable. So the set of all cosets of those groups gives an at most countable basis $\mathcal{B} := \{U_1, U_2, \dots\}$ that is closed under left multiplication. If $\mathcal{B}$ is infinite, we define the map $\xi \colon G \to \mathrm{Sym}(\mathbb{N})$ by setting

$$\xi(g)(n) = m \;\Leftrightarrow\; gU_n = U_m .$$

If $|\mathcal{B}| = n_0$ is finite, we define the map $\xi\colon G \to \mathrm{Sym}(\mathbb{N})$ similarly, but set $\xi(g)(n) = n$ for all $n > n_0$.

Claim 1. ξ is a homomorphism. We have $\xi(fg) = \xi(f)\xi(g)$ since

$$\xi(f)\xi(g)(n) = m \Leftrightarrow f(gU_n) = U_m \Leftrightarrow fgU_n = U_m \Leftrightarrow \xi(fg)(n) = m\,.$$

Claim 2. ξ is injective: when $f, g \in G$ are distinct, then there are disjoint open subsets U and V with $f \in U$ and $g \in V$, because the topology is metrisable and therefore Hausdorff; since $\mathcal{B}$ is a basis, we can assume that $U = U_n$ for some $n \geq 1$. If $fU_n = gU_n$, then $g \in U_n = U$ since $f \in U_n$, contradicting the assumption that U and V are disjoint. Hence, $\xi(f)(n) \neq \xi(g)(n)$, and so $\xi(f) \neq \xi(g)$. So ξ is indeed an isomorphism between G and a subgroup of $\mathrm{Sym}(\mathbb{N})$.

Claim 3. ξ is continuous. Let $V \subseteq \mathrm{Sym}(\mathbb{N})$ be an open set. Then V is a union of basic open sets $S(\bar{a}, \bar{b})$ for some $\bar{a}, \bar{b} \in \mathbb{N}^n$. Let $i \leq n$ and $g, h \in G$ be such that $g \circ h \in U_{b_i}$. Since composition in G is continuous and U_{b_i} is open, there is an open subset $G_{g,h}$ of G containing g and an open set $H_{g,h}$ of G containing h such that $(g, h) \in G_{g,h} \times H_{g,h} \subseteq \circ^{-1}(U_{b_i})$. We then have

$$\begin{aligned} \xi^{-1}(S(\bar{a}, \bar{b})) &= \bigcap_{i \leq n} \{g \in G \mid gU_{a_i} = U_{b_i}\} \\ &= \bigcap_{i \leq n} \bigcup_{g \in G, h \in U_{a_i}} G_{g,h}. \end{aligned}$$

This set is a finite intersection of a union of open sets and thus open. Hence, $\xi^{-1}(V)$ is a union of open sets and therefore open as well, which concludes the proof that ξ is continuous.

Claim 4. The map ξ is open and the image of ξ is closed in $\mathrm{Sym}(\mathbb{N})$. Let

- d_1 be the left-invariant compatible metric on G (Theorem 4.2.8),
- d_1' be the compatible complete metric on the Polish group G defined as $d_1'(g, h) = d_1(g, h) + d_1(g^{-1}, h^{-1})$ (see Lemma 4.2.15), and
- d_2' be the compatible complete metric on $\mathrm{Sym}(\mathbb{N})$ from Example 54.

We will show that ξ^{-1} is Cauchy-continuous as a map from $(\xi(G), d_2')$ to (G, d_1'). This clearly implies both parts of the claim.

Let $g_1, g_2, \ldots$ be a sequence in G such that $\xi(g_1), \xi(g_2), \ldots$ converges against $h \in \mathrm{Sym}(\mathbb{N})$. We have to show that $g_1, g_2, \ldots$ is d_1'-Cauchy. Since d_1 is left-invariant, $\lim_{n,m\to\infty} d_1(g_n, g_m) = 0$ if and only if $\lim_{n,m\to\infty} d_1(g_m^{-1} g_n, 1) = 0$. Let $\epsilon > 0$ be arbitrary. Since $\mathcal{B}$ is a basis, there exists $U_k \in \mathcal{B}$ such that

$$U_k \subseteq \{g \in G \mid d_1(g, 1) < \epsilon/2\}$$

and $U_k U_k^{-1} \subseteq \{g \in G \mid d_1(g, 1) < \epsilon\}$. Since $\lim_{n\to\infty} \xi(g_n) = h$, there exists an n_0 such that $\xi(g_n)(k) = \xi(g_m)(k) = h(k)$ for all $n, m > n_0$. Then $g_n U_k = g_m U_k$, and so

$$g_m^{-1} g_n \in U_k U_k^{-1} \subseteq \{g \in G \mid d_1(g, 1) < \epsilon\}\,.$$

Hence, $d_1(g_m g_n^{-1}, 1) < \epsilon$, and $\lim_{n,m\to\infty} d_1(g_m^{-1} g_n, 1) = 0$. Similarly one can show that $\lim_{n,m\to\infty} d_1(g_m g_n^{-1}, 1) = 0$, using the fact that $\xi(g_n^{-1}) = \xi(g_n)^{-1}$, and hence $\lim_{n\to\infty} \xi(g_n^{-1}) = h^{-1}$. Thus, $\lim_{n,m\to\infty} d_1(g_n, g_m) = 0 = \lim_{n,m\to\infty} d_1(g_n^{-1}, g_m^{-1})$, and therefore $\lim_{n,m\to\infty} d_1'(g_n, g_m) = 0$. $\square$

4.4. Open Subgroups

We present a simple but useful characterisation of the open subgroups of permutation groups. Recall the definition of point stabilisers from Definition 1.2.5.

LEMMA 4.4.1. *Let G be subgroup of* $\mathrm{Sym}(\mathbb{N})$ *and let U be a subgroup. Then the following are equivalent.*

(1) *U is open in G;*
(2) *U contains G_t for some $t \in \mathbb{N}^n$, $n \in \mathbb{N}$;*
(3) *U contains an open subset of G.*

PROOF. (1) $\Rightarrow$ (2): Since U is open in G it must contain $G \cap S(a,b)$ for some $a, b \in \mathbb{N}^n$ since these sets form a basis of the topology of $\mathrm{Sym}(\mathbb{N})$. Every element of G_a can be written as $\alpha\beta$ with $\alpha \in G \cap S(b,a) \subseteq U$ and $\beta \in G \cap S(a,b) \subseteq U$. Hence, U contains G_a.

(2) $\Rightarrow$ (3): trivial since G_t is open in G.

(3) $\Rightarrow$ (1): Let H be an open subset of U. Then $U = \bigcup_{\beta \in U} \beta H$. Since βH are open, it follows that U is open, too. $\square$

It follows that all open subgroups of S_ω have countable index.

REMARK 4.4.2. *We have seen in Proposition 4.2.1 that every open subgroup of a topological group is closed. The converse is false: for example, when E is an equivalence relation on a countably infinite set B with two infinite classes, then* $\mathrm{Aut}(B; E)$ *is a closed subgroup of* $\mathrm{Sym}(B)$ *(we already saw in Proposition 1.2.2 that the closed subgroups of the automorphism group of a structure $\underline{A}$ correspond precisely to arbitrary expansions of $\underline{A}$), but does not contain the point stabiliser of some finite subset of B.*

Next, we present another characterisation of the open subgroups $\underline{U}$ of subgroups $\underline{G}$ of S_ω which more explicitly describes all elements of U. Let $\underline{U} \leq \underline{G}$, $n \in \mathbb{N}$, and $a \in B^n$ be such that $\underline{U}$ contains G_a. Define $E_{U,a}$ to be

$$\{(\alpha(a), \alpha\beta(a)) \mid \beta \in U, \alpha \in G\}.$$

Then $E_{U,a}$ is a congruence of the componentwise action of G on B^n: it is preserved by G, and an equivalence relation on B^n. Reflexivity is clear. For symmetry, let $\alpha \in G$ and $\beta \in U$, so that $(\alpha(a), \alpha\beta(a)) \in E_{U,a}$; we have to show that $(\alpha\beta(a), \alpha(a)) \in E_{U,a}$. Let $\gamma := \alpha\beta \in G$. Then $(\alpha\beta(a), \alpha(a)) = (\gamma(a), \gamma\beta^{-1}(a)) \in E_{U,a}$ since $\beta^{-1} \in U$. For transitivity, let $(u,v), (v,w) \in E := E_{U,a}$. Then $u = \alpha(a)$, $v = \alpha\beta(a) = \alpha'(a)$, and $w = \alpha'\beta'(a)$, for $\alpha, \alpha' \in G, \beta, \beta' \in U$. Hence, $\beta^{-1}\alpha^{-1}\alpha' \in G_a \subseteq U$ and $\alpha^{-1}\alpha' \in U$. So we obtain that $(u,w) = (\alpha(a), \alpha'\beta'(a)) = (\alpha(a), \alpha(\alpha^{-1}\alpha'\beta')(a)) \in E_{U,a}$.

LEMMA 4.4.3. *Let $\underline{G} \leq$* $\mathrm{Sym}(B)$ *and $\underline{U} \leq \underline{G}$. Then the following are equivalent.*

(1) *U is open in G.*
(2) *There is $a \in B^n$, $n \in \mathbb{N}$, such that $G_a \leq U = \{\alpha \in G \mid (a, \alpha(a)) \in E_{U,a}\}$.*
(3) *$U = G_{\{S\}}$ is the set stabiliser of a block S of the componentwise action of G on B^n for some $n \in \mathbb{N}$.*

PROOF. (1) $\Rightarrow$ (2). Let U be an open subgroup of G. Then Lemma 4.4.1 implies that U must contain G_a for some $n \in \mathbb{N}$ and some $a \in B^n$. Moreover, we claim that

$$U = \{\alpha \in G \mid (a, \alpha(a)) \in E_{U,a}\}. \tag{8}$$

Indeed, if $\beta \in U$, then

$$(a, \beta(a)) \in E_{U,a} = \{(\alpha(a), \alpha\gamma(a)) \mid \gamma \in U, \alpha \in G\}$$

as witnessed by $\gamma = \beta$ and $\alpha = \mathrm{id}_B$. This proves $\subseteq$ in (8). Conversely, if $\alpha \in G$ is such that $(a, \alpha(a)) \in E_{U,a}$, then $(a, \alpha(a)) = (\alpha'(a), \alpha'\beta(a))$ for some $\beta \in U$ and $\alpha' \in G$. Hence, $\alpha' \in G_a$ and $\alpha^{-1}\alpha'\beta \in G_a$, and since $G_a \subseteq U$ we have that $\alpha \in U$. This proves $\supseteq$ in (8).

(2) $\Rightarrow$ (3). Let S be the congruence class of $E_{U,a}$ which contains a. Lemma 1.4.2 shows that S is a block of the componentwise action of $\underline{G}$ on B^n. We claim that

$$G_S = \{\alpha \in G \mid (a, \alpha(a)) \in E_{U,a}\}.$$

Let $\beta \in G_S$. Then $\beta(a) \in S$ and hence $(a, \beta(a)) \in E_{U,a}$. This shows that $\beta \in \{g \in G \mid (a, \alpha(a)) \in E_{U,a}\}$. Conversely, suppose that $\alpha \in G$ is such that $(a, \alpha(a)) \in E_{U,a}$; then $\alpha(a)$ must be in the same congruence class of $E_{U,a}$ as a, which is S, and hence $\alpha \in G_S$.

(3) $\Rightarrow$ (1). Let $S \subseteq B^n$ be a block and let C be the corresponding congruence of G. Arbitrarily pick an $s \in S$. To show that $U = G_{\{S\}}$ is open it suffices by Lemma 4.4.1 that $G_{\{S\}}$ contains G_s. Let $\alpha \in G_s$ and $t \in S$. Then $(s, t) \in C$ and hence $(\alpha s, \alpha t) \in C$. Since $\alpha s = s \in S$ we conclude that $\alpha t \in S$. So $\alpha \in G_{\{S\}}$. $\square$

Lemma 4.4.3 has the following consequence.

Corollary 4.4.4. *Every oligomorphic subgroup of* Sym($\mathbb{N}$) *has countably many open subgroups.*

Proof. An oligomorphic group $\underline{G}$ has for each n finitely many congruences of the componentwise action of $\underline{G}$ on B^n, and at most countably many congruence classes for each congruence. $\square$

So in particular Sym($\mathbb{N}$) itself has only countably many open subgroups. We mention another fact about open subgroups of oligomorphic permutation groups.

Corollary 4.4.5 (Lemma 2.4 in [**74**]). *Let $\underline{G}$ be an oligomorphic subgroup of* Sym($\mathbb{N}$) *and let $\underline{U}$ be an open subgroup of $\underline{G}$. Then $\underline{U}$ is contained in only finitely many subgroups of $\underline{G}$.*

Proof. By Lemma 4.4.1, $\underline{U}$ contains $\underline{G}_a$ for some $a \in \mathbb{N}^n$ and $n \in \mathbb{N}$. So it suffices to show the statement for $\underline{U} = \underline{G}_a$. Let $\underline{H}$ be any subgroup of $\underline{G}$ that contains $\underline{G}_a$. By (8), $\underline{H}$ is of the form

$$\{\alpha \in G \mid (a, \alpha(a)) \in E_{H,a}\}.$$

As $\underline{G}$ is oligomorphic, there are finitely many congruence relations of $\underline{G}$ on B^n, which implies the statement. $\square$

Exercises.

(107) Let $\underline{A}$ be a countable ω-categorical structure. Let $B \subseteq A$ be finite and let $C := \mathrm{acl}_{\underline{A}}(B)$. Then $\mathrm{Aut}(\underline{A})_{\{C\}}$ is open in $\mathrm{Aut}(\underline{A})$.

4.5. Compact Subgroups

Proposition 4.5.1. *Let G be a compact subgroup of* Sym($\mathbb{N}$). *Then all orbits of G are finite.*

Proof. Let O be an infinite orbit of G, and fix $a \in O$. Then the sets $S(a, b)$ for $b \in O$ form an open partition of G, and hence no finite sub-collection of those sets can cover G. Hence, if O has infinite orbits, then G is not compact. $\square$

Proposition 4.5.2. *Let G be a closed subgroup of* Sym($\mathbb{N}$). *Then G is compact if and only if all orbits of G are finite.*

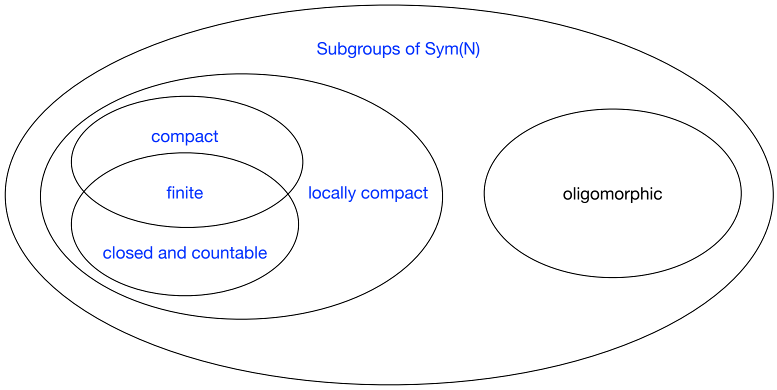


Figure 4.1. An illustration of the relationship between basic finiteness properties of subgroups of Sym($\mathbb{N}$): finite, countable, compact, locally compact, and oligomorphic.

Proof. The forward implication follows from Proposition 4.5.1. For the other direction, suppose that the orbits $O_1, O_2, \dots$ of G are finite. We write $G|_{O_i}$ for the permutation group formed by the restrictions of G to the finite set O_i. Then $G = \prod_i G|_{O_i}$ is a closed subset of a product of finite subgroups of G, and hence compact by Tychonoff's theorem (Theorem 4.1.14) and Proposition 4.1.13. We do not use the entire strength of Tychonoff's theorem, and show two alternative proofs.

Second Proof. Let $\{U_i\}_{i \in A}$ be an open cover of G. Since Sym($\mathbb{N}$) and hence G are second-countable, we can assume that $A = \mathbb{N}$ (Proposition 4.1.17). Suppose for contradiction that for no finite $B \subseteq A$ we have that $G \subseteq \bigcup_{i \in B} U_i$. Consider the following rooted tree. The vertices on level n are the restrictions of the permutations in G to $\{1, \dots, n\}$. Adjacency between a vertex on level n and vertices on level $n+1$ is defined by restriction. Clearly, for every n there are finitely many vertices on level n since the orbit of $(1, \dots, n)$ with respect to the componentwise action of G is finite. A vertex on level n is *good* if it is the restriction of a function from $G \setminus \bigcup_{i \leq n} U_n$. Clearly, the restriction of a good vertex is good. Moreover, by assumption there are good vertices on all levels. By König's tree lemma, there is an infinite branch of good vertices in the tree. This branch defines an injection from $\mathbb{N}$ to $\mathbb{N}$. In fact, it must be a bijection since the finiteness of the orbits implies that the map is surjective onto each orbit. This map is from G since G is closed, but it does not lie in any of the U_i, a contradiction.

Third proof. Our final proof uses Theorem 4.1.19. Clearly G is complete since it is closed in Sym($\mathbb{N}$). To prove total boundedness of G, let $\epsilon > 0$. Choose $n \in \mathbb{N}$ such that $1/2^n < \epsilon$. Since all orbits of G are finite, the orbit O of $(1, \dots, n)$ with respect to the componentwise action of G on $\mathbb{N}^n$ is finite too; choose $f_1, \dots, f_k \in G$ so that $O = \{f_1(1, \dots, n), \dots, f_k(1, \dots, n)\}$. Then by construction $B_{f_1}(\epsilon), \dots, B_{f_k}(\epsilon)$ covers all of G. $\square$

It follows from Proposition 4.5.1 that a compact subgroup $\underline{G}$ of Sym($\mathbb{N}$) must have infinitely many orbits, and in particular cannot be oligomorphic. In fact, oligomorphicity is already ruled out by local compactness; this can be seen from Lemma 3.0.1 and the following.

Corollary 4.5.3. *Let $\underline{G} \leq \mathrm{Sym}(\mathbb{N})$ be locally compact. Then there exists $a \in \mathbb{N}^n$, $n \in \mathbb{N}$ such that $\underline{G}_a$ has only finite orbits. If $\underline{G}$ is additionally closed, then also the converse holds.*

Proof. If G is locally compact there exists an open set $U \subseteq G$ that contains $1^{\underline{G}}$ and that is contained in a compact set $K \subseteq G$. Since U is open and contains $1^{\underline{G}}$ there exists a finite tuple $\bar{a} \in \mathbb{N}^n$, $n \in \mathbb{N}$, such that $G_a \subseteq U$. Then G_a is a closed subset of the compact set K and hence compact by Proposition 4.1.13. The statement then follows from Proposition 4.5.1.

Conversely, if G is closed, then so is G_a, and hence G_a is compact by Theorem 4.5.2. To show local compactness of G, let $\alpha \in G$. Then αG_a is an open and compact set which contains α, and hence G is locally compact. $\square$

Also note that locally compact subgroups of $\mathrm{Sym}(\mathbb{N})$ contain all closed countable subgroups of $\mathrm{Sym}(\mathbb{N})$ by Theorem 1.2.6; see Figure 4.1.

Exercises.

(108) Prove that every countable compact subgroup of $\mathrm{Sym}(\mathbb{N})$ is finite.

(109) Prove that every closed subgroup of $\mathrm{Sym}(\mathbb{N})$ which is not locally compact has a homeomorphism ξ to $\mathrm{Sym}(\mathbb{N})$ such that ξ and ξ^{-1} are uniformly continuous (with respect to the metric inherited from the Baire space).

4.6. Closed Normal Subgroups

Example 73. Let E be an equivalence relation on a countably infinite set D such that all equivalence classes $B_1, B_2, \dots$ of E have finite size. Then the oligomorphic group $\mathrm{Aut}(D; E)$ has the closed normal subgroup $\mathrm{Aut}(D; B_1, B_2, \dots)$ (which is not oligomorphic). $\triangle$

Proposition 4.6.1. *Let $\underline{G}$ be a closed subgroup of $\mathrm{Sym}(B)$. If E is an equivalence relation on B^n, $n \in \mathbb{N}$, which is preserved by G, then the subgroup of $\underline{G}$ that preserves each equivalence class of E is closed and normal. Conversely, every closed normal subgroup of $\underline{G}$ is the intersection of closed normal subgroups that arise in this way.*

Proof. Let $\underline{C}$ be the expansion of $\underline{B}$ by a unary relation for each equivalence class of E. Then $\mathrm{Aut}(\underline{C})$ is closed by Proposition 1.2.2, and it is a normal subgroup of $\mathrm{Aut}(\underline{B})$: when $g \in \mathrm{Aut}(\underline{B})$ and $h \in \mathrm{Aut}(\underline{C})$, then $g \circ h \circ g^{-1}$ preserves each equivalence class of E, and thus is an automorphism of $\underline{C}$. Normality of $\mathrm{Aut}(\underline{C})$ follows from Proposition 1.5.1.

For the second part, suppose that $\underline{G}$ has a closed normal subgroup $\underline{N}$. Consider the relation

$$R_n := \{(x, y) \mid x, y \in B^n \text{ and there is } h \in N \text{ such that } h(x) = y\} .$$

This relation is obviously an equivalence relation, and it is preserved by all the automorphisms of $\underline{B}$. For this, we have to show that for all $g \in G$ and all $(x, y) \in R_n$ we have that $(g(x), g(y)) \in R_n$. So suppose that $x, y \in B^n$ such that $h(x) = y$ for some $h \in N$. Then $g(y) = g(h(x)) \in (gN)(x) = (Ng)(x) = N(g(x))$ by normality of $\underline{N}$. Hence there exists an $h' \in N$ such that $h'(g(x)) = g(y)$, which shows that $(g(x), g(y)) \in R_n$.

Let $\underline{C}$ be the structure that contains for all n the n-ary relations given by the equivalence classes of the relations R_n for all $n \geq 0$. We claim that $\underline{N}$ is precisely the automorphism group of $\underline{C}$. As in the first part we can verify that every $h \in N$ is an automorphism of $\underline{C}$. The converse follows by local closure as follows. Let g be an automorphism of $\underline{C}$, and let x, y be from B^n so that $g(x) = y$. Since g preserves the equivalence classes of R_n, there exists an $h \in N$ such that $h(x) = y$. Hence, g lies in the closure of $\underline{N}$, which implies that g is from $\underline{N}$ since $\underline{N}$ is closed. $\square$

EXAMPLE 74. The automorphism group $\underline{G}$ of the structure $\underline{B} = (\mathbb{Q}; \mathrm{Betw})$, where $\mathrm{Betw} = \big\{(x,y,z) \mid (x < y < z) \vee (z < y < x)\big\}$, is 2-transitive and therefore primitive. However, the relation

$$\big\{((x_1,x_2),(y_1,y_2)) \mid (x_1 < x_2 \wedge y_1 < y_2) \vee (x_1 > x_2 \wedge y_1 > y_2) \vee (x_1 = x_2 \wedge y_1 = y_2)\big\}$$

is an equivalence relation on $\mathbb{Q}^2$ that is preserved by G. And indeed, $\underline{G}$ has a closed normal subgroup $\underline{N}$ that is isomorphic to the automorphism group of $(\mathbb{Q};<)$, and $\underline{G}/\underline{N}$ has two elements, corresponding to the automorphisms that reverse the order $<$, and the automorphisms that preserve the order. $\triangle$

THEOREM 4.6.2. *The group* $\mathrm{Sym}(\mathbb{N})$ *is* topologically simple*, i.e., it has no proper non-trivial closed normal subgroups.*

PROOF. The statement follows from a stronger statement about all the normal subgroups of $\mathrm{Sym}(\mathbb{N})$. There are four such subgroups, which is an old result that has been discovered independently by several authors [**5**, **127**, **141**]: besides the trivial subgroup $\{\mathrm{id}_{\mathbb{N}}\}$ and the full subgroup $\mathrm{Sym}(\mathbb{N})$, there is only the finitary alternating group A (Exercise 10) and the subgroup P of permutations with finite support (Example 9, Exercise 27). Clearly, the latter two are not closed.

A self-contained proof of Sebastian Meyer goes as follows. Suppose that $\underline{H}$ is a closed non-trivial normal subgroup of $\mathrm{Sym}(\mathbb{N})$. Then there exists $f \in H \setminus \{\mathrm{id}_{\mathbb{N}}\}$ because $\underline{H}$ is non-trivial. Let $a \in \mathbb{N}$ be such that $f(a) \neq a$. Let $c \in \mathbb{N} \setminus \{a, f^{-1}(a), f(a)\}$. Since $\underline{H}$ is a normal subgroup of $\mathrm{Sym}(\mathbb{N})$, for any $g \in \mathrm{Sym}(\mathbb{N})$ we have that $g^{-1}fg \in H$ and $(g^{-1}f^{-1}g)f \in H$. Let $g = g^{-1} \in \mathrm{Sym}(\mathbb{N})$ be the map that exchanges a and c and fixes all other elements of $\mathbb{N}$. Then

$$g^{-1}f^{-1}gf(a) = g^{-1}(a) = c$$

and for $x \in \mathbb{N} \setminus \{a, f^{-1}(a), c, f^{-1}(c)\}$ we have

$$g^{-1}f^{-1}gf(x) = g^{-1}(x) = x.$$

To show that $\overline{H} = \mathrm{Sym}(\mathbb{N})$, let $k \in \mathrm{Sym}(\mathbb{N})$ and $M \subseteq \mathbb{N}$ be finite; we have to show that H contains an operation h such that $h(x) = k(x)$ for every $x \in M$. Choose $h \in H$ such that $M' := \{x \in M \mid h(x) = k(x)\}$ is largest possible. If $M' = M$ then we are done; otherwise there is $b \in M$ such that $h(b) \neq k(b)$. Let p and q be distinct elements from

$$\mathbb{N} \setminus \{k(x), h(x) \mid x \in M\}.$$

Let $g' \in \mathrm{Sym}(\mathbb{N})$ be any permutation that maps $h(b)$ to a, $k(b)$ to c, and p to $f^{-1}(a)$. Moreover, if $c \neq f(c)$ then g' maps q to $f^{-1}(c)$. Then $(g')^{-1}(g^{-1}f^{-1}gf)g'h \in H$. Moreover, for each $x \in M'$ we have that $h(x) \notin \{p, q, h(b), k(b)\}$. This implies for every $x \in M'$ that $g'h(x) \notin \{a, f^{-1}(a), c, f^{-1}(c)\}$ and

$$(g')^{-1}(g^{-1}f^{-1}gf)g'h(x) = h(x) = k(x)$$

because $g^{-1}f^{-1}gf$ is the identity on $g'h(x) \notin \{a, f(a), c, f(c)\}$. We also have that

$$(g')^{-1}(g^{-1}f^{-1}gf)g'h(b) = (g')^{-1}(g^{-1}f^{-1}gf)(a) = (g')^{-1}(c) = k(b)$$

which contradicts the choice of h so that M' is largest possible. $\square$

Exercises.

(110) Let $G \leq \mathrm{Sym}(\mathbb{N})$ be oligomorphic, and let G° be the intersection of the closed subgroups of Δ of finite index in G. Show that G° is oligomorphic and that $(G^\circ)^\circ = G^\circ$.

(111) Show that for an oligomorphic permutation group G, the quotient G/G° (see Exercise 110) is *profinite*, i.e., compact and totally disconnected.

(112) Can the group $\mathrm{Aut}(\underline{A})$ from Example 67 be written as a semidirect product $\underline{N} \rtimes_\theta \mathbb{Z}_2^{\mathbb{N}}$? Here, $\underline{N}$ is the normal subgroup of $\mathrm{Aut}(\underline{A})$ consisting of all automorphisms that fix all the equivalence classes of all the equivalence relations $E_0, E_1, \ldots$ (see Proposition 4.6.1 and Proposition 1.5.4).

CHAPTER 5

Birkhoff's Theorem and Permutation Groups

Birkhoff's theorem from universal algebra, specialised to permutation groups, describes how a given group can act on a given set (Section 5.1). In this chapter we also present a topological generalisation where we will be interested in *continuous actions* of topological groups (Section 5.2). As an application, this gives rise to a topological characterisation of bi-interpretability for ω-categorical structures (Section 5.3; bi-interpretability has already been defined in Section 3.6.2). More specifically, we obtain the following corollary which has been credited to Coquand by Ahlbrandt and Ziegler [**3**].

THEOREM 5.0.1. *Let $\underline{A}$ and $\underline{B}$ be countable ω-categorical structures, each with at least two elements. Then $\mathrm{Aut}(\underline{A})$ and $\mathrm{Aut}(\underline{B})$ are isomorphic as topological groups if and only if $\underline{A}$ and $\underline{B}$ are bi-interpretable.*

5.1. Birkhoff's Theorem

In order to apply Birkhoff's theorem to permutation groups, we represent permutation groups G as G-sets as in Example 4 and 7. If $\mathcal{K}$ is a class of τ-structures, then we write

- $\mathrm{P}(\mathcal{K})$ for the class of all products of structures in $\mathcal{K}$;
- $\mathrm{S}(\mathcal{K})$ for the class of all substructures of structrues in $\mathcal{K}$;
- $\mathrm{H}(\mathcal{K})$ for the class of all homomorphic images of structures in $\mathcal{K}$.

If $\underline{A}$ is a G-set, then we also write $\mathrm{Gr}(\underline{A})$ for G; recall that by definition G equals the permutation group on A consisting of all unary term functions over $\underline{A}$.

THEOREM 5.1.1 (Birkhoff's theorem for permutation groups). *Let $\underline{G}$ be a subgroup of $\mathrm{Sym}(B)$, for $|B| \geq 2$, and let $\underline{B}$ be a G-set with signature τ.*

- *For every homomorphism $\xi\colon \underline{G} \to \mathrm{Sym}(A)$ there exists a G-set $\underline{A}$ on A such that $\underline{A} \in \mathrm{HSP}(\{\underline{B}\})$ and $t^{\underline{A}} = \xi(t^{\underline{B}})$ for every τ-term t.*
- *If $\underline{A} \in \mathrm{HSP}(\{\underline{B}\})$ then the function ξ that maps $t^{\underline{B}}$ to $t^{\underline{A}}$ for every τ-term t is well defined and a homomorphism from $\underline{G}$ onto $\mathrm{Gr}(\underline{A})$.*

PROOF. Let $\xi\colon \underline{G} \to \mathrm{Sym}(A)$ be a homomorphism. Let I be a well-ordered set such that $B^A = \{c^i \mid i \in I\}$. For $a \in A$, define $c_a := (c^i(a))_{i \in I}$. Let $\underline{S}$ be the smallest substructure of $\underline{B}^{B^A}$ that contains $\{c_a \mid a \in A\}$. So the elements of S are precisely those that can be written as $t^{\underline{S}}(c_a)$ for some τ-term $t(x)$ and some $a \in A$. Define $\mu\colon S \to A$ by

$$\mu(t^{\underline{S}}(c_a)) := \xi(t^{\underline{B}})(a).$$

Claim 1. μ is well-defined. Suppose that $t^{\underline{S}}(c_a) = r^{\underline{S}}(c_{a'})$. We first show that $t^{\underline{B}} = r^{\underline{B}}$. Let $b \in B$. Note that there is some $i \in I$ such that $c^i(a) = b$ and $c^i(a') = b$. Hence,

$$\begin{aligned} t^{\underline{B}}(b) = t^{\underline{B}}(c^i(a)) &= t^{\underline{S}}(c_a)_i \\ &= r^{\underline{S}}(c_{a'})_i = r^{\underline{B}}(c^i(a'))) = r^{\underline{B}}(b). \end{aligned}$$

Hence, $t^{\underline{B}} = r^{\underline{B}}$ and therefore $t^{\underline{S}} = r^{\underline{S}}$. Since $r^{\underline{S}}$ is injective, $t^{\underline{S}}(c_a) = r^{\underline{S}}(c_{a'})$ implies that $c_a = c_{a'}$. Since $|B| \geq 2$ this implies that $a = a'$. Therefore

$$\xi(t^{\underline{B}})(a) = \xi(r^{\underline{B}})(a) = \xi(r^{\underline{B}})(a')$$

and hence μ is well-defined.

Claim 2. μ is surjective. Let $a \in A$ and choose the τ-term $t := x$ (just a variable) so that $r^{\underline{B}} = 1^{\underline{G}}$; then

$$\mu(c_a) = \mu(t^{\underline{S}}(c_a)) = \xi(t^{\underline{B}})(a) = a$$

since $\xi(1^{\underline{G}}) = 1^{\mathrm{Sym}(A)}$.

Let $\underline{A}$ be the τ-algebra where $g \in \tau$ denotes $\xi(g^{\underline{B}})$.

Claim 3. μ is a homomorphism from $\underline{S}$ to $\underline{A}$. Let $f \in \tau$ and let $s \in S$, and let $s = t^{\underline{S}}(c_a)$ for some τ-term t and some $a \in A$. Then

$$\begin{aligned}\mu(f^{\underline{S}}(s)) = \mu(f^{\underline{S}}(t^{\underline{S}}(c_a))) &= \mu(f(t)^{\underline{S}}(c_a)) \\ &= f(t)^{\underline{A}}(a) \\ &= f^{\underline{A}}(t^{\underline{A}}(a)) \\ &= f^{\underline{A}}(t^{\underline{S}}(c_a)) = f^{\underline{A}}(\mu(s))\end{aligned}$$

Hence, $\underline{A}$ is the homomorphic image of the subalgebra $\underline{S}$ of $\underline{B}^{B^A}$, so $\underline{A} \in \mathrm{HSP}(\underline{B})$.

For the second statement, we first show that ξ is well-defined. Suppose that r and t are τ-terms such that $r^{\underline{B}} = t^{\underline{B}}$, and let $\mu\colon \underline{S} \to \underline{A}$ be a homomorphism from a subalgebra $\underline{S}$ of a power of $\underline{B}$ to $\underline{A}$. Then $r^{\underline{S}} = t^{\underline{B}}$. Hence, for all $s \in S$ we have $\mu(r^{\underline{S}}(s)) = \mu(t^{\underline{S}}(s))$ and $r^{\underline{A}}(\mu(s)) = t^{\underline{A}}(\mu(s))$ since μ is a homomorphism. Since μ is surjective it follows that $r^{\underline{A}} = t^{\underline{A}}$.

To show that ξ is a homomorphism, we have to prove that

$$\xi(1^{\underline{G}}) = 1^{\mathrm{Sym}(A)} \tag{9}$$

$$\xi((g^{\underline{B}})^{-1}) = \xi(g^{\underline{B}})^{-1} \tag{10}$$

$$\text{and } \xi(g^{\underline{B}}h^{\underline{B}}) = \xi(g^{\underline{B}})\xi(h^{\underline{B}}). \tag{11}$$

To show (9), consider the τ-term x that just consists of a variable. Then

$$\xi(1^{\underline{G}}) = \xi(\mathrm{id}_B) = \xi(x^{\underline{B}}) = x^{\underline{A}} = \mathrm{id}_A = 1^{\mathrm{Sym}(A)}.$$

To show (11), let g and h be τ-terms. Then

$$\xi(g^{\underline{B}}h^{\underline{B}}) = \xi((gh)^{\underline{B}}) = (gh)^{\underline{A}} = g^{\underline{A}}h^{\underline{A}} = \xi(g^{\underline{B}})\xi(h^{\underline{B}}).$$

Finally, since $\mathrm{Gr}(\underline{B})$ is a permutation group, there exists a τ-term t such that $t^{\underline{B}} = (g^{\underline{B}})^{-1}$. Then

$$\mathrm{id}_A = \xi(\mathrm{id}_B) = \xi(g^{\underline{B}}t^{\underline{B}}) = \xi(g^{\underline{B}})\xi(t^{\underline{B}}) = g^{\underline{A}}t^{\underline{A}}$$

and hence $t^{\underline{A}} = (g^{\underline{A}})^{-1}$. We then have

$$\xi(g^{\underline{B}})^{-1} = \xi(t^{\underline{B}}) = t^{\underline{A}} = (g^{\underline{A}})^{-1} = \xi(g^{\underline{B}})^{-1}. \qquad \square$$

COROLLARY 5.1.2. *Let $\underline{G} \leq \mathrm{Sym}(B)$ and $\underline{H} \leq \mathrm{Sym}(C)$, with $|B|, |C| \geq 2$. Then the following are equivalent:*

(1) *$\underline{G}$ and $\underline{H}$ are isomorphic as abstract groups;*
(2) *There exists a G-set $\underline{B}$ and an H-set $\underline{C}$ such that*

$$\mathrm{HSP}(\underline{B}) = \mathrm{HSP}(\underline{C}).$$

5.2. Topological Birkhoff

In the following, uniform continuity will refer to the left-invariant ultrametric of $\mathrm{Sym}(A)$.

THEOREM 5.2.1 (Topological Birkhoff for permutation groups). *Let $\underline{G}$ be a subgroup of $\mathrm{Sym}(B)$, for $|B| \geq 2$, and let $\underline{B}$ be a G-set with signature τ.*

- *For every continuous homomorphism $\xi\colon \underline{G} \to \mathrm{Sym}(A)$ such that $\xi(G)$ has finitely many orbits, there is an $\underline{A} \in \mathrm{HSP}^{\mathrm{fin}}(\{\underline{B}\})$ such that $t^{\underline{A}} = \xi(t^{\underline{B}})$ for every τ-term t.*
- *If $\underline{A} \in \mathrm{HSP}^{\mathrm{fin}}(\{\underline{B}\})$ then the function ξ that maps $t^{\underline{B}}$ to $t^{\underline{A}}$ for every τ-term t is well-defined and a (uniformly) continuous surjective homomorphism from $\underline{G}$ to $\mathrm{Gr}(\underline{A})$.*

PROOF. The proof is an adaptation of the proof of Theorem 5.1.1. Let $F = \{a_1, \ldots, a_k\} \subseteq A$ be a finite set that contains one element from each orbit of $\xi(G)$ on A. By Proposition 4.2.10, ξ is uniformly continuous, and hence there exists a finite $F' \subseteq B$ such that

$$\text{for all } f, g \in G \text{ if } f|_{F'} = g|_{F'} \text{ then } \xi(f)|_F = \xi(g)|_F. \tag{12}$$

Choose F' to be smallest possible; note that this implies that F' contains at most one element from each orbit of G. Let C be $(F')^F$ and let $m := |C| = |F'|^k$. Let $c^1, \ldots, c^m$ be the elements of C, and for $j \leq k$ define $c_j = (c^1(a_j), \ldots, c^m(a_j))$. Let $\underline{S}$ be the substructure of $\underline{B}^m$ generated by $c_1, \ldots, c_k$; so the elements of $\underline{S}$ are precisely those of the form $t^{\underline{S}}(c_j)$ for a τ-term t and $j \leq k$. Define a function $\mu\colon S \to A$ by setting

$$\mu(t^{\underline{S}}(c_j)) := \xi(t^{\underline{B}})(a_j).$$

Claim 1. μ is well-defined. Suppose that $t^{\underline{S}}(c_j) = r^{\underline{S}}(c_l)$ for $j, l \leq k$. We first show that $t^{\underline{B}}|_{F'} = r^{\underline{B}}|_{F'}$. Let $b \in F'$. Note that there is some $i \leq m$ such that $c^i(a_j) = b$ and $c^i(a_l) = b$. Hence,

$$\begin{aligned} t^{\underline{B}}(b) = t^{\underline{B}}(c^i(a_j)) &= t^{\underline{B}}((c_j)_i) \\ &= r^{\underline{B}}((c_l)_i) && \text{(since } t^{\underline{S}}(c_j) = r^{\underline{S}}(c_l)) \\ &= r^{\underline{B}}(c^i(a_l)) = s^{\underline{B}}(b). \end{aligned}$$

Hence, $t^{\underline{B}}|_{F'} = r^{\underline{B}}|_{F'}$. Therefore, using (12), we obtain

$$\xi(t^{\underline{B}})|_F = \xi(r^{\underline{B}})|_F.$$

It also follows from $t^{\underline{S}}(c_j) = r^{\underline{S}}(c_l)$ that for all $i \leq m$ the elements $(c_j)_i$ and $(c_l)_i$ lie in the same orbit of G (here we use the assumption that $|B| \geq 2$). By our assumption on F' this means that $l = j$. Hence,

$$\xi(t^{\underline{B}})(a_j) = \xi(r^{\underline{B}})(a_j) = \xi(r^{\underline{B}})(a_l),$$

and μ is indeed well-defined.

Claim 2. μ is surjective. This follows immediately from the assumption that F contains an element from each orbit of $\xi(G)$ on A.

Let $\underline{A}$ be the τ-structure where $g^{\underline{A}} := \xi(g^{\underline{B}})$ for every $g \in \tau$.

Claim 3. μ is a homomorphism. Let $f \in \tau$ and let $s \in S$. Since $\underline{S}$ is generated by $c_1, \ldots, c_k$ there exists a τ-term t and a $j \leq k$ such that $s = t^{\underline{S}}(c_j)$. Then

$$\begin{aligned}\mu(f^{\underline{S}}(s)) &= \mu(f^{\underline{S}}(t^{\underline{S}}(c_j))) = \mu((f(t))^{\underline{S}}(c_j)) \\ &= (f(t))^{\underline{A}}(a_j) \\ &= f^{\underline{A}}(t^{\underline{A}}(a_j)) \\ &= f^{\underline{A}}(\mu(t^{\underline{S}}(c_j))) = f^{\underline{A}}(\mu(s)).\end{aligned}$$

It follows that $\underline{A}$ is the homomorphic image of the subalgebra $\underline{S}$ of $\underline{B}^m$, and so $\underline{A} \in \mathrm{HSP}^{\mathrm{fin}}(\underline{B})$.

For the second statement we have already seen in Theorem 5.1.1 that if $\underline{A} \in \mathrm{HSP}^{\mathrm{fin}}(\underline{B}) \subseteq \mathrm{HSP}(\underline{B})$ then the natural homomorphism $\xi \colon \underline{G} \to \mathrm{Gr}(\underline{A})$ exists. It thus remains to show that ξ is uniformly continuous.

Let F be a finite subset of A. We have to find a finite subset F' of B such that for all $k \in \mathbb{N}$ and $f, g \in \underline{G}$ if $f|_{F'} = g|_{F'}$ then $\xi(f)|_F = \xi(g)|_F$. By assumption, there exists a surjective homomorphism μ from a subalgebra $\underline{S}$ of $\underline{B}^n$, for some $n \in \mathbb{N}$, to $\underline{A}$. For each $a \in F$ pick an $s \in S$ such that $\mu(s) = a$; let $F' \subseteq B$ be the (finite) set of all entries of all the tuples in s. Now let $f, g \in \mathrm{Gr}(\underline{B})$ be such that $f|_{F'} = g|_{F'}$. Choose τ-terms r, t such that $r^{\underline{B}} = f$ and $t^{\underline{B}} = g$. Clearly, $r^{\underline{S}}|_{S \cap G^n} = t^{\underline{S}}|_{G^n}$. Since $F \subseteq \mu(S \cap G^n)$ it follows that $r^{\underline{A}}|_F = t^{\underline{A}}|_F$, which proves the statement since $r^{\underline{A}} = \xi(f)$ and $t^{\underline{A}} = \xi(g)$. □

Corollary 5.2.2 (Topological Birkhoff for permutation groups). *Let $\underline{G}$ be a subgroup of $\mathrm{Sym}(B)$, for $|B| \geq 2$, and let $\underline{B}$ be a G-set with signature τ. Then the following are equivalent.*

- *There exists a continuous homomorphism $\xi \colon \underline{G} \to \mathrm{Sym}(A)$ such that $\xi(G)$ has finitely many orbits.*
- *There exists an $\underline{A} \in \mathrm{HSP}^{\mathrm{fin}}(\{\underline{B}\})$ such that $\mathrm{Gr}(\underline{A}) = \xi(G)$.*

Corollary 5.2.3. *Let $\underline{G} \leq \mathrm{Sym}(B)$ and $\underline{H} \leq \mathrm{Sym}(C)$, with $|B|, |C| \geq 2$. Then the following are equivalent:*

(1) *$\underline{G}$ and $\underline{H}$ are isomorphic as topological groups;*
(2) *There exists a G-set $\underline{B}$ and an H-set $\underline{C}$ such that*

$$\mathrm{HSP}^{\mathrm{fin}}(\underline{B}) = \mathrm{HSP}^{\mathrm{fin}}(\underline{C}).$$

5.3. Continuous Homomorphisms and Interpretability

In this section we prove Theorem 5.0.1: we give a characterisation of topological isomorphism of automorphism groups of ω-categorical structures in terms of bi-interpretability. We first establish the link between pseudo-varieties and full interpretations.

Definition 5.3.1. *Let I be a d-dimensional interpretation of $\underline{A}$ in $\underline{B}$. Then I is called* full *if a relation $R \subseteq A^k$ is first-order definable in $\underline{A}$ if and only if the relation $h^{-1}(R)$, defined as*

$$\begin{aligned}\{(b_1^1, \ldots, b_1^d, \ldots, b_k^1, \ldots, b_k^d) \in B^{d_1 d_2} \mid {} & (b_1^1, \ldots, b_1^d), \ldots, (b_k^1, \ldots, b_k^d) \in S \text{ and} \\ & (h(b_1^1, \ldots, b_1^d), \ldots, h(b_k^1, \ldots, b_k^d)) \in R\}\end{aligned}$$

is first-order definable in $\underline{B}$.

Observe that any structure with an interpretation in a structure $\underline{B}$ is a reduct of a structure with a full interpretation in $\underline{B}$.

THEOREM 5.3.2. *Let $\underline{B}$ be a structure and let $\underline{B}'$ be an* $\mathrm{Aut}(\underline{B})$*-set. If $\underline{A}$ has a full interpretation in $\underline{B}$ then there is an $\underline{A}' \in \mathrm{HSP}^{\mathrm{fin}}(\underline{B}')$ such that* $\overline{\mathrm{Gr}(\underline{A}')} = \mathrm{Aut}(\underline{A})$. *If $\underline{B}$ is finite or countably infinite ω-categorical, then the converse implication holds as well.*

PROOF. Suppose first that $\underline{A}$ has a d-dimensional full interpretation I in $\underline{B}$. Since $I^{-1}(A)$ is definable in $\underline{B}$, it is preserved by all operations in $\underline{B}'$, and therefore induces a subalgebra $\underline{C}'$ of ${\underline{B}'}^d$. Let K be the kernel of I. Since $I^{-1}(=_A)$ is definable in $\underline{B}$, all operations of $\underline{B}'$ preserve $K = I^{-1}(=_A)$, so K is a congruence of $\underline{C}'$. Thus, I is a surjective homomorphism from $\underline{C}'$ to $\underline{A}' := \underline{C}'/K$. We verify that $\overline{\mathrm{Gr}(\underline{A}')} = \mathrm{Aut}(\underline{A})$. We have to show that for every $f \in \tau$, the relation R is preserved by $f^{\underline{A}'}$. Since R is definable in $\underline{A}$ by the assumption that the interpretation I is full, we have that $I^{-1}(R)$ is definable in $\underline{B}$. This in turn implies that $f^{\underline{B}'}$ preserves $I^{-1}(R)$, which implies the claim.

For the converse implication, suppose that $\underline{B}$ is finite or countably infinite and ω-categorical. Then there is an algebra $\underline{A}' \in \mathrm{HSP}^{\mathrm{fin}}(\underline{B}')$ such that $\mathrm{Gr}(\underline{A}') = \mathrm{Aut}(\underline{A})$. So there exists a finite number $d \geq 1$, a subalgebra $\underline{C}'$ of ${\underline{B}'}^d$, and a surjective homomorphism I from $\underline{C}'$ to $\underline{A}'$. We claim that I is a d-dimensional interpretation of $\underline{A}$ in $\underline{B}$. All operations of $\underline{B}'$ preserve C (viewed as a d-ary relation over $\underline{B}$) since $\underline{C}'$ is a substructure of ${\underline{B}'}^d$. By Theorem 3.1.1, this implies that C has a definition in $\underline{B}$, which becomes the domain formula of the interpretation. Since I is an algebra homomorphism, the kernel K of I is a congruence of $\underline{C}'$. It follows that K, viewed as a $2d$-ary relation over B, is preserved by all operations from $\underline{B}'$. Theorem 3.1.1 implies that K has a definition in $\underline{B}$. This definition becomes the interpreting formula of the equality relation on A.

To see that I is a full interpretation, let $R \subseteq A^k$ be a relation of $\underline{A}$, let τ be the signature of $\underline{B}'$, and let $f \in \tau$ be arbitrary. By assumption, $f^{\underline{A}'}$ preserves R. Therefore, $f^{\underline{B}'}$ preserves $I^{-1}(R)$. Hence, all automorphisms of $\underline{B}$ preserve $I^{-1}(R)$, and because $\underline{B}$ is ω-categorical, the relation $I^{-1}(R)$ has a definition in $\underline{B}$ (Theorem 3.1.1), which becomes the interpreting formula for $R(x_1, \ldots, x_k)$. We have verified that I is an interpretation of $\underline{A}$ in $\underline{B}$. To see that I is a full interpretation, let $R \subseteq A^k$ be a relation such that $I^{-1}(R)$ is definable in $\underline{B}$. Then $I^{-1}(R)$ is preserved by $\mathrm{Gr}(\underline{B}')$ and R is preserved by $\mathrm{Gr}(\underline{A}')$. By assumption $\overline{\mathrm{Gr}(\underline{A}')} = \mathrm{Aut}(\underline{A})$, and hence R is preserved by all automorphisms of $\underline{A}$ and definable in $\underline{A}$ by Theorem 3.1.1. □

The following corollary is a direct consequence of Theorem 5.3.2 and Theorem 5.2.1.

COROLLARY 5.3.3. *Let $\underline{B}$ be a countable ω-categorical, with $|B| \geq 2$, and $\underline{A}$ an arbitrary structure. Then $\underline{A}$ has a full interpretation in $\underline{B}$ if and only if $\underline{A}$ is at most countable ω-categorical and there exists a continuous homomorphism from* $\mathrm{Aut}(\underline{B})$ *to* $\mathrm{Aut}(\underline{A})$ *whose image is dense in* $\mathrm{Aut}(\underline{A})$.

PROOF. Let $\underline{B}'$ be a $\mathrm{Aut}(\underline{B})$-set. By Theorem 5.3.2, there is a full interpretation of $\underline{A}$ in $\underline{B}$ if and only if there is an $\underline{A}' \in \mathrm{HSP}^{\mathrm{fin}}(\underline{B}')$ such that $\overline{\mathrm{Gr}(\underline{A}')} = \mathrm{Aut}(\underline{A})$. Corollary **??** shows that this is the case if and only if there exists a continuous homomorphism from $\mathrm{Aut}(\underline{B})$ to $\mathrm{Aut}(\underline{A})$ whose image is dense in $\mathrm{Aut}(\underline{A})$. □

REMARK 5.3.4. *In Corollary 5.3.3 we cannot in general require surjectivity of the homomorphism (instead of requiring that the image being dense) as we have seen in Example 70.*

5.4. Topological Isomorphism and Bi-interpretability

We continue to state some consequences of the topological Birkhoff theorem.

THEOREM 5.4.1. *Let $\underline{C}, \underline{D}$ be ω-categorical structures with $|C|, |D| \geq 2$. Them* $(1) \Leftrightarrow (2) \Leftrightarrow (3)$*:*

(1) $\mathrm{Aut}(\underline{C})$ *and* $\mathrm{Aut}(\underline{D})$ *are isomorphic as topological groups.*
(2) *There is an* $\mathrm{Aut}(\underline{C})$*-set* $\underline{C}'$ *and an* $\mathrm{Aut}(\underline{D})$*-set* $\underline{D}'$ *such that*
$$\mathrm{HSP}^{\mathrm{fin}}(\underline{C}') = \mathrm{HSP}^{\mathrm{fin}}(\underline{D}').$$
(3) $\underline{C}$ *and* $\underline{D}$ *are bi-interpretable.*

PROOF. The equivalence between (1) and (2) follows from Theorem 5.2.1. For the implication from (2) to (3), we assume that there is a $d_1 \geq 1$, a substructure $\underline{S}_1$ of $\underline{C}^{d_1}$, and a surjective homomorphism h_1 from $\underline{S}_1$ to $\underline{D}$. Moreover, we assume that there is a $d_2 \geq 1$, a subalgebra $\underline{S}_2$ of $\underline{D}^{d_2}$, and a surjective homomorphisms h_2 from $\underline{S}_2$ to $\underline{C}$. The proof of Theorem 5.3.2 shows that h_1 is an interpretation of $\underline{D}$ in $\underline{C}$ and h_2 is an interpretation of $\underline{C}$ in $\underline{D}$. Because the statement is symmetric it suffices to show that the (graph of the) function $h_1 \circ h_2 \colon (S_2)^{d_1} \to D$ defined by
$$(y_{1,1}, \ldots, y_{1,d_2}, \ldots, y_{d_1,1}, \ldots, y_{d_1,d_2}) \mapsto h_1(h_2(y_{1,1}, \ldots, y_{1,d_2}), \ldots, h_2(y_{d_1,1}, \ldots, y_{d_1,d_2}))$$
is definable in $\underline{D}$. Theorem 3.1.1 asserts that this is equivalent to showing that $h_1 \circ h_2$ is preserved by $\mathrm{Aut}(\underline{D}) = \mathrm{Gr}(\underline{D}')$. So let b be an element of $(S_2)^{d_1}$. Then indeed
$$\begin{aligned} f^{\underline{D}'}((h_1 \circ h_2)(b)) &= h_1\big(f^{\underline{C}'}(h_2(b_1)), \ldots, f^{\underline{C}'}(h_2(b_{d_1}))\big) \\ &= h_1\big(h_2(f^{\underline{D}'}(b_1)), \ldots, h_2(f^{\underline{D}'}(b_{d_1}))\big) \\ &= (h_1 \circ h_2)(f^{\underline{D}'}(b)) \; . \end{aligned}$$
For the implication from (3) to (1), suppose that $\underline{C}$ and $\underline{D}$ are bi-interpretable via an interpretation $I_1 \colon C^{d_1} \to D$ and $I_2 \colon D^{d_2} \to C$. Let $\underline{A}'$ be an $\mathrm{Aut}(\underline{A})$-set. As we have seen in the proof of Theorem 5.3.2, the domain S_1 of I_1 induces a structure $\underline{S}_1$ in $(\underline{A}')^{d_1}$ and I_1 is a surjective homomorphism from $\underline{S}_1$ onto an $\mathrm{Aut}(\underline{D})$-set $\underline{D}'$ with the same signature τ as $\underline{C}'$. Similarly, the domain S_2 of I_2 induces in $(\underline{D}')^{d_2}$ a structure $\underline{S}_2$ and I_2 is a homomorphism from $\underline{S}_2$ onto an $\mathrm{Aut}(\underline{C})$-set $\underline{A}''$ with the same signature as $\underline{D}'$.

We claim that $\mathrm{HSP}^{\mathrm{fin}}(\underline{C}') = \mathrm{HSP}^{\mathrm{fin}}(\underline{D}')$. The inclusion '$\supseteq$' is clear since $\underline{D}' \in \mathrm{HSP}^{\mathrm{fin}}(\underline{C}')$. For the reverse inclusion it suffices to show that $\underline{C}' = \underline{A}''$ since $\underline{A}'' \in \mathrm{HSP}^{\mathrm{fin}}(\underline{D}')$. Let $f \in \tau$; we show that $f^{\underline{C}'} = f^{\underline{A}''}$. Let $a \in C$. Since $I_2 \circ I_1$ is surjective onto C, there are $c = (c_{1,1}, \ldots, c_{d_1,d_2}) \in C^{d_1 d_2}$ such that $a = I_2(I_1(c))$. Then
$$\begin{aligned} f^{\underline{A}''}(a) &= f^{\underline{A}''}(I_2 \circ I_1(c)) \\ &= I_2\big(f^{\underline{D}'}(I_1(c_{1,1}, \ldots, c_{d_1,1})), \ldots, f^{\underline{D}'}(I_1(c_{1,d_2}, \ldots, c_{d_1,d_2}))\big) \\ &= I_2 \circ I_1(f^{\underline{C}'}(c)) \\ &= f^{\underline{C}'}(I_2 \circ I_1(c)) = f^{\underline{C}'}(a) \end{aligned}$$
where the second and third equations hold since I_2 and I_1 are algebra homomorphisms, and the fourth equation holds because $f^{\underline{C}'}$ preserves $h_2 \circ h_1$, because $I_2 \circ I_1$ is homotopic to the identity. $\square$

EXAMPLE 75. The structures
$$\underline{C} := (\mathbb{N}^2; \{(u_1, u_2), (v_1, v_2) \mid u_1 = v_1\}) \quad \text{and} \quad \underline{D} := (\mathbb{N}; =)$$
are mutually interpretable, but *not* bi-interpretable. To see this, observe that $\mathrm{Aut}(\underline{C})$ has a proper non-trivial closed normal subgroup $\underline{N}$. To specify $\underline{N}$, let $P_i := \{(u_1, u_2) \mid$

$u_1 = i\}$ for $i \in \mathbb{N}$. Then $\mathrm{Aut}(\underline{C}, P_1, P_2, \dots)$ is a non-trivial ($\mathrm{Aut}(\underline{C})/\underline{N}$ is isomorphic to $\mathrm{Aut}(\underline{D})$) proper closed subgroup, and it can be verified that $\underline{N}$ is normal (see Proposition 4.6.1), whereas $\mathrm{Aut}(\underline{D})$, the symmetric permutation group of a countably infinite set, has no proper non-trivial closed normal subgroups (it has exactly three proper non-trivial normal subgroups [**141**], none of which is closed). $\triangle$

We also mention that Theorem 5.4.1 in combination with Proposition 3.6.5 shows that for every ω-categorical structure $\underline{B}$, having essentially infinite signature only depends on the automorphism group of $\underline{B}$ viewed as a topological group.

Example 76. Let $\underline{B}$ be the graph with domain $\binom{\mathbb{N}}{k}$ and the edge relation

$$E^{\underline{B}} = \big\{(S,T) \mid |S \cap T| = 1\big\}.$$

We claim that $\underline{B}$ and $(\mathbb{N}; =)$ are bi-interpretable. Let I be the k-dimensional interpretation of $\underline{B}$ in $(\mathbb{N}; =\})$ whose domain is $\mathbb{N}^k_{\neq}$ and which maps $(x_1, \dots, x_k) \in \mathbb{N}^k_{\neq}$ to $\{x_1, \dots, x_k\}$. Clearly, the relations

$$\begin{aligned}
I^{-1}(B) &= \mathbb{N}^k_{\neq} \\
I^{-1}(=_B) &= \big\{(x,y) \mid x,y \in \mathbb{N}^k_{\neq} \text{ and } \{x_1,\dots,x_n\} = \{y_1,\dots,y_n\}\big\} \\
I^{-1}(E^{\underline{B}}) &= \big\{(x,y) \mid x,y \in \mathbb{N}^k_{\neq} \text{ and } \{x_1,\dots,x_n\} \cap \{y_1,\dots,y_n\} = 1\big\}
\end{aligned}$$

are definable in $(\mathbb{N}; =)$.

Let J be the 2-dimensional interpretation of $(\mathbb{N}; =)$ in $\underline{B}$ whose domain is $E^{\underline{B}}$ and which maps $(S,T) \in E^{\underline{B}}$ to $S \cap T$. Let $n \in \mathbb{N}$ be at least 2. Let $m := max(n, k^2 - k + 2)$. Let $R_n \subseteq B^n$ be the relation with the definition

$$R_n(U_1, \dots, U_n) :\Leftrightarrow \exists U_{n+1}, \dots, U_m \bigwedge_{i,j \in \{1,\dots,m\}, i \neq j} E(U_i, U_j).$$

Claim. If $R_m(U_1, \dots, U_m)$ then $|U_1 \cap \dots \cap U_n| = 1$. To see this, choose $u_1 \in \mathbb{N}$ such that $\mathcal{S}_1 := \{U \subseteq \{U_1, \dots, U_m\} \mid u_1 \in U\}$ is maximal. If $|\mathcal{S}_1| = m$, then in particular $U_1 \cap \dots \cap U_n = \{u_1\}$ and then there is nothing to be shown. Otherwise, if $U_j \notin \mathcal{S}_1$, then $|\mathcal{S}_1| \leq k$ because U_j has to intersect every $V \in S$ in an element u_V which is distinct from u_1. If $V, V' \in S$ are distinct then $u_V \neq u_{V'}$ because $V \cap V' = \{u_1\}$. Hence, $|U_j| = k$ implies that $|\mathcal{S}_1| \leq k$. We now choose $u_2 \in \mathbb{N}$ such that $\mathcal{S}_2 := \{U \subseteq \{U_1, \dots, U_n\} \setminus \mathcal{S}_1 \mid u_2 \in U\}$ is maximal. We claim that $|\mathcal{S}_2| \leq k-1$. To see this, pick any $V \in \mathcal{S}_1$. Then every $U \in \mathcal{S}_2$ must intersect V in a distinct point which is different from u_1. We may now repeat the argument $k-1$ times, and obtain that $m \leq k + (k-1) + \dots + (k-1) = k(k-1)+1$, which is a contradiction to our choice of $m \geq k^2 - k + 2$.

Note that if $(S,T), (U,V) \in E^{\underline{B}}$, then $S \cap T = U \cap V$ if and only if

$$\exists P, Q\big(R_4^{\underline{B}}(U,V,P,Q) \wedge R_4^{\underline{B}}(S,T,P,Q)\big).$$

It follows that the relations

$$\begin{aligned}
J^{-1}(\mathbb{N}) &= E^{\underline{B}} \\
J^{-1}(=_{\mathbb{N}}) &= \big\{\big((S,T),(U,V)\big) \mid (S,T),(U,V) \in E^{\underline{B}}, S \cap T = U \cap V\big\}
\end{aligned}$$

are definable in $\underline{B}$.

Finally, note that for all $(x_1, \dots, x_k), (y_1, \dots, y_k) \in \mathbb{N}^k_{\neq}$ and $z \in \mathbb{N}$ we have

$$J\big(I(x_1, \dots, x_k), I(y_1, \dots, y_k)\big) = z$$

if and only if $z = \{x_1, \dots, x_k\} \cap \{y_1, \dots, y_k\}$, which is definable in $(\mathbb{N}; =)$.

Moreover, for all $(S_1, T_1), \dots, (S_k, T_k) \in E^{\underline{B}}$ and $V \in B$ we have

$$I\big(J(S_1, T_1), \dots, J(S_k, T_k)\big) = V$$

if and only if there are $U_1, \dots, U_k$ such that $(S_i, T_i, U_i, V) \in R_4^{\underline{B}}$ for all $i \in [k]$ and $(U_i, U_j) \notin E^{\underline{B}}$ for all $\{i, j\} \in \binom{[k]}{2}$, and hence is definable in $\underline{B}$. This concludes the proof of the claim above.

Then Theorem 5.4.1 implies that there exists an injective (continuous) homomorphism from $\mathrm{Aut}(\mathbb{N}; =)$ to $\mathrm{Aut}(\underline{B})$. Thus, the action of S_ω on $\binom{\mathbb{N}}{k}$ is faithful, as claimed earlier in Example 11. The same argument works for any permutation group instead of S_ω. △

Exercises.

(113) Show that the line graph of an undirected graph G has an interpretation in G.

(114) Show that every finite structure has an interpretation in every structure with at least two elements.

(115) Show that the automorphism group of infinitely many disjoint copies of the 2-element clique K_2 is not topologically isomorphic to infinitely many disjoint copies of the three-element clique K_3.

CHAPTER 6

Reconstruction of Topology and Automatic Continuity

The question what information about a structure can be recovered from its automorphism group when considered as an abstract group, has long attracted the attention of research in model theory and the theory of infinite permutation groups; a very incomplete list of references is [**6**, **47**, **50**, **52**, **54**, **60**, **69**, **74**, **88**, **104**, **130**, **138**, **139**, **158**, **162**].

6.1. Reconstruction Notions

We study the question whether we can reconstruct the topology of closed subgroups of $\mathrm{Sym}(\mathbb{N})$ from the abstract group.

DEFINITION 6.1.1. *Let $\underline{G}$ be a closed subgroup of* $\mathrm{Sym}(\mathbb{N})$. *We say that*

- $\underline{G}$ *is* reconstructible *(or that $\underline{G}$ has* reconstruction*) iff for every other closed subgroup $\underline{H}$ of* $\mathrm{Sym}(\mathbb{N})$, *if there exists an isomorphism between $\underline{H}$ and $\underline{G}$, then there also exists a group isomorphism between $\underline{H}$ and $\underline{G}$ which is a homeomorphism;*
- $\underline{G}$ *has* automatic homeomorphicity (AH) *iff every group isomorphism between $\underline{G}$ and a closed subgroup of* $\mathrm{Sym}(\mathbb{N})$ *is a homeomorphism;*
- $\underline{G}$ *has* automatic continuity (AC) *iff every homomorphism from $\underline{G}$ to* $\mathrm{Sym}(\mathbb{N})$ *is continuous.*

Obviously, automatic homeomorphicity implies reconstruction. Less obviously, we will later see in Proposition 6.3.18 that automatic continuity implies automatic homeomorphicity.

EXAMPLE 77. We have seen an example of a closed oligomorphic permutation group without automatic continuity in Example 67 (clearly, there are closed subgroups of $\mathrm{Sym}(\mathbb{N})$ that are isomorphic to $\mathbb{Z}_2$ with the discrete topology). The example still has reconstruction; this can be shown using the results of Rubin (see Remark 5.4.3 in [**110**]). △

A more involved example of a closed oligomorphic permutation group without reconstruction has been found by Evans and Hewitt [**54**]. In the next section we present a fundamental concept for proving automatic continuity (and hence automatic homeomorphicity and reconstruction), the so-called small index property. A very powerful method to show the small index property is the concept of *ample generics*, introduced in Section 6.4. To verify that an automorphism group has ample generics, we need to study the important combinatorial property of classes of finite structures, called *EPPA* (or the *Hrushovski property*; Section 6.5). The structure of this chapter is therefore organised along the following chain of implications.

	Reconstruction	
$\Leftarrow$	Automatic Homeomorphicity	
$\Leftarrow$	Automatic Continuity	(Proposition 6.3.18)
$\Leftrightarrow$	The Small Index Property	(Proposition 6.2.1)
$\Leftarrow$	Ample Generics	(Theorem 6.4.4)
$\Leftarrow$	EPPA / the Hrushovski property	(Section 6.5)

For many automorphism groups (e.g., for the automorphism group of the Rado graph), this chain of implications is to the best of my knowledge the only known way to prove reconstruction.

6.2. The Small Index Property

Recall that a subgroup of $\mathrm{Sym}(\mathbb{N})$ is open if it contains the stabiliser G_a for some $a \in \mathbb{N}^n$, $n \in \mathbb{N}$ (Lemma 4.4.1). Clearly, these groups have countable index, so all open subgroups of $\mathrm{Sym}(\mathbb{N})$ have countable index. A topological group $\underline{G}$ has the *small index property (SIP)* if the converse holds, i.e., if every subgroup of $\underline{G}$ of at most countable index is open.

Proposition 6.2.1 (Folklore). *Let $\underline{G}$ be a closed subgroup of* $\mathrm{Sym}(\mathbb{N})$. *Then $\underline{G}$ has automatic continuity if and only if it has the small index property.*

Proof. Let $\underline{G}$ be a topological group with automatic continuity, and let $\underline{U}$ be a subgroup of $\underline{G}$ of at most countable index. We have to show that $\underline{U}$ is open. Let $\xi\colon \underline{G} \to \underline{G}/U$ be the action of $\underline{G}$ on the left cosets of $\underline{U}$ in $\underline{G}$ by left translation (Example 12), where $\underline{G}/U$ is equipped with the discrete topology. By automatic continuity, ξ is continuous. In particular, the pre-image of the open set $\{\alpha \in \mathrm{Sym}(\underline{G}/U) \mid \alpha(U) = U\}$ is open. But this pre-image is precisely U, which proves the SIP.

Now suppose that $\underline{G}$ has the SIP, and let $\xi\colon \underline{G} \to S_\omega$ be a homomorphism. We show that for every basic open set $U := S(a, b) \cap \xi(G)$ for $a, b \in \mathbb{N}^n$, $n \in \mathbb{N}$, the set $\xi^{-1}(U)$ is open, too. Writing S_a for the stabiliser subgroup $\{\alpha \in \xi(G) \mid \alpha(a) = a\}$, we have $U = \beta \mathrm{S}_a$ for $\beta \in U$. Since ξ is a homomorphism, $\xi^{-1}(U) = \xi^{-1}(\beta)\xi^{-1}(\underline{S}_a)$. The subgroup $\underline{S}_a$ of $\xi(G)$ has countable index, and therefore $\xi^{-1}(\underline{S}_a)$ is a subgroup of $\underline{G}$ of countable index, too. By assumption, $\xi^{-1}(\underline{S}_a)$ is open, and since multiplication by $\xi^{-1}(\beta)$ is continuous, $\xi^{-1}(\beta)\xi^{-1}(\underline{S}_a) = \xi^{-1}(U)$ is open, which establishes continuity of ξ. □

The small index property has been verified for the following groups:

(1) $\mathrm{Sym}(\mathbb{N})$ [**47**, **137**, **142**];
(2) the automorphism groups of countable vector spaces over finite fields [**52**];
(3) $\mathrm{Aut}(\mathbb{Q}; <)$ and the automorphism group of the atomless Boolean algebra [**158**];
(4) the automorphism group of the ω-categorical dense semi-linear order giving rise to a meet-semilattice [**51**];
(5) the automorphism group of the countable random graph [**74**] (see Example 29);
(6) all automorphism groups of ω-categorical ω-stable structures [**74**];
(7) the automorphism groups of the Henson graphs [**69**] (see Example 31).

On the other hand, the small index property is not known for the automorphism groups of the countable universal homogeneous tournament, the countable universal poset, or the homogeneous universal permutation (see [**110**]).

6.3. Consistency of Automatic Continuity

In this section we prove that it is consistent with Zermelo-Fraenkel set theory and the axiom of dependent choice (see Appendix A.2) that *every* closed subgroup of $\mathrm{Sym}(\mathbb{N})$ has then SIP and automatic continuity (Theorem 6.3.17) and hence automatic homeomorphicity (Proposition 6.3.18). We first introduce basic concepts from descriptive set theory; these concepts will be relevant again in Secrtion 6.4 and Section 7.2.1.

6.3.1. Nowhere dense and meager.

Let X be a topological space.

Definition 6.3.1. *A set $A \subseteq X$ is called* somewhere dense *if its closure contains a nonempty open set, and* nowhere dense *otherwise.*

The following is immediate from the definitions.

Proposition 6.3.2. *Let $A \subseteq X$. Then the following are equivalent.*

(1) *A is nowhere dense;*
(2) *For every nonempty open set $U \subseteq X$ there is a nonempty open set $V \subseteq U$ such that V and A are disjoint;*
(3) *$X \setminus \overline{A}$ is dense.*

Proof. (1) $\Rightarrow$ (3): Let U be an open subset of X. Since A is nowhere dense, $\bar{A}$ does not contain U. So $X \setminus \bar{A}$ contains an element of U. This shows that $X \setminus \bar{A}$ is dense.

(3) $\Rightarrow$ (2): Let $U \subseteq X$ be nonempty open. Since $X \setminus \bar{A}$ is dense there exists $x \in U \setminus \bar{A}$. So there exists a non-empty open set $V \subseteq U$ such that V and A are disjoint.

(2) $\Rightarrow$ (1). Let U be non-empty open. Then by assumption there is a nonempty open set $V \subseteq U$ which is disjoint from A. Hence, A cannot be dense in U, showing that A is nowhere dense. $\square$

Clearly, A is nowhere dense if and only if its closure $\overline{A}$ is nowhere dense, and subsets of nowhere dense sets are nowhere dense.

Definition 6.3.3. *Let X be a topological space. A set $A \subseteq X$ is* meager (in X)[1] *if it is a countable union of nowhere dense sets. Otherwise, A is called* non-meager[2] *The complement of a meager set is called* comeager.

Clearly, every subset of a meager set is meager, and every countable union of meager sets is meager.

Remark 6.3.4. *The intuition will be that meager sets are 'negligible', and correspondingly that comeager sets capture the notion of 'almost all'. This methaphor works particularly well if the topology is Baire, as we will see below (Section 6.3.2, Exercise 118).*

Example 78. The set of meager subsets of $\mathbb{R}$ with respect to the standard topology is strictly larger than the set of nowhere dense subsets of $\mathbb{R}$. Consider for example the set $\mathbb{Q}$ of rational numbers, which is a meager subset of $\mathbb{R}$, because it is a countable union of one-element sets (which are nowhere dense). However, the rational numbers are dense in $\mathbb{R}$ and therefore in particular somewhere dense. $\triangle$

Lemma 6.3.5. *A set is comeager if and only if it contains a countable intersection of dense open sets.*

[1] Sometimes, meager sets are also called *of first category.*
[2] Non-meager sets are also called *of second category.*

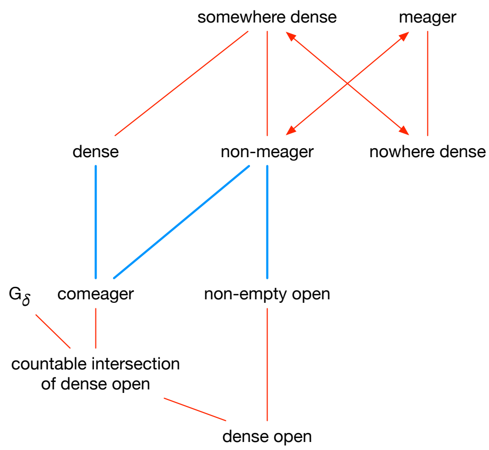


FIGURE 6.1. Properties of subsets of X from Section 6.3.1 and Section 6.3.2, ordered by inclusion; we assume that X is non-empty. Blue edges hold in Baire spaces. Double edges indicate negation.

PROOF. If $A \subseteq X$ is comeager, then $X \setminus A$ is meager and hence $A = \bigcap_{i \in \mathbb{N}} (X \setminus A_i)$ for nowhere dense sets A_i. Hence, $\bigcap_{i \in \mathbb{N}} (X \setminus \overline{A_i}) \subseteq A$, and $X \setminus \overline{A}_i$ is open and dense.

Conversely, suppose that A contains $\bigcap_i U_i$ where U_i is open and dense. Then $X \setminus A$ is contained in $X \setminus \bigcap_i U_i = \bigcup_i (X \setminus U_i)$. Note that $X \setminus U_i$ is nowhere dense, so $X \setminus A$ is meager since it is a subset of a meager set. Hence, A is comeager. □

COROLLARY 6.3.6. *The intersection of countably many comeager sets is comeager.*

6.3.2. Baire Spaces. Recall from Theorem 4.1.8. that a topological space X is called *Baire* if every countable intersection of open dense sets is dense.

EXAMPLE 79. The standard topology on $\mathbb{R}$ is Baire. This follows from Theorem 4.1.8 since $\mathbb{R}$ is Polish (Example 51). Similarly, the Baire space and the Cantor space (Example 53) are Baire. △

EXAMPLE 80. Clearly, the empty space is Baire. It is the only space which is Baire and meager (see Proposition 6.3.7 below). △

PROPOSITION 6.3.7. *Let X be a topological space. Then the following statements are equivalent:*

(1) *X is Baire.*
(2) *Every comeager set in X is dense.*
(3) *Every nonempty open set in X is non-meager.*

PROOF. (1) implies (2). Let $A \subseteq X$ be comeager. By Lemma 6.3.5, A contains a countable intersection of dense open sets, which is dense since X is Baire.

(2) implies (3). If U is meager, then $X \setminus U$ is comeager and by (2) it is dense. Hence, U is either empty or not open.

(3) implies (1). Let $A = \bigcap_{i \in \mathbb{N}} A_i$. If A is not dense, $X \setminus A$ contains a nonempty open set U. By (3), U is non-meager. If all of the A_i would be dense open, then $X \setminus A$ and hence U would be meager. □

Exercises.

(116) Let S be a countable Polish space.
Is S meager or non-meager?

(117) Show that $\mathbb{R} \setminus \mathbb{Q}$ is non-meager.

(118) Show that in non-empty Baire spaces comeager sets are non-meager.[3]

(119) Prove Lemma 6.3.6.

(120) Let $S_1, S_2, \dots$ be comeager subsets of S_ω. Show that $\cap_{i \in \mathbb{N}} S_i$ is non-empty.

(121) Show that a dense G_δ set (Definition 4.1.5) is comeager.

6.3.3. The Baire Property. Let X be a topological space (typically a Baire space).

DEFINITION 6.3.8. *A set $A \subseteq X$ is said to have the* Baire property[4] (with respect to X) *if for some open set $U \subseteq X$ the symmetric difference $U \Delta A$ between U and A is meager.*

EXAMPLE 81. Using the Axiom of Choice (AC), we construct a subset of $\mathbb{R}$ which does not have the Baire property. For $x, y \in \mathbb{R}$, define $x \sim y$ if $x - y \in \mathbb{Q}$; clearly, $\sim$ is an equivalence relation. Let $A \subseteq \mathbb{R}$ be such that it contains exactly one member from each $\sim$-class (here we use AC; such a set is called a *Vitali set*). Clearly, $\{A+q \mid q \in \mathbb{Q}\}$ forms a countable partition of $\mathbb{R}$. The map $x \mapsto x + q$ is a homeomorphism of $\mathbb{R}$, and hence the sets $A + q$ for $q \in \mathbb{Q}$ cannot be meager (otherwise, $\mathbb{R}$ would be meager, contrary to Theorem 4.1.8). Suppose for contradiction that A has the Baire property, and let U be an open set and M a meager set such that $U \Delta A = M$. Then $U \neq \emptyset$ since A is not meager, so there are $a, b \in \mathbb{R}$ such that $a < b$ and $(a, b) \subseteq U$. Let $r = b - a$ and let $x \in \mathbb{R}$ be such that $|x| < r$. Then $(a + x, b + x) \cap (a, b)$ is an open interval. Since $M + x \cup M$ is meager, so

$$S := ((a + x, b + x) \cap (a, b)) \setminus ((M + x) \cup M) \neq \emptyset.$$

Observe that S is contained in $(A + x) \cap A$. Hence, if we choose $x > 0$ to be rational we obtain a contradiction to the fact that the sets of the form $A + q$ for $q \in \mathbb{Q}$ are pairwise distinct. △

EXAMPLE 82. Non-principal ultrafilters $\mathcal{U}$ on $\mathbb{N}$, viewed as a subset of the Cantor space $2^{\mathbb{N}}$, do not have the Baire property. First note that $X \mapsto \mathbb{N} \setminus X$ is a homeomorphism of $2^{\mathbb{N}}$ that sends $\mathcal{U}$ to its complement $2^{\mathbb{N}} \setminus \mathcal{U}$. Hence, if $\mathcal{U}$ would be meager, then $2^{\mathbb{N}} \setminus \mathcal{U}$ would be meager, and hence $2^{\mathbb{N}}$ would be meager as a union of meager sets, in contradiction to the Cantor space being meager (this follows from Proposition 6.3.7 since the Cantor space is Baire; Example 79). For the same reason, $\mathcal{U}$ cannot be comeager. Note that the symmetric difference of a set in the non-principal ultrafilter $\mathcal{U}$ with a finite set must again be in $\mathcal{U}$. Then Theorem 8.46 in [**88**] implies that $\mathcal{U}$ cannot have the Baire property. △

[3] Note that this is certainly a property that we wish for the metaphor from Remark 6.3.4.

[4] As pointed out in [**118**], instead of the expression 'A has the Baire property' it would be more suggestive to call A *Baire measurable*, to indicate an analogy with measure theory. But we are stuck with the standard terminology here.

Let $U, A \subseteq X$. We say that A is *meager in* U if $A \cap U$ is meager in the subspace U of X, and otherwise that A is *non-meager in* U. If $U = A$ we then also say that A is *non-meager in its relative topology.* Note that if U is open, then A is meager in U if and only if $A \cap U$ is meager in X. The assumption that U is open is necessary, as demonstrated by $\mathbb{R} \subseteq \mathbb{C}$, because $\mathbb{R}$ is non-meager in its relative topology, but $\mathbb{R}$ is nowhere dense in $\mathbb{C}$. Similarly, we say that A is *comeager in* U if $U \setminus A$ is meager in X.

PROPOSITION 6.3.9 ('Localisation'). *Let X be a topological space and suppose that $A \subseteq X$ has the Baire property. Then A is meager or there is a non-empty open set $U \subseteq X$ such that A is comeager in U. If X is a Baire space, then the two alternatives are mutually exclusive.*

PROOF. Let $U \Delta A = M$ where U is open and M is meager. If U is empty then $A = M$ is meager. So suppose that U is non-empty. Since $U \setminus A \subseteq M$ is meager in X, it is also meager in U, and hence A is comeager in U.

For the second statement, suppose that X is a Baire space, that $A \subseteq X$ is meager, and that $U \subseteq X$ is open such that A is comeager in U. Then $C := U \setminus A$ is meager in X, and hence $U \subseteq A \cup C$ is meager. By item (3) of Proposition 6.3.7 we can therefore conclude that $U = \emptyset$. This shows that the second alternative does not apply. $\square$

It is sometimes convenient to use the following logical notation:

$$\forall^* x. A(x)$$

stands for A is comeager, and is pronounced 'A holds for comeagerly many x'. Similarly,

$$\exists^* x. A(x)$$

stands for A is non-meager, and pronounced 'A holds for non-meagerly many x'. The notation has the obvious extension to localised expressions of the form

$$\forall^* x \in U. A(x) \quad \text{and} \quad \exists^* x \in U. A(x)$$

The following result is sometimes called the *Fubini theorem for category* because of the analogy to Fubini's theorem about the order of integration in double integrals in analysis.

THEOREM 6.3.10 (Kuratowski-Ulam). *Let X, Y be second-countable and suppose that $A \subseteq X \times Y$ has the Baire property. Then*

$$\exists^*(x,y). A(x,y) \quad \Leftrightarrow \quad \forall^* x \exists^* y. A(x,y) \quad \Leftrightarrow \quad \forall^* y \exists^* x. A(x,y) \tag{13}$$

$$\textit{and} \quad \forall^*(x,y). A(x,y) \quad \Leftrightarrow \quad \forall^* x \forall^* y. A(x,y) \quad \Leftrightarrow \quad \forall^* y \forall^* x. A(x,y) \tag{14}$$

PROOF. Clearly, (13) and (14) are equivalent by taking complements. Moreover, the two equivalences in (13) are symmetric, so we only show the first equivalence.

If $S \subseteq X \times Y$ and $x \in X$, then S_x denotes the set $\{y \in Y \mid (x,y) \in S\}$. To show the forward implication in (13), we first prove that if a set $B \subseteq X \times Y$ is nowhere dense, then $\forall^* x (B_x$ is nowhere dense in $Y)$. We may assume that B is closed, since replacing B by its closure can only increase the sets B_x; hence, if we can prove that for comeagerly many $x \in X$ some supersets of the B_x are nowhere dense in Y, then the same holds for the sets B_x. Let $U := (X \times Y) \setminus B$. It suffices to show that $\forall^* x (U_x$ is dense$)$, because if U_x is dense then $\overline{B}_x = B_x = X \setminus U_x$ does not contain a non-empty open set and hence $\forall^*(B_x$ is nowhere dense$)$. Let $\{V_1, V_2, \dots\}$ be a basis of non-empty open sets for Y. Then

$$U_n := \{x \in X \mid \exists y \in V_n. (x,y) \in U\}$$

is the projection of an open set and hence open. Moreover, U_n is dense in X: if $G \subseteq X$ is nonempty open, then $U \cap (G \times V_n) \neq \emptyset$, because otherwise B would contain the nonempty open set $G \times V_n$, contrary to being nowhere dense. If $x \in \bigcap_{n\in\mathbb{N}} U_n$, then for every $n \in \mathbb{N}$ we have $U_x \cap V_n \neq \emptyset$, i.e., U_x is dense. We are done because $\bigcap_{n\in\mathbb{N}} U_n$ is comeager by Corollary 6.3.6.

Let A be meager. Then $A = \bigcup_{n\in\mathbb{N}} B_n$ where B_n is nowhere dense. For each $n \in \mathbb{N}$, let X_n be the set of all $x \in X$ such that $(B_n)_x$ is nowhere dense; by what we have seen above, X_n is comeager. By Corollary 6.3.6, we have that $\bigcap_{n\in\mathbb{N}} X_n$ is comeager. It follows that

$$\forall^* x \; \forall n \in \mathbb{N} \big((B_n)_x \text{ is nowhere dense} \big)$$

which implies that

$$\forall^* x (A_x \text{ is meager})$$

since $A_x = \bigcup_{n\in\mathbb{N}} (B_n)_x$.

We now prove the converse implication in (13). Suppose that A is non-meager. Since A has the Baire Property, there exists an open set U and a meager set M such that $A = U \Delta M$. Since A is non-meager, U is non-meager. By the definition of the product topology, U is a union of sets of the form $S \times T$ for $S \subseteq X$ open and $T \subseteq Y$ open. Since X and Y are second-countable, we may assume that this union is countable. Hence, there must exist some open $S \subseteq X$ and some open $T \subseteq Y$ such that $S \times T \subseteq U$ and $S \times T$ is non-meager (otherwise U would be meager).

Then both S and T are non-meager: otherwise, if S is meager and $S = \bigcup_{n\in\mathbb{N}} F_n$ where F_n is nowhere dense, then $S \times T = \bigcup_n (F_n \times T)$, so it is enough to show that $F_n \times T$ is nowhere dense. Indeed, since $U = X \setminus \overline{F_n}$ is dense open in X (Proposition 6.3.2), we have that $U \times Y$ is dense open in $X \times Y$. Since $X \times Y \setminus \overline{F_n \times T}$ contains $U \times Y$, it is dense, and hence $F_n \times T$ is nowhere dense.

For every $x \in S$ we have $A_x \supseteq V \setminus M_x$. By the forward implication, M_x is meager for comeagerly many $x \in X$. Therefore, A_x is non-meager for comeagerly many $x \in S$. This implies that A_x is non-meager for non-meagerly many $x \in X$, i.e., $\exists^* x \exists^* y. A(x, y)$. $\square$

Remark 6.3.11. *Note that the assumption that A has the Baire property was only used in the proof of the converse implication in* (13)*; the forward implication holds in general.*

The following is beyond the scope of this text, but we need it in the important Proposition 6.3.18 below.

Theorem 6.3.12 (Lusin-Sierpiński; see Kechris [**88**] (21.6)). *Let X be Polish, let Y be a metric space, and let $f \colon X \to Y$ be continuous. Then the image of every open set in X under f has the Baire property in Y.*

The *axiom of dependent choices (DC)* is a weak form of the axiom of choice (AC) that is still sufficient to develop most of real analysis (see Appendix A.2). Over Zermelo-Fraenkel set theory (ZF), the Axiom of Dependent Choices is equivalent to the version of the Baire Category Theorem (Theorem 4.1.8) where we only require that S is completely metrisable (instead of Polish). The relevance of the following for reconstruction has been pointed out by Lascar [**104**, Theorem 2.7].

Theorem 6.3.13 (Solovay [**150**], Shelah [**143**]). *If ZF is consistent, then it is consistent with ZF+DC that every subset of $\mathbb{R}$ has the Baire property.*

Recall that the irrational numbers are homeomorphic to the Baire space (Theorem 4.1.3). Hence, we obtain the following.

Corollary 6.3.14. *If ZF is consistent, then it is consistent with ZF+DC that every subset of* $\mathbb{N}^{\mathbb{N}}$ *has the Baire property.*

6.3.4. The Baire property and permutation groups. We now discuss the concepts from the previous section (nowhere dense, meager, Baire) in the context of permutation groups. Every closed subgroup of S_ω is Polish (Theorem 4.3.2) and hence Baire (Theorem 4.1.8). In particular, G itself is non-meager; this follows from Proposition 6.3.7 (3) since G is non-empty.

Lemma 6.3.15. *Let* $\underline{G}$ *be a Polish group. If* $\underline{U}$ *is a subgroup of* $\underline{G}$ *of countable index, then* $\underline{U}$ *is non-meager.*

Proof. If $\underline{U}$ is meager, then all cosets of $\underline{U}$ are meager, too. The set G is the union of all the cosets of $\underline{U}$, but is non-meager. Since a countable union of a meager set is meager, $\underline{U}$ must have uncountable index. □

The following lemma is sometimes called Pettis' lemma; it is Lemma 2.6 in [**104**] and a consequence of Theorem 2.3.2 in [**60**] or Lemma 9.9 in [**88**].

Lemma 6.3.16. *Let* $\underline{G}$ *be a closed subgroup of* Sym($\mathbb{N}$) *and let* $\underline{H}$ *be a subgroup of* $\underline{G}$*. Then* $\underline{H}$ *is meager, open, or for every non-empty open* $U \subseteq G$ *the set* $U \setminus H$ *is non-meager (in particular,* $G \setminus H$ *is dense). Consequently, if* H *has the Baire property then* H *is either meager or open.*

Proof. The second statement clearly follows from the first: if U has the Baire Property then there exists an open $U \subseteq \text{Sym}(\mathbb{N})$ such that $H \Delta U$ is meager, and in particular $U \setminus H$ is meager, so either U is empty and H is meager, or H is open by the first statement.

To show the first statement, let $\underline{H}$ be non-meager and suppose that there exists a non-empty open $U \subseteq G$ such that $U \setminus H$ is meager. Then U contains fG_a for some $f \in G$, $a \in \mathbb{N}^n$, and $n \in \mathbb{N}$.

Claim. The subgroup $H \cap G_a$ of G_a is comeager in G_a. The set fG_a is non-meager. If $fG_a \subseteq U \setminus H$, then $U \setminus H$ would not be meager, contrary to our assumptions. Hence, there exists exists an $h \in H \cap fG_a$. Since $fG_a \setminus H \subseteq U \setminus H$ is meager, we have that $h^{-1}fG_a \setminus h^{-1}H$ is meager. But $h^{-1}H = H$ and $h^{-1}fG_a = G_a$ and hence $G_a \setminus H$ is meager, which proves the claim.

The claim implies that all cosets of $H \cap G_a$ in G_a are comeager. Since intersections of comeager sets are comeager (Corollary 6.3.6) and in particular non-empty, there can be only one coset. Therefore, H contains G_a and hence is open by Lemma 4.4.1. □

Theorem 6.3.17 (Lascar [**104**]). *Assume that ZF is consistent. Then it is consistent with ZF+DC that every closed subgroup* G *of* S_ω *the SIP and hence has automatic continuity.*

Proof. Assume that every subset of G has the Baire property; this is consistent with ZF+DC by Corollary 6.3.14. Let $\underline{U}$ be a subgroup of $\underline{G}$ of countable index. Then U cannot be meager by Lemma 6.3.15, so it must be open by Lemma 6.3.16. Automatic continuity follows by Proposition 6.2.1. □

So we need the Axiom of Choice to find closed subgroups of S_ω without automatic continuity. The following statement demonstrates that the machinery of this section can also be used for permutation groups to prove statements in ZF.

Proposition 6.3.18 (Corollary 2.8 in [**104**]). *Let* ϕ *be an continuous isomorphism between closed subgroups of* S_ω*. Then* ϕ *is a homeomorphism.*

Proof. Let $\phi\colon G \to H$ for closed $\underline{G}, \underline{H} \leq S_\omega$. Let $\underline{U}$ be an open subgroup of $\underline{G}$. We have to show that $\phi(U)$ is open in H (Remark 4.3.1). Theorem 6.3.12 asserts that $\phi(U)$ has the Baire property. Recall that $\underline{U}$ has countable index in $\underline{G}$, so $\phi(U)$ has countable index in H. Hence, it cannot be meager according to Lemma 6.3.15 and hence must be open because of Lemma 6.3.16. □

Remark 6.3.19. *Proposition 6.3.18 shows that automatic continuity of closed subgroups of S_ω is a property of the abstract group in the sense that if two closed subgroups of S_ω are isomorphic as abstract groups, and one has automatic continuity, then so has the other.*

Exercises.

(122) Prove the statement in Remark 6.3.19.

6.4. Ample Generics

In this section we introduce a powerful method for proving that certain automorphism groups G have the small index property (without any set-theoretic assumptions).

6.4.1. Generic automorphisms. Intuitively, a *generic element* of an automorphism group G is one that *looks typical*; the notion of automorphisms *looking the same* is captured by conjugation (Example 13).

Definition 6.4.1 (Generic elements). *An element of a Polish group G is called* generic *if it lies in a comeager orbit with respect to the action of G on G by conjugation.*

Note that there is always at most one comeager orbit, since the intersection of two comeager sets is comeager (Corollary 6.3.6) and in particular non-empty. The following example is taken from Truss [**159**].

Example 83. S_ω has a generic element. In fact, $g \in S_\omega$ is generic if and only if g has no infinite cycles and infinitely many cycles of any given finite length. To see this, let C be the set of permutations of the specified cycle type. Clearly, all elements in C are conjugate in S_ω; moreover, conjugate elements have the same cycle type (see Exercise 12). Hence, for the backwards implication we have to prove that C is comeager. Observe that for every $x \in \mathbb{N}$

$$D_x = \{g \in S_\omega \mid x \text{ lies in a finite cycle of } g\}$$

is dense and open. Similarly, for all $m, n \geq 1$ the set

$$D(m,n) := \{g \in S_\omega \mid g \text{ has at least } m \text{ cycles of length } n\}$$

is dense and open. Now

$$C = \bigcap_{x \in \mathbb{N}} D_x \cap \bigcap_{m,n \geq 1} D(m,n)$$

and it follows that C is comeager. For the forward implication, recall that there is at most one comeager orbit, which shows that the elements of C are the only generic elements of S_ω. △

Example 84. The automorphism group G of an equivalence relation on a countably infinite set with two infinite classes has *no* generic elements. Clearly, G has a normal subgroup of index two. But Polish groups with a generic element have no proper normal subgroups of countable index. To see this, let C be the conjugacy class of the generic element g. Suppose for contradiction that G has a proper normal subgroup N of countable index. If $g \in N$, then by the normality of N we have $C \subseteq N$, so N is comeager. Let $f \in G \setminus N$. Then fN is comeager and disjoint from N, contradicting the fact that the intersection of two comeager sets is comeager (Corollary 6.3.6). Hence, $C \cap N = \emptyset$ and N is meager. Since every coset of N is meager, G is a countable union of meager sets and hence meager, a contradiction. $\triangle$

EXAMPLE 85. The automorphism group of $\mathrm{Aut}(\mathbb{Q};<)$ has generic elements. They can be described explicitly; this is beyond the scope of this text but can be found in [**159**] (Theorem 4.1). Their existence can also be derived from the techniques that will be developed in the following sections; see Example 88. $\triangle$

Exercises.

(123) Discuss whether and why the adjective *generic* in the name of the concept of a *generic superposition* (Section 3.4.2) is appropriate.

(124) Show that the set D_x from Example 83 is indeed dense and open.

(125) Show that the set $D(m,n)$ from Example 83 is dense and open.

(126) Let G be a closed subgroup of S_ω with a generic element g; then
- G has no proper normal subgroup of countable index.
- Every element of G is a product of two conjugates of g.
- every element of G is a *commutator*, i.e., can be written as $aba^{-1}b^{-1}$ for some $a, b \in G$.

6.4.2. n-generic automorphisms. The *diagonal conjugacy action* of G on G^n is the action given by

$$g \cdot (g_1, \dots, g_n) := (gg_1g^{-1}, \dots, gg_ng^{-1}).$$

An element $\overline{g} \in G^n$ is called *n-generic* if the orbit of $\overline{g}$ under this action is comeager. A Polish group G has *ample generics* if G^n contains an n-generic element for each $n \in \mathbb{N}_{>0}$. We will see many examples of structures with ample generics later.

EXAMPLE 86. Let $G = \mathrm{Aut}(\mathbb{Q};<)$. Then G does not have a 2-generic element; this is due to Hodkinson and can be found in [**160**] (Theorem 2.4). Hence, G does not have ample generics. Another proof of this fact was found by Siniora (Lemma 6.1.1 in [**146**]; see Corollary 6.4.13). $\triangle$

A proof of the following result can be found e.g. in [**60**] (Theorem 3.2.4).

THEOREM 6.4.2 (Effros). *Let $\underline{G}$ be a Polish group which acts continuously on a Polish space X. Then for every $x \in X$ the following are equivalent:*

(1) *$G \cdot x$ is non-meager in its closure.*
(2) *The map $g \mapsto g \cdot x$ is open from G onto $G \cdot x$.*
(3) *The map from $\underline{G}/\underline{G}_x$ onto $\underline{G} \cdot x$ that maps gG_x to $g \cdot x$, for every $g \in G$, is a homeomorphism.*
(4) *$G \cdot x$ is G_δ, i.e., a countable intersection of open subsets of X.*
(5) *$G \cdot x$ is comeager in its closure.*

LEMMA 6.4.3 (Lemma 6.6 in [**92**]). *Let $\underline{G}$ be a closed subgroup of S_ω with ample generics. Let $A \subseteq G$ be non-meager and $B \subseteq G$ be non-meager in any non-empty open set. Let $n \in \mathbb{N}$ and $\bar{x} \in G^n$ be n-generic and $V \subseteq G$ be open such that $1 \in V$. Then there are $a \in A$, $b \in B$, and $v \in V$ such that $(\bar{x}, a)$ and $(\bar{x}, b)$ are $(n+1)$-generic and $v \cdot (\bar{x}, a) = (\bar{x}, b)$.*

Proof. Since $\underline{G}$ has ample generics, the action of $\underline{G}$ on G^{n+1} by conjugation has a comeager orbit C, i.e., $\forall^*(\bar{z}, y)C(\bar{z}, y)$. By the theorem of Kuratowski-Ulam (Theorem 6.3.10; see Remark 6.3.11), we have that

$$Z := \{\bar{z} \mid \forall^* y.C(\bar{z}, y)\}$$

is comeager, and hence Z must intersect the comeager orbit of $\bar{x}$. Moreover, Z is preserved by the action of G on G^n, so the orbit of $\bar{x}$ is contained in Z. It follows in particular that $\forall^* y.C(\bar{x}, y)$, i.e., the set $C_{\bar{x}} := \{y \in G \mid C(\bar{x}, y)\}$ is comeager.

Let $y \in C_{\bar{x}}$. Let $G_{\bar{x}}$ be the stabiliser of $\bar{x}$ for the conjugation action of G on G^n. Note that $C_{\bar{x}} = G_{\bar{x}} \cdot y$: indeed, since C is an orbit of the conjugation action of G on X^{n+1}, for any $z \in G$ we have $(\bar{x}, z) \in C$ if and only if there exists $g \in G$ such that $g \cdot (\bar{x}, z) = (\bar{x}, y)$, i.e., $g \in G_{\bar{x}}$ such that $g \cdot z = y$. It follows that for any $y \in C_{\bar{x}}$ the set $G_{\bar{x}} \cdot y$ is comeager. Fix $a \in A \cap C_{\bar{x}}$.

By Effros' theorem (Theorem 6.4.2, (2)), the set $(G_{\bar{x}} \cap V) \cdot a$ is open in $G_{\bar{x}} \cdot a$. By assumption, A is non-meager in $(G_{\bar{x}} \cap V) \cdot a$. Hence, $(G_{\bar{x}} \cap V) \cdot a \cap B \neq \emptyset$. Fix $b \in (G_{\bar{x}} \cap V) \cdot a \cap B$. Then for some $v \in G_{\bar{x}} \cap V$ we have $v \cdot a = b$. □

Theorem 6.4.4 (Theorem 5.3 in [**74**], Theorem 6.9 in [**92**]). *Let $\underline{G}$ be a closed subgroup of S_ω with ample generics. Then $\underline{G}$ has the small index property.*

Proof. Suppose that $\underline{H} \leq \underline{G}$ has countable index. Then H is non-meager by Lemma 6.3.15. If there exists a non-empty open set U such that $U \setminus H$ is meager, then H is open by Lemma 6.3.16.

Otherwise, $G \setminus H$ is non-meager in every non-empty open set. In this case we will reach a contradiction as follows. We will apply Lemma 6.4.3 to $A = H$ and $B = G \setminus H$ to construct for every $a \in 2^{\mathbb{N}}$ an element $h_a \in G$ such that for all $a, b \in 2^{\mathbb{N}}$, if $a \neq b$ then h_a and h_b lie in different cosets of $\underline{H}$ in $\underline{G}$, contradicting the assumption that $\underline{H}$ has countable index.

Let $a \in 2^{\mathbb{N}}$. For every $n \in \mathbb{N}$ and $s = (s_0, s_1, \dots, s_n) \in \{0,1\}^{n+1}$ we inductively define $x_s, f_s, h_s \in G$ such that

(1) x_s is generic,
(2) $x_s \in H$ if $s_n = a_n$ and $x_s \in G \setminus H$ if $s_n = 1 - a_n$,
(3) $f_s = 1^{\underline{G}}$ if $s_n = 1 - a_n$,
(4) $h_s = f_{(s_0)} \circ f_{(s_0,s_1)} \circ \cdots \circ f_{(s_0,\dots,s_n)}$,
(5) $d(h_s, h_s f_{(s,1-a_n)}) < 2^{-n}$ (we use any compatible complete metric d on G, e.g., the metric d' from Lemma 4.2.15),
(6) if $s_n = (s', a_n)$ for $s' \in \mathbb{N}^n$, then $f_s x_{(s',a_n)} f_{s'}^{-1} = x_{(s',1-a_n)}$.

For $n = 1$, and if $a_0 = 0$, we apply Lemma 6.4.3 for $A := H$, $B := G \setminus H$, and $V = G$ and set $x_0 := a$, $x_1 := b$, $f_1 := 1^{\underline{G}}$, and $f_1 := v$. If $a_0 = 1$, we set $x_0 := b$, $x_1 = a$, $f_1 := v^{-1}$, and $f_0 := 1^{\underline{G}}$.

For $n > 1$, suppose that x_s and f_s, for $s \in \{0,1\}^n$, are already defined. First consider the case that $a_n = 0$. Again by Lemma 6.4.3, this time applied to $V := \{f \in G \mid d(h_s, h_s f_{(s,1)}) < 2^{-n}\}$, there are $x_{(s,0)} \in H$ and $x_{(s,1)} \in G \setminus H$ such that $(x_s, s_{(s,0)})$ and $(x_s, s_{(s,1)})$ are generic, and $f_{(s,0)} \cdot x_{(s,0)} = x_{(s,1)}$. The inductive step for $a_n = 1$ is analogous. By (3) and (4), the sequence $(h_{s_n})_{n \in \mathbb{N}}$ is Cauchy, and by the completeness of d converges to some $h_a \in G$.

Claim 1. $a \mapsto h_a$ is a continuous map from $2^{\mathbb{N}}$ to G.

Claim 2. If $a, b \in 2^{\mathbb{N}}$ are such that for some $n \in \mathbb{N}$ we have $a_1 = b_1, \ldots, a_n = b_n$, $a_{n+1} = 0$, and $b_{n+1} = 1$, then $h_a \cdot H \cap h_b \cdot (G \setminus H) \neq \emptyset$. Indeed, we have

$$\begin{aligned} h_a \cdot x_{(a_1,\ldots,a_n,0)} &= h_{(a_0,\ldots,a_n,0)} \cdot x_{(a_1,\ldots,a_n,0)} \\ &= h_{(a_0,\ldots,a_n)} f_{(a_0,\ldots,a_n,1)} \cdot x_{(a_1,\ldots,a_n,0)} \\ &= h_{(a_0,\ldots,a_n)} \cdot x_{(a_1,\ldots,a_n,0)} \qquad \text{(as } f_{(a_0,\ldots,a_n,1)} = 1^{\underline{G}}\text{)} \end{aligned}$$

and

$$\begin{aligned} h_b \cdot x_{(a_1,\ldots,a_n,1)} &= h_{(a_0,\ldots,a_n,1)} \cdot x_{(a_1,\ldots,a_n,1)} \\ &= h_{(a_0,\ldots,a_n)} f_{(a_0,\ldots,a_n,0)} \cdot x_{(a_1,\ldots,a_n,1)} \\ &= h_{(a_0,\ldots,a_n)} \cdot x_{(a_1,\ldots,a_n,0)}. \end{aligned}$$

So $h_a \cdot x_{(a_1,\ldots,a_n,0)} = h_b \cdot x_{(a_1,\ldots,a_n,1)}$ which proves the claim since $x_{(a_1,\ldots,a_n,0)} \in H$ and $x_{(a_1,\ldots,a_n,1)} \in G \setminus H$.

Claim 3. If $a, b \in 2^{\mathbb{N}}$ are distinct, then h_a and h_b lie in different cosets of $\underline{H}$. Let $n \in \mathbb{N}$ be smallest so that $a_n \neq b_n$. By the previous claim we have

$$h_a \circ H \circ h_a^{-1} \cap h_b \circ (G \setminus H) \circ h_b^{-1} \neq \emptyset$$

and hence

$$h_b^{-1} h_a H h_a^{-1} h_b \cap G \setminus H \neq \emptyset,$$

so $h^{-1} h_a \notin H$. This shows that h_a and h_b are in different cosets of $\underline{H}$.

Claim 3 contradicts the assumption that $\underline{H}$ has countable index in $\underline{G}$. ☐

Exercises.

(127) Let V be a countably infinite set and let E be an equivalence relation on V with countably many countable classes. Prove that $\mathrm{Sym}(V; E)$ does not have ample generics.

6.4.3. Dense conjugacy classes and the JEP. We have seen in the previous section that if $\underline{G} \leq S_\omega$ has ample generics, then it has the small index property: but how do we prove that $\underline{G}$ has ample generics? To this end, we present in this section and the following sections an elegant characterisation of the existence of ample generics of Kechris and Rosendal [**92**] which builds on ideas from [**74**] and [**159**].

If $\underline{G}$ has a generic element α, then the orbit of α is in particular dense (since G is Polish and by Proposition 6.3.7). So we will first focus on understanding whether $\underline{G}$ has a dense conjugacy class.

Definition 6.4.5. *Let $\mathcal{K}$ be an amalgamation class. Then $\mathcal{K}_p$ denotes the class of all pairs*

$$(\underline{A}, e \colon B \to C)$$

such that $\underline{A} \in \mathcal{K}$ and e is an isomorphism between substructures $\underline{B}$ and $\underline{C}$ of $\underline{A}$.

An embedding of $(\underline{A}, e \colon B \to C) \in \mathcal{K}_p$ into $(\underline{A}', e' \colon B' \to C') \in \mathcal{K}_p$ is an embedding $f \colon \underline{A} \hookrightarrow \underline{A}'$ such that $f(B) \subseteq B'$, $f(C) \subseteq C'$, and $f \circ e = e' \circ (f|_B)$. Note that the definition of the joint embedding property (JEP) and the amalgamation property (AP) were purely categorical in the sense that their definition only requires a notion of embedding; so JEP and AP are naturally defined not only for structures and embeddings, but also for classes of the form $\mathcal{K}_p$ as introduced above.

We say that a continuous action $\xi \colon \underline{G} \to \mathrm{Sym}(X)$, for some set X, is *topologically transitive* if for any two non-empty open subsets $U, V \subseteq X$ there exists $g \in G$ such that $g(U) \cap V \neq \emptyset$. The implication from (1) to (2) in the following theorem can already be found in [**158**]; the equivalence of (1) and (2) is Theorem 2.1 in [**92**].

Theorem 6.4.6. *Let $\mathcal{K}$ be an amalgamation class and let $\underline{L}$ be its Fraïssé limit. Then the following are equivalent.*

1. (1) $\underline{G} := \mathrm{Aut}(\underline{L})$ *has a dense conjugacy class.*
2. (2) $\mathcal{K}_p$ *has the JEP.*
3. (3) *The action of $\underline{G}$ on G by conjugation is topologically transitive.*

Proof. (1) $\Rightarrow$ (2). Fix an element $\alpha \in G$ having a dense conjugacy class in $\underline{G}$. To show that $\mathcal{K}_p$ satisfies the JEP, let $(\underline{A}_i, e_i \colon B_i \to C_i) \in \mathcal{K}_p$ for $i \in \{1,2\}$; we assume that $\underline{A}_i$ is a substructure of $\underline{L}$. By the homogeneity of $\underline{L}$ the embedding e_i has an extension in G, so by the density of the conjugacy class of α there is a $\beta_i \in G$ such that $e_i = \beta_i^{-1}\alpha\beta_i|_B$. Let

$$\underline{A} := \underline{L}[\beta_1 A_1 \cup \beta_2 A_2] \qquad \underline{B} := \underline{L}[\beta_1 B_1 \cup \beta_2 B_2] \qquad e := \alpha|_B.$$

Then $h_i := \beta_i|_{A_i}$ is an embedding of $(\underline{A}_i, e_i)$ into $(\underline{A}, e)$.

(2) $\Rightarrow$ (3). Suppose that $\mathcal{K}_p$ has the JEP and let $U_1, U_2 \subseteq G$ be non-empty open. For $i \in \{1,2\}$, the set U_i contains a set of the form $\{\alpha \in G \mid e_i = \alpha|_{B_i}\}$ for some isomorphism $e_i \colon B_i \to C_i$ between finite substructures of $\underline{L}$. Let $A_i := B_i \cup C_i$. Since $\mathcal{K}_p$ has the JEP, there exists $(\underline{A}, e)$ and embeddings $f_1 \colon (\underline{A}_1, e_1) \hookrightarrow (\underline{A}, e)$ and $f_2 \colon (\underline{A}_2, e_2) \hookrightarrow (\underline{A}, e)$. By the homogeneity of $\underline{L}$ there exist $\beta, \gamma_1, \gamma_2 \in G$ such that β extends e, γ_1 extends f_1, and γ_2 extends f_2. Then $(\gamma_2^{-1}\gamma_1 \cdot U_1) \cap U_2 \neq \emptyset$, because

- $\gamma_2^{-1}\beta\gamma_2 \in U_2$ since it extends e_2, and
- $\gamma_2^{-1}\beta\gamma_2 \in \gamma_2^{-1}\gamma_1 \cdot U_1 = \gamma_2^{-1}\gamma_1 U_1 \gamma_1^{-1}\gamma_2$ since $\gamma_1^{-1}\beta\gamma_1$ extends e_1.

(3) $\Rightarrow$ (1). There are countably many basic open sets of the form $S(a,b)$ in G, for $a, b \in L^n$, $n \in \mathbb{N}$. For each of them, the set

$$D_{a,b} := G \cdot S(a,b) = \{\alpha \in G \mid \exists g \in G \text{ such that } g\alpha g^{-1}(a) = b\}$$

is clearly open, and dense since the conjugation action is topologically transitive. Since G is a Baire space, the countable intersection C over all the $D_{a,b}$ is dense, and in particular non-empty. Let $f \in C$; then the conjugacy class of f is dense, because for every $a, b \in L^n$, $n \in \mathbb{N}$, we have that $f \in D_{a,b}$, and hence there is $g \in G$ such that $g \cdot f \in S_{a,b}$. $\square$

Corollary 6.4.7. *S_ω, the automorphism group of the random graph, and more generally the automorphism group of Fraïssé-limits of classes with free amalgamation have a dense conjugacy class.*

Proof. Let $\mathcal{K}$ be the class of all finite structures with the empty signature so that the automorphism group of the Fraïssé-limit of $\mathcal{K}$ is isomorphic (as a permutation group) to $\mathrm{Sym}(\mathbb{N})$. By Theorem 6.4.6, it suffices to verify that $\mathcal{K}_p$ has the JEP. So let $(\underline{A}_i, e_i \colon B_i \to C_i) \in \mathcal{K}_p$ for $i \in \{1,2\}$. We may assume that $A_1 \cap A_2 = \emptyset$ and define $A := A_1 \cup A_2$, $B := B_1 \cup B_2$, $C := C_1 \cup C_2$, and $e \colon B \to C$ as the common extension of both e_1 and e_2. Then the identity map $f_i \colon A_i \to A$ is an embedding of $(\underline{A}_i, e)$ into $(\underline{A}, e)$ showing the JEP. The same proof works for the other groups in the statement. $\square$

Exercises.

(128) Let E^2 be the equivalence relation on $\mathbb{N}$ with two infinite classes, and let $\mathcal{K} := \mathrm{Age}(\mathbb{N}; E^2)$. Show that $\mathcal{K}_p$ does not have the JEP.

(129) Directly show that $\mathrm{Aut}(\mathbb{N}; E^2)$ does not have a dense conjugacy class (without using Theorem 6.4.6).

(130) Show that $\mathrm{Aut}(\mathbb{Q}; \mathrm{Cycl})$ (see Section 2.4) does not have a dense conjugacy class.

(131) Let $\underline{V}$ be the countably infinite vector space over $\mathbb{F}_2$.
Show that $\mathrm{Aut}(\underline{V})$ has a dense conjugacy class.

6.4.4. Non-meager Conjugacy Classes and the WAP. Suppose that $\underline{G}$ is a closed subgroup of S_ω with a generic element α, i.e., the orbit of α with respect to the conjugation action is comeager. This means in particular that the orbit of α is non-meager. If the orbit of α is dense, then the converse is true as well, by Effros' theorem (Theorem 6.4.2): indeed, if $G \cdot x$ is nonmeager in its closure, then $G \cdot \alpha$ is is comeager in its closure by the implication (1) $\Rightarrow$ (5) in Effros' theorem. Thus, if $\overline{G \cdot \alpha} = G$, then $G \cdot \alpha$ is comeager in G.

This section presents an equivalent characterisation of non-meager orbits of continuous actions of automorphism groups on Polish spaces; in the next section, this will be applied to characterise the existence of generic elements.

Proposition 6.4.8 (Proposition 3.2 in [**92**]). *Let $\underline{G} \leq \mathrm{Sym}(\mathbb{N})$ be closed and $\xi \colon G \to X$ be a continuous action of G on a Polish space X. Let $x \in X$. Then the following are equivalent.*

1. *The orbit $\underline{G} \cdot x$ is non-meager in X.*
2. *for every open subgroup $\underline{V}$ of $\underline{G}$, the set $V \cdot x$ is non-meager in X.*
3. *For every open subgroup $\underline{V}$ of $\underline{G}$, the set $V \cdot x$ is somewhere dense in X.*
4. *For every open subgroup $\underline{V}$ of $\underline{G}$, we have $x \in \mathrm{Int}(\overline{V \cdot x})$.*

Proof. (1) implies (2). Suppose that $\underline{G} \cdot x$ is non-meager in X and $\underline{V} \leq \underline{G}$ is an open subgroup of G. Since open subgroups of $\underline{G}$ have countable index, we finde $g_0, g_1, \dots$ such that $G = \bigcup_{i \in \mathbb{N}} g_i V$. So some $g_n V \cdot x$ is non-meager. Hence, $V \cdot x$ is non-meager in X.

Clearly, (2) implies (3).

(3) implies (4). Let $\underline{V} \leq \underline{G}$ be open. Then $V \cdot x$ is somewhere dense, i.e., $\overline{V \cdot x}$ contains a non-empty open set U. Take $g \in V$ such that $g \cdot x \in U$. Then $g^{-1} V \cdot x = V \cdot x$ is dense in the open set $g^{-1} \cdot U$ which contains x. Hence, $x \in \mathrm{Int}(\overline{V \cdot x})$.

(4) implies (1). We prove the contraposition and suppose that $G \cdot x$ is meager, i.e., $G \cdot x \subseteq \bigcup_{n \in \mathbb{N}} F_n$ where $F_0, F_1, F_2, \dots$ are closed nowhere dense subsets of X. Then for each $n \in \mathbb{N}$ the set $K_n := \{g \in G \mid g \cdot x \in F_n\}$ is closed. Note that $\bigcup_{n \in \mathbb{N}} K_n = G$ and hence some K_n must be non-meager, i.e., $\overline{K_n}$ must have non-empy interior. So there is a non-empty open subgroup $\underline{V} \leq \underline{G}$ and $g \in G$ such that $\overline{K_n}$ contains gV. Then $gV \cdot x \subseteq F_n$ and hence V is nowhere dense. Therefore $V \cdot x$ is nowhere dense and $\overline{V \cdot x}$ has empty interior, which proves that (4) does not hold. $\square$

As in the case of the JEP, also the WAP only depends on the notion of embeddings, and hence makes also sense for the class $\mathcal{K}_p$ from Definition 6.4.5. We now study the question whether classes of the form $\mathcal{K}_p$ from Section 6.4.3 have the AP or the WAP.

Example 87. Let $\mathcal{K}$ be the class of finite structures with the empty signature. Then $\mathcal{K}_p$ does not have the AP, but it has the WAP. To see why the amalgamation property fails, let $\underline{A} \in \mathcal{K}$ be such that A contains a single element a, and let e be the isomorphism with the empty domain. Let $\underline{B}_1$ be the extension of $\underline{A}$ by two elements and let $e_1 \colon B_1 \to B_1$ be the bijection which exchanges the two elements, and let $\underline{B}_2$ be the extension of $\underline{A}$ with three elements and $e_2 \colon B_2 \to B_2$ be the bijection which cyclically shifts the three elements. Then $(\underline{A}, e)$ embeds via the inclusion map into $(\underline{B}_i, e_i)$, for $i \in \{1, 2\}$. If $(\underline{C}, g)$ is such that there is an embedding f_i from $(\underline{B}_i, e_i)$ into $(\underline{C}, g)$, then $f_1(a) \neq f_2(a)$ because $f_1(a)$ must lie in a cycle of g of length 2 and $f_2(a)$ must lie in a cycle of g of length 3.

To verify that $\mathcal{K}_p$ has the WAP, let $(\underline{A}, e\colon B \to C) \in \mathcal{K}_p$. Let e' be an extension of e to a permutation of A. Clearly, $(\underline{A}, e)$ embeds into $(\underline{A}, e')$. For $i \in \{1,2\}$, let $(\underline{A}_i, e_i\colon B_i \to C_i) \in \mathcal{K}_p$ be such that there exists an embedding $f_i\colon (\underline{A}, e') \to (\underline{A}_i, e_i)$; we may assume without loss of generality that $A_1 \cap A_2 = A$ and that $e_1|_A = e_2|_A = \mathrm{id}_A$. Let $\underline{A}^*$ be the free amalgam of $\underline{A}_1$ and $\underline{A}_2$, and for $i \in \{1,\dots,n\}$ let e^* be the union of e_1 and e_2; this is well-defined by the stipulation that $e_1|_A = e_2|_A$. Note that id_{A_i}, for $i \in \{1,2\}$, is an embedding of $(\underline{A}_i, e_i)$ into $(\underline{A}^*, e^*)$, proving that $(\underline{A}, e')$ is determined on $(\underline{A}, e)$. This shows the WAP for $\mathcal{K}_p$. △

The proof of the WAP in the previous example works more generally if every $(\underline{A}, e) \in \mathcal{K}_p$ can be embedded into $(\underline{A}', e')$ such that e' is an automorphism of $\underline{A}'$, since $(\underline{A}', e')$ is always determined on $(\underline{A}, e)$. The following example shows a situation where the WAP cannot be shown in this way by extending e to an automorphism.

Example 88. Let $\mathcal{K} := \mathrm{Age}(\mathbb{Q}; <)$. Then $\mathcal{K}_p$ does not have the AP: to see this, suppose that $\underline{A} \in \mathcal{K}$ contains a single element a, and let e be the isomorphism with the empty domain, just as in Example 87. Let $\underline{B} \in \mathcal{K}$ be with two elements a, b, let $e_1\colon \{a\} \to \{a\}$, and let $e_2\colon \{a\} \to \{b\}$ be partial isomorphisms. Then for $i \in \{1,2\}$ the identity $\mathrm{id}_{\{a\}}$ is an embedding of $(\underline{A}, e)$ into $(\underline{B}, e_i)$. If $\underline{C}$ is such that there are embeddings $g_i\colon \underline{B} \to \underline{C}$, for $i \in \{1,2\}$, and $g_1(a) = g_2(a)$, then $g_1(e_1(a)) = g_1(a) \neq g_2(b) = g_2(e_2(a))$.

To see that $\mathcal{K}_p$ has the WAP, let $(\underline{A}, e) \in \mathcal{K}_p$. To find $(\underline{A}', e') \in \mathcal{K}_p$ which is determined on $(\underline{A}, e)$, we want to extend e so that it is defined on all of A; to this end, we might have to choose $\underline{A}'$ to be a proper extension of $\underline{A}$. We may suppose that $\underline{A}$ is a substructure of $(\mathbb{Q}; <)$ and by the homogeneity of $(\mathbb{Q}; <)$ we find $\alpha \in \mathrm{Aut}(\mathbb{Q}; <)$ which extends e. Let $\underline{A}' := (\mathbb{Q}; <)[A \cup \alpha(A)]$ and $e' := \alpha|_A$. Then id_A is an embedding of $(\underline{A}, e)$ into $(\underline{A}', e')$. To verify that $(\underline{A}', e')$ is determined on $(\underline{A}, e)$, let f_i, for $i \in \{1,2\}$, be an embedding of $(\underline{A}', e')$ into $(\underline{B}_i, e_i)$. Let $\underline{C}$ be the free amalgam of $\underline{B}_1$ and $\underline{B}_2$ over $\underline{A}$. Then repeatedly identify $e_1(x)$ with $e_2(x)$ in $\underline{C}$ for all $x \in B_1 \cap B_2$; the resulting structure $\underline{C}'$ is well-defined by the property that both e_1 and e_2 are order-preserving. In $\underline{C}$ the relation symbol $<$ denotes an acyclic relation which can be extended to a linear order; let $\underline{D}$ be the resulting structure in $\mathcal{K}$. Then there exists an embedding g_i of $\underline{B}_i$ into $\underline{D}$ and $g_1 \circ f_1|_A = g_2 \circ f_2|_A$. △

Theorem 6.4.9 (Theorem 3.4 in [**92**], [**80**]). *Let $\mathcal{K}$ be an amalgamation class and let $\underline{L}$ be its Fraïssé limit. Then the following are equivalent.*

- *$\underline{G} := \mathrm{Aut}(\underline{L})$ has a generic element, i.e., a comeager conjugacy class.*
- *$\mathcal{K}_p$ has the JEP and the WAP.*

Proof. Suppose that the orbit of α with respect to the action of $\underline{G}$ on G by conjugation is comeager. Then the orbit is in particular dense (since G is Polish and by Proposition 6.3.7) and hence Theorem 6.4.6 shows that $\mathcal{K}_p$ satisfies the JEP.

To show that $\mathcal{K}_p$ satisfies the WAP, let $(\underline{A}, e) \in \mathcal{K}_p$ be given. We may suppose that $\underline{A}$ is a substructure of $\underline{L}$. Since α is in a dense conjugacy class we may assume that α is an extension of e. For the open subgroup $V := \{\beta \in G \mid \beta|_A = \mathrm{id}_A\}$ of $\underline{G}$, we know by Proposition 6.4.8 (4) that $\overline{V \cdot \alpha}$ contains an open set U that contains α. We may suppose that U is of the form $S(a, f(a))$ for some $a \in L^n$, $n \in \mathbb{N}$, whose entries contain all elements of A, and some isomorphism f between finite substructures of $\underline{L}$. Since α extends e, we know that f extends e as well.

Let $\underline{A}'$ be the substructure induced by $\underline{L}$ on $\mathrm{dom}(f) \cup \mathrm{im}(f)$. For $i \in \{1,2\}$, let $e_i\colon (\underline{A}', f) \to (\underline{B}_i, f_i)$ for some $(\underline{B}_i, f_i) \in \mathcal{K}_i$. Since $\underline{L}$ is homogeneous we may assume that $\underline{B}_i$ is a substructure of $\underline{L}$ and e_i is the identity on A', and consequently f_i extends f. By the density of $V \cdot \alpha$ in U there is $\beta_i \in V$ such that $\beta_i^{-1} \alpha \beta$ extends f_i. Let

$\underline{C} := \underline{L}[\beta(B_1) \cup \beta(B_2)]$ and let g be the restriction of α to $\beta_1(\mathrm{dom}(f_1)) \cup \beta_2(\mathrm{dom}(f_2))$. Then $\beta_i|_{B_i}$ is an embedding of $(\underline{B}_i, f_i)$ into $(\underline{C}, g)$. Finally, for every $a \in A$ we have $\beta_1(f_1(a)) = \beta_2(f_2(a))$, so $(\underline{C}, g)$ is indeed an amalgam of $(\underline{B}_1, f_1)$ and $(\underline{B}_2, f_2)$ over $(\underline{A}, f)$.

Conversely, suppose that $\mathcal{K}_p$ has the JEP and the WAP. We will construct an element $\alpha \in G$ such that $G \cdot \alpha$ is dense in G and non-meager in its closure.

Let $e_1, e_2, \dots$ be an enumeration of all isomorphisms e between finite substructures of $\underline{L}$ such that any two $f_i \colon (\underline{L}[\mathrm{dom}(e) \cup \mathrm{im}(e)], e) \hookrightarrow (\underline{B}, g)$ can be amalgamated.

Claim. For every open subgroup $\underline{V} \leq \underline{G}$ TODO.

By Proposition 6.4.8 (4) $\Rightarrow$ (1) we have that $G \cdot \alpha$ is non-meager in $\overline{G \cdot \alpha}$. Theorem 6.4.2 (1) $\Rightarrow$ (5) now implies that $G \cdot \alpha$ is comeager in $\overline{G \cdot \alpha} = G$. □

6.4.5. WAP and Ample Generics. This section finally presents the characterisation of those homogeneous structures $\underline{L}$ whose automorphism group has ample generics. Let $\underline{L}$ be the Fraïssé-limit of the amalgamation class $\mathcal{K}$. We introduce the class $\mathcal{K}_p^n$ for $n \geq 1$, which consists of tuples $(\underline{A}, e_1 \colon B_1 \to C_1, \dots, e_n \colon B_n \to C_n)$ where $\underline{A} \in \mathcal{K}$ and e_i an isomorphism between substructures $\underline{B}_i$ and $\underline{C}_i$ of $\underline{A}$. Embeddings between elements of $\mathcal{K}_p^n$ are defined analogously as embeddings between elements of $\mathcal{K}_p$, and again properties like the JEP and the AP make sense.

Theorem 6.4.10 (Theorem 6.2 in [**92**]). *Let $\mathcal{K}$ be an amalgamation class and let $\underline{L}$ be its Fraïssé limit. Then the following are equivalent.*

- $\mathrm{Aut}(\underline{L})$ *has an n-generic element.*
- *The class $\mathcal{K}_p^n$ has the JEP and the WAP.*

Proof. The proof is similar to the proof of Theorem 6.4.9. □

Lemma 6.4.11. $\mathrm{Sym}(\mathbb{N})$ *has ample generics.*

Proof. Let $\mathcal{K}$ be the class of all finite structures over the empty signature. We have already seen in Corollary 6.4.7 that $\mathcal{K}_p$ has the JEP; the proof that $\mathcal{K}_p^n$ has the JEP is analogous. Moreover, we have already seen in Example 87 that $\mathcal{K}_p$ has the WAP; the proof that $\mathcal{K}_p^n$ has the WAP is analogous: again we may find for every $(\underline{A}, e) \in \mathcal{K}_p^n$ an extension $(\underline{A}', e') \in \mathcal{K}_p^n$ which is determined on $(\underline{A}, e)$ by choosing e' to be an automorphism of $\underline{A}'$. Now the statement follows from Theorem 6.4.10. □

Corollary 6.4.12. $\mathrm{Sym}(\mathbb{N})$ *has the small index property, automatic continuity, and automatic homeomorphicity.*

Proof. We have just seen that $\mathrm{Sym}(\mathbb{N})$ has ample generics, so it follows from Theorem 6.4.4 that $\mathrm{Sym}(\mathbb{N})$ has the small index property. Automatic continuity follows from Proposition 6.2.1, and automatic homeomorphicity follows from Proposition 6.3.18. □

Corollary 6.4.13. $\mathrm{Aut}(\mathbb{Q}; <)$ *does not have a 2-generic element.*

Proof. Let $\mathcal{K} := \mathrm{Age}(\mathbb{Q}; <)$. By Theorem 6.4.10 it suffices to show that $\mathcal{K}_p^2$ does not have the WAP. Let $a_1, a_2 \in \mathbb{Q}$ be such that $a_1 < a_2$, let $\underline{A} := (\mathbb{Q}; <)[\{a_1, a_2\}]$, let $e_1(a_1) = a_2$ and $e_2(a_1) = a_2$. We claim that there is no $(\underline{A}', e_1', e_2') \in \mathcal{K}_p^2$ which is determined on $(\underline{A}, e_1, e_2)$. Note that in $\underline{A}'$, for $i \in \{1, 2\}$ we have $a_1 < e_i'(a_1)$, and if $(e_i')^k(a_1)$ is defined for $k \geq 1$, then $(e_i')^{k-1}(a_1) < (e_i')^k(a_1)$. Let $k \geq 1$ be smallest so that $(e_1')^k(a_1)$ is undefined, and let $b := (e_1')^{k-1}(a_1)$. Let $\underline{B}_1$ be the extension of $\underline{A}'$ by one new element c which is larger than all elements in $\underline{A}'$, and let $e_{1,1}$ be the extension of e_1' given by $e_{1,1}(b) = c$, and let $e_{1,2}$ be the extension of e_2' given by $e_{1,2}(c) = c$. Let $\underline{B}_2$ be the extension of $\underline{A}'$ by two new elements d_1

and d_2 such that d_1 is larger than all elements in $\underline{A}'$ and d_2 is larger than d_1. Let $e_{2,1}$ be the extension of e'_1 given by $e_{2,1}(b) = d_1$ and let $e_{2,2}$ be the extension of e'_2 given by $e_{2,2}(d_1) = d_2$. Note that $(\underline{B}_1, e_{1,1}, e_{1,2}), (\underline{B}_2, e_{2,1}, e_{2,2}) \in \mathcal{K}^2_p$. Now suppose for contradiction that there exists $(\underline{C}, e''_1, e''_2)$ and embeddings $f_i \colon (\underline{B}_i, e_{i,1}, e_{i,2}) \hookrightarrow (\underline{C}, e''_1, e''_2)$ such that $f_1|_A = f_2|_A$. Then in particular $f_1(a_1) = f_2(a_1)$, and hence $f_1(b) = f_1((e_{1,1})^k(a_1)) = f_2((e_{2,1})^k(a_1)) = f_2(b)$, and $f_1(e_{1,1}(b)) = f_2(e_{2,1}(b))$. However, $e_{1,2}(e_{1,1}(b)) = e_{1,1}(b) = c$, whereas $e_{2,1}(b) = d_1 < d_2 = e_{2,2}(e_{2,1}(b))$. Hence,

$$f_1(e_{1,1}(b)) = f_2(e_{2,1}(b)) < f_2(e_{2,2}(e_{2,1}(b)) = f_1(e_{1,2}(e_{1,1}(b)) = f_1(e_{1,1}(b))$$

a contradiction. □

Exercises.

(132) Let $\mathcal{K}$ be the class of all finite partial orders. Show that the automorphism group of the Fraïssé-limit of $\mathcal{K}$ does not have ample generics.

6.5. The Extension Property for Partial Automorphisms

The results in this section can be used to prove that for some class of finite structures $\mathcal{K}$, the class $\mathcal{K}^n_p$ from the previous section has the WAP. If a class of finite structures has the so-called *Extension Property for Partial Automorphisms*, then this is a combinatorial statement that is often of independent interest, and found e.g. applications in theoretical computer science [**64**].

Definition 6.5.1 (EPPA / Hrushovski property). *A class $\mathcal{C}$ of finite relational τ-structures has the* EPPA *(the* extension property for partial automorphisms*, also known as the* Hrushovski property*) if every structure $A \in \mathcal{C}$ has an extension $B \in \mathcal{C}$ such that every partial isomorphism of A extends to an automorphism of B.*

Clearly, the class of all finite linear orders, or the class of all finite partial orders do not have the EPPA.

Theorem 6.5.2 (of [**77**]). *The class of all finite undirected graphs has the EPPA.*

Proof. Let X be a finite set and n a positive integer. Let $G(X, n)$ denote the graph with vertex set $\binom{X}{n}$, the set of n-element subsets of X, and where $a, b \in \binom{X}{n}$ are adjacent iff $a \cap b \neq \emptyset$. An induced subgraph G_0 of $G(X, n)$ is called *poor* if

- every $x \in X$ in contained in at most two vertices of G_0, and
- any two vertices of G_0 intersect in at most one point.

Note that every permutation α of X induces an automorphism of $G(X, n)$, which will be denoted by α^*.

Claim 1. For every finite graph G there exists a finite set X and a positive integer n such that G is isomorphic to a poor subgraph of $G(X, n)$. We only prove the claim for d-regular graphs G; the argument can be adapted to the general case. Let X be the edge set of G. Define $f \colon V(G) \to \binom{X}{d}$ by $a \mapsto \{x \in X \mid a \in x\}$; this map is an isomorphism between G and a poor subgraph of $G(X, d)$.

Claim 2. For every isomorphism g between two poor subgraphs G_0 and G_1 of $G(X, n)$ there exists a permutation α of X such that α^* extends g.

First define $\alpha(x)$ for $x \in X$ belonging to two elements $a, b \in V(G_0)$ to be the unique element of $g(a) \cap g(b)$; then define $\alpha(x)$ for the elements of X belonging to one element of $V(G_0)$, and then for the others. □

Theorem 6.5.3. *Let $\mathcal{K}$ be an amalgamation class with the EPPA. Then the Fraïssé limit of $\mathcal{K}$ has ample generics.*

Proof. We use the EPPA to verify the assumptions of Theorem 6.4.10. Let $(\underline{A}, e_1 \colon B_1 \to C_1, \dots, e_n \colon B_n \to C_n) \in \mathcal{K}_p^n$ be given. Let $\underline{D} \in \mathcal{K}$ be the structure from Definition 6.5.1, i.e., a structure such that every partial isomorphism of $\underline{A}$ extends to an automorphism of $\underline{D}$. In particular, for each $i \in \{1, \dots, n\}$, there exists an extension e_i' of e_i to an automorphism of $\underline{D}$. Then $(\underline{A}, e_1, \dots, e_n)$ embeds into $(\underline{D}, e_1', \dots, e_n')$, and we may amalgamate embeddings from $(\underline{D}, e_1', \dots, e_n')$ into other structures of $\mathcal{K}_n^p$, which shows that $\mathcal{K}_p^n$ has the WAP. $\square$

Corollary 6.5.4. *The automorphism group of the random graph has ample generics and the small index property.*

Proof. Combine Theorem 6.5.2 and Theorem 6.5.3 to obtain ample generics, and Theorem 6.4.4 to obtain the SIP. $\square$

Herwig [**68**] showed EPPA for the class of all finite τ-structures, for any finite relational signature τ (also see [**70**] and [**81**] for other proofs), properly generalising Corollary 6.5.4. Herwig [**69**] showed EPPA for the class of all K_n-free graphs. More generally, Lascar and Herwig [**70**, Theorem 3.2] showed EPPA for all classes that are described by homomorphically forbidding finitely many structures, i.e., classes $\mathcal{C}$ such that there exists a finite set of structures $\mathcal{F}$ such that $\underline{A} \in \mathcal{C}$ if no structure in $\mathcal{F}$ admits a homomorphism to $\underline{A}$. Hodkinson and Otto [**75**] (also see Corollary 4.6 in [**145**]) obtained the following, also properly generalising Corollary 6.5.4.

Theorem 6.5.5. *Let $\underline{B}$ be a homogeneous structure with finite relational signature whose age $\mathcal{C}$ has the free amalgamation property. Then $\mathcal{C}$ has the EPPA. Consequently,* $\mathrm{Aut}(\underline{B})$ *has ample generics, the small index property, automatic continuity and homeomorphicity, and reconstruction.*

Exercises.

(133) Let $\underline{B}$ be a homogeneous structure with finite relational signature. Show that the age of $\underline{B}$ has the EPPA if and only if $\mathrm{Aut}(\underline{B})$ is the closure of the union of a countable chain $G_1 \leq G_2 \leq \cdots$ of compact subgroups of $\mathrm{Aut}(\underline{B})$.

(134) Let $\underline{A}$ be a finite substructure of a τ-structure $\underline{B}$. We write $\underline{A} \leq_{\mathrm{homog}} \underline{B}$ if for every $n \in \mathbb{N}$ and any two $a, b \in A^n$ we have that a and b lie in the same orbit of (the componentwise action of) $\mathrm{Aut}(\underline{B})$ on B^n if and only if they lie in the same orbit of (the componentwise action of) $\mathrm{Aut}(\underline{A})$ on A^n.

(a) Prove that if $\underline{A} \leq_{\mathrm{homog}} \underline{B}$ and $\underline{B} \leq_{\mathrm{homog}} \underline{C}$, then $\underline{A} \leq_{\mathrm{homog}} \underline{C}$. 

(b) Provide a counterexample to the transitivity of $\leq$ from the previous exercise if we replace the action of $\mathrm{Aut}(\underline{A})$ on A^n by the action of $\mathrm{Aut}(\underline{B})_A$ on B^n. 

(c) Show that if $\underline{A} \leq_{\mathrm{homog}} \underline{B}$ and $\underline{B}$ is ω-categorical, then for every relation $R \subseteq B^n$ with a definition in $\underline{B}$ the relation $R \cap A^n$ is definable in $\underline{A}$. 

Let $(\underline{A}_i)_{i \in \mathbb{N}}$ be such that $\underline{A}_i \leq_{\mathrm{homog}} \underline{A}_{i+1}$ for all $i \in \mathbb{N}$. We write $\lim_{i \in \mathbb{N}} \underline{A}_i$ for the τ-structure with domain $\bigcup_{i \in \mathbb{N}} A_i$ whose relations are the unions of the respective relations of the $\underline{A}_i$. Write $\mathcal{A}$ for the class of all countably infinite ω-categorical structures of the form $\lim_{i \in \mathbb{N}} \underline{A}_i$ such that $\underline{A}_i \leq_{\mathrm{homog}} \underline{A}$ for all $i \in \mathbb{N}$.

(d) Show that no structure in $\mathcal{A}$ contains a linear order. 

(e) Give three examples of structures in $\mathcal{A}$ with pairwise non-isomorphic automorphism group. 

(f) Show that if $\underline{A} \in \mathcal{A}$, then the expansion of $\underline{A}$ by all definable relations is also in $\mathcal{A}$ and homogeneous. 

(g) Show that if $\underline{A}$ is homogeneous, then $\underline{A} \in \mathcal{A}$ if and only if $\underline{A}$ is ω-categorical and for every first-order τ-sentence ϕ such that $\underline{A} \models \phi$ there exists a finite τ-structure $\underline{B}$ such that $\underline{B} \models \phi$ and $\underline{B} \leq_{\text{homog}} \underline{A}$. 

(h) Show that no homogeneous $\underline{A} \in \mathcal{A}$ is *finitely axiomatisable*, i.e., there is no finite set of τ-sentences which is equivalent to $\text{Th}(\underline{A})$ (in the sense that it has the same models as $\text{Th}(\underline{A})$). 

(i) Show that for every homogeneous and ω-categorical $\underline{A}$, we have $\underline{A} \in \mathcal{A}$ if and only if $\text{Age}(\underline{A})$ has EPPA. 

6.6. The Strong Small Index Property

A permutation group $\underline{G} \leq \text{Sym}(\mathbb{N})$ has the *strong small index property (SSIP)* if every countable index subgroup of $\underline{G}$ lies between the pointwise and the set stabiliser of a finite subset of $\mathbb{N}$. The strong small index property for $\text{Sym}(\mathbb{N})$ itself was shown in [**47**]; it also follows from the Small Index Property of $\text{Sym}(\mathbb{N})$ (Corollary 6.4.12) via a more general result, which we present below (Theorem 6.6.6).

Example 89. An example of an oligomorphic permutation group which has the small index property, but not the strong small index property is the automorphism group G of an equivalence relation E on a set V with (at least) two infinite classes: it has the open subgroup H which fixes one equivalence class of E. Then H has countable index, but is not contained in the set-stabiliser of G of some finite subset of V. On the other hand, G has the small index property. △

We first start with some results that illustrate what the SSIP can be useful for.

Proposition 6.6.1. *Let $\xi \colon \text{Sym}(\mathbb{N}) \to \text{Sym}(\mathbb{N})$ be a homomorphism such that $\xi(\text{Sym}(\mathbb{N}))$ is a primitive permutation group G. Then there exists an $n \in \mathbb{N}$ such that G is isomorphic (as a permutation group) to the setwise action of $\text{Sym}(\mathbb{N})$ on $\binom{\mathbb{N}}{n}$.*

Proof. Recall from Corollary 1.4.8 that the primitivity of G implies that for any $a \in \mathbb{N}$ the point stabiliser G_a is a maximal subgroup of G, and hence $H := \xi^{-1}(G_a)$ is a maximal subgroup of $\text{Sym}(\mathbb{N})$. The strong small index property of $\text{Sym}(\mathbb{N})$ implies that H is contained in the set-wise stabiliser $\text{Sym}(\mathbb{N})_{\{F\}}$ for some finite $F \subseteq \mathbb{N}$. By the maximality of H, this means that H equals $\text{Sym}(\mathbb{N})_{\{F\}}$. Let $n := |F|$. Let i be the map from $\binom{\mathbb{N}}{n} \to \mathbb{N}$ that maps for each $\alpha \in \text{Sym}(\mathbb{N})$ the set $\alpha(F) \in \binom{\mathbb{N}}{n}$ to $\xi(\alpha)(a) \in \mathbb{N}$. Note that i is well-defined because if $\alpha, \beta \in \text{Sym}(\mathbb{N})$ are such that $\alpha(F) = \beta(F)$, then $\alpha^{-1}\beta \in \text{Sym}(\mathbb{N})_{\{F\}} = H$, and hence $\xi(\alpha^{-1}\beta) \in G_a$. Thus, $\alpha^{-1}\beta(a) = a$ and $\alpha(a) = \beta(a)$. Moreover, i is an isomorphism between the image of the setwise action of $\text{Sym}(\mathbb{N})$ on $\binom{\mathbb{N}}{n}$ and G: for $\alpha \in \text{Sym}(\mathbb{N})$ and $S \in \binom{\mathbb{N}}{n}$, let $\beta \in \text{Sym}(\mathbb{N})$ be such that $\beta(F) = S$. Then we have

$$i(\alpha(S)) = i(\alpha\beta(F)) = \xi(\alpha\beta)(a) = \xi(\alpha)\xi(\beta)(a) = \xi(\alpha)i(\beta(F)) = \xi(\alpha)i(S). \qquad \square$$

In fact, under the additional assumption of having no algebraicity, the strong small index property allows a very strong form of reconstruction of countable ω-categorical structures from their abstract automorphism group.

THEOREM 6.6.2 (Paolini and Shelah [**130**] (Corollary 2)). *Let $\underline{A}$ and $\underline{B}$ be countable ω-categorical structures with the strong small index property and without algebraicity. Then* $\mathrm{Aut}(\underline{A})$ *and* $\mathrm{Aut}(\underline{B})$ *are isomorphic as abstract groups if and only if $\underline{A}$ and $\underline{B}$ are bi-definable (Definition 3.1.8). Moreover, if $\xi\colon \mathrm{Aut}(\underline{A}) \to \mathrm{Aut}(\underline{B})$ is an abstract group isomorphism, then there is a bijection $f\colon A \to B$ witnessing the bi-definability of $\underline{A}$ and $\underline{B}$ such that $\xi(\alpha) = f\alpha f^{-1}$.*

6.6.1. Weak Elimination of Imaginaries. The difference between the strong small index property and the small index property is closely related to an important property in model theory.

DEFINITION 6.6.3. *A structure $\underline{A}$ has* weak elimination of imaginaries *if for every $R \subseteq A^k$ which is parameter-definable over $\underline{A}$ there exists an inclusion-wise smallest algebraically closed subset $B \subseteq A$ such that R is definable in $\underline{A}$ over B.*

EXAMPLE 90. Let $\underline{A}$ be the relational structure with a single equivalence relation with two infinite classes C_1 and C_2. Clearly, $\underline{A}$ has no algebraicity. Let $a, b \in A$ be distinct but from the same equivalence class C_1. Then C_1 is definable in $\underline{A}$ over $\{a\}$, and definable in $\underline{A}$ over $\{b\}$, but not definable in $\underline{A}$ without parameters. So $\underline{A}$ does not have weak elimination of imaginaries. △

The next example illustrates that whether an ω-categorical structure $\underline{A}$ has weak elimination of imaginaries is not captured by the topological automorphism group of $\underline{A}$. Weak elimination of imaginaries of $\underline{A}$ is rather a property of the action of the automorphism group.

EXAMPLE 91. Let $\underline{B}$ be the relational structure obtained from the structure $\underline{A}$ in the previous example (Example 90) as follows: add two new elements p and q, and add a new binary relation $\{(a,p) \mid a \in C_1\} \cup \{(b,q) \mid b \in C_2\}$. Note that $\underline{A}$ and $\underline{B}$ are bi-interpretable, and hence $\mathrm{Aut}(\underline{A})$ and $\mathrm{Aut}(\underline{B})$ are isomorphic as topological groups. However, $\underline{B}$ has weak elimination of imaginaries. As in Example 91, if a, b are distinct elements of C_1, then C_1 is definable in $\underline{B}$ over $\{a\}$ and definable in $\underline{B}$ over $\{b\}$, but neither $\{a\}$ nor $\{b\}$ is algebraically closed in $\underline{B}$. Instead, $\mathrm{acl}_{\underline{B}}(\{a\}) = \{a, p, q\}$ and $\mathrm{acl}_{\underline{B}}(\{b\}) = \{b, p, q\}$, and there is a unique smallest algebraically closed subset of B over which C_1 is definable, namely $\{p, q\}$. (By adding the elements p and q we have 'eliminated' the 'imaginary elements' C_1 and C_2.) △

LEMMA 6.6.4. *Let $\underline{A}$ be a countable ω-categorical structure. Then the following are equivalent.*

1. *$\underline{A}$ has weak elimination of imaginaries.*
2. *Every $R \subseteq A^k$ which is definable in $\underline{A}$ over $B_1 \subseteq A$ and definable in $\underline{A}$ over $B_2 \subseteq A$ is also definable in $\underline{A}$ over* $\mathrm{acl}(B_1) \cap \mathrm{acl}(B_2)$.
3. *For all finite $B_1, B_2 \subseteq A$ that are algebraically closed in $\underline{A}$ we have*
$$\langle \mathrm{Aut}(\underline{A})_{(B_1)} \cup \mathrm{Aut}(\underline{A})_{(B_2)} \rangle = \mathrm{Aut}(\underline{A})_{(B_1 \cap B_2)}. \tag{15}$$
4. *For every open subgroup H of $\mathrm{Aut}(\underline{A})$ there exists a unique finite algebraically closed $B \subseteq A$ of smallest cardinality such that $\mathrm{Aut}(\underline{A})_{(B)} \subseteq H$.*
5. *For every open subgroup H of $\mathrm{Aut}(\underline{A})$ there exists a finite algebraically closed finite set $B \subseteq A$ such that $\mathrm{Aut}(\underline{A})_{(B)} \leq H \leq \mathrm{Aut}(\underline{A})_{\{B\}}$.*
6. *Every open subgroup $H \leq \mathrm{Aut}(\underline{A}) =: G$ has a finite index subgroup which is the point stabiliser $G_{(B)}$ for some finite $B \subseteq A$.*

PROOF. We show (1) ⇔ (2), (2) ⇔ (3), (3) ⇒ (4) ⇒ (5) ⇒ (6) ⇒ (3).

(1) ⇒ (2). Let B be the inclusion-wise smallest algebraically closed subset of A such that R is definable in $\underline{A}$ with parameters from B. Then $B \subseteq \mathrm{acl}(B_1)$ and $B \subseteq \mathrm{acl}(B_2)$, so R is also definable in $\underline{A}$ over $\mathrm{acl}(B_1) \cap \mathrm{acl}(B_2)$.

(1) $\Leftarrow$ (2). Every parameter-definable set is definable over a finite set B. Since in ω-categorical structures, the algebraic closure of a finite set is finite (Lemma 3.4.3), we may assume that B is algebraically closed. Suppose that B_1 and B_2 are inclusion-wise minimal algebraically closed sets such that R is first-order definable in $\underline{A}$ over each of them. By assumption, R is also definable over $\mathrm{acl}(B_1) \cap \mathrm{acl}(B_2) = B_1 \cap B_2$. The minimality of B_1 and B_2 then implies that $B_1 = B_2$ is the smallest algebraically closed set such that R is definable in $\underline{A}$ over B.

(2) $\Rightarrow$ (3). Let $B_1, B_2 \subseteq A$ be finite and algebraically closed in $\underline{A}$. The inclusion $\subseteq$ in (15) holds in general. For the converse inclusion, first observe that the group on the left is a closed subgroup of $\mathrm{Sym}(A)$. So by Exercise 11, it suffices to show that every relation that is preserved by $\langle \mathrm{Aut}(\underline{A})_{(B_1)} \cup \mathrm{Aut}(\underline{A})_{(B_2)} \rangle$ is also preserved by $\mathrm{Aut}(\underline{A})_{(B_1 \cap B_2)}$. Every relation R that is preserved by $\langle \mathrm{Aut}(\underline{A})_{(B_1)} \cup \mathrm{Aut}(\underline{A})_{(B_2)} \rangle$ is in particular preserved by $\mathrm{Aut}(\underline{A})_{(B_1)}$, and by Theorem 3.1.1 it is first-order definable in $\underline{A}$ over B_1. Similarly, R is first-order definable in $\underline{A}$ over B_2. By assumption, R is also definable in $\underline{A}$ over $B_1 \cap B_2$, and hence preserved by $\mathrm{Aut}(\underline{A})_{(B_1 \cap B_2)}$.

(2) $\Leftarrow$ (3). If $R \subseteq A^k$ is definable in $\underline{A}$ over $B_1 \subseteq A$ and over $B_2 \subseteq A$, then it is also definable over finite subsets $B_1' \subseteq B_1$ and $B_2' \subseteq B_2$. Since in ω-categorical structures the algebraic closure of a finite set is finite, we may assume that B_1' and B_2' are algebraically closed. The relation R is preserved by $\mathrm{Aut}(A)_{(B_1')}$ and by $\mathrm{Aut}(A)_{(B_2')}$. Then (2) implies that R is preserved by $\mathrm{Aut}(\underline{A})_{(B_1' \cap B_2')}$. By Theorem 3.2.3 and Theorem 3.1.1, R is definable in $\underline{A}$ over $B_1' \cap B_2' \subseteq \mathrm{acl}(B_1) \cup \mathrm{acl}(B_2)$.

(3) $\Rightarrow$ (4). Let $H \leq \mathrm{Aut}(\underline{A})$ be open. Then there exists a finite $B \subseteq A$ such that $\mathrm{Aut}(\underline{A})_{(B)} \leq H$ (Lemma 4.4.1). By ω-categoricity, the algebraic closure of B is finite (Lemma 3.4.3), so we may assume without loss of generality that B is algebraically closed. Suppose for contradiction that there are two distinct algebraically closed finite sets B_1, B_2 of minimal cardinality with $\mathrm{Aut}(\underline{A})_{(B_1)} \leq H$ and $\mathrm{Aut}(\underline{A})_{(B_2)} \leq H$. By assumption, $\mathrm{Aut}(\underline{A})_{(B_1 \cap B_2)} = \langle G_{(B_1)} \cup G_{(B_2)} \rangle \subseteq H$. Note that $B := \mathrm{acl}(B_1 \cap B_2) \subseteq \mathrm{acl}(B_1) = B_1$ and $\mathrm{acl}(B_1 \cap B_2) \subseteq \mathrm{acl}(B_2) = B_2$, so $|B| < |B_1| = |B_2|$, a contradiction.

(4) $\Rightarrow$ (5). It suffices to show that $H \subseteq \mathrm{Aut}(\underline{A})_{\{B\}}$. Let $h \in H$. Then $G_{(h(B))} = hG_{(B)}h^{-1} \subseteq hHh^{-1} = H$. Clearly, $h(B)$ is finite and algebraically closed, so by assumption we have that $B \subseteq h(B)$. Hence, $h(B) = B$ and $h \in G_{\{B\}}$.

(5) $\Rightarrow$ (6) Clearly, $G_{(B)}$ is an open subgroup of H and has finite index in $G_{\{B\}}$.

(6) $\Rightarrow$ (3). The inclusion $\subseteq$ is clear. The group $K := \langle \mathrm{Aut}(\underline{A})_{(B_1)} \cup \mathrm{Aut}(\underline{A})_{(B_2)} \rangle$ is open in $\mathrm{Aut}(\underline{A})$ (Lemma 4.4.1). By assumption, there exists a finite $C \subseteq A$ which is algebraically closed in $\underline{A}$ such that $\mathrm{Aut}(\underline{A})_{(C)} \leq K$ and the index of $\mathrm{Aut}(\underline{A})_{(C)}$ in K is finite. We will prove that $C \subseteq B_1 \cap B_2$. First note that every orbit of $c \in C$ in K is finite by the assumption that the index of $\mathrm{Aut}(\underline{A})_{(C)}$ in K is finite. However, the orbit of every $x \in A \setminus B_1$ in $\mathrm{Aut}(\underline{A})_{B_1}$ is infinite since B_1 is algebraically closed. Therefore, $C \subseteq B_1$. Similarly, $C \subseteq B_2$. Thus, $\mathrm{Aut}(\underline{A})_{B_1 \cap B_2} \leq \mathrm{Aut}(\underline{A})_C \leq K = \langle \underline{A}_{(B_1)} \cup \mathrm{Aut}_{(B_2)} \rangle$. $\square$

Lemma 6.6.5. *Let $\underline{A}$ be a countable ω-categorical structure such that* $\mathrm{Aut}(\underline{A})$ *has the SIP. Then $\underline{A}$ has weak elimination of imaginaries if and only if* $\mathrm{Aut}(\underline{A})$ *has the SSIP.*

Proof. Let $H \leq \mathrm{Aut}(\underline{A})$ be of countable index. By the SIP, H is open. The equivalence then follows from the equivalence (1) $\Leftrightarrow$ (5) in Lemma 6.6.4. $\square$

We close this section by verifying weak elimination of imaginaries for a large class of homogeneous structures $\underline{A}$, namely for those whose age has the free amalgamation property. In the next proof, if $\underline{B}_1, \underline{B}_2, \underline{C}$ are finite substructures of $\underline{A}$, then we write $B_1 \downarrow_C B_2$ if $\underline{A}[B_1 \cup B_2]$ equals $\underline{B}_1 \cup \underline{B}_2$ and $B_1 \cap B_2 \subseteq C$.

THEOREM 6.6.6 (Lemma 2.7 in [**111**], Corollary 1 in [**129**]). *Let $\underline{A}$ be a homogeneous ω-categorical structure whose age has the free amalgamation property. Then* $\mathrm{Aut}(\underline{A})$ *has the strong small index property.*

PROOF. By Theorem 6.5.5, $G := \mathrm{Aut}(\underline{A})$ has the small index property. To verify weak elimination of imaginaries, we verify (2) in Lemma 6.6.4. Let $B_1, B_2 \subseteq A$ be finite, and suppose that $R \subseteq A^k$ is definable in $\underline{A}$ over B_1 and over B_2. Since $\underline{A}$ has no algebraicity, we have to show that R is also definable over $B_1 \cap B_2$. By Theorem 3.1.1, it suffices to show that R is preserved by $G_{(B_1 \cap B_2)}$, so let $g \in G_{(B_1 \cap B_2)}$. There exists $h_1 \in G_{(B_1)}$ with $h_1 g(B_1) \downarrow_{B_1} B_2$ (see Exercise 75). There is $h_2 \in G_{(B_2)}$ with $h_2 h_1 g(B_1) \downarrow_{B_2} B_1$, and $h_3 \in G_{(B_1)}$ so that $h_3(B_1) \downarrow_{B_2} B_1$. By homogeneity, there is $h_4 \in G_{(B_1)}$ so that $h_4 h_2 h_1 g|_{B_1} = h_3|_{B_1}$. Thus, $h_3^{-1} h_4 h_2 h_1 g \in G_{(B_1)}$, so $g \in \langle G_{(B_1)} \cup G_{(B_2)} \rangle$. $\square$

6.6.2. *G*-finiteness. This section is under construction.

DEFINITION 6.6.7 (Lascar [**103**]). *An oligomorphic group $H \leq \mathrm{Sym}(\mathbb{N})$ is called* G-finite *if for every open subgroup $U \leq H$, the intersection of the open subgroups of finite index in U is of finite index in U.*

EXAMPLE 92. Let $\underline{A}$ be the structure from Example 67. Then $\mathrm{Aut}(\underline{A})$ is *not* G-finite: $\mathrm{Aut}(\underline{A})$ itself has the stabilisers of the equivalence classes of E_i, for $i \in \mathbb{N}$, as open subgroups of index two (see Lemma 4.4.3). Their intersection has infinite index. $\triangle$

Exercises.

(135) Let $\underline{A}$ be a countable ω-categorical structure. Show that $\underline{A}$ has no algebraicity and weak elimination of imaginaries if and only if $\mathrm{Aut}(\underline{A})$ has no fixed points and for all finite $B_1, B_2 \subseteq A$ we have that $\langle G_{B_1} \cup G_{B_2} \rangle = G_{B_1 \cap B_2}$.

(136) If G is an oligomorphic permutation group on a set B, then let's write G^* for the closed normal subgroup of G consisting of all $\alpha \in G$ that fix all blocks of congruences of the action of G on B^n, for some $n \in \mathbb{N}$, that have finitely many classes. Show that an oligomorphic permutation group on a set B is G-finite if and only if for every finite $A \subseteq B$, the index of $(G_A)^*$ in G_A is finite.

(137) Show that there are uncountably many closed oligomorphic permutation groups, up to isomorphism of abstract groups.

Hint. Show that the automorphism groups of the countable homogeneous digraphs from Example 31 are pairwise non-isomorphic as permutation groups, then as topological groups, and finally as abstract groups. Use statements of this text (even the ones that have not been proven here but just cite the literature).

(138) Is there a theorem for extending partial homomorphisms between finite graphs to endomorphisms of supergraphs (similarly as Theorem 6.5.2 for partial isomorphisms and automorphisms)?

(139) Show that if $\underline{A}$ is a structure with a first-order interpretation in $(\mathbb{N}; =)$, and $B \subseteq A$ is finite, then $G_{(\mathrm{acl}(B))}$ is open in $\mathrm{Aut}(\underline{A})$.

(140) Show that every structure with a first-order interpretation in $(\mathbb{N}; =)$ has an automorphism group which is G-finite.

Structure	Reconstruction	SIP	Ample Generics	EPPA
$(\mathbb{N}; \neq)$	Yes	Yes	Yes	Yes
Rado	Yes	Yes	Yes [**74**]	Yes [**77**]
K_3-free	Yes	Yes	Yes	Yes [**69**]
Henson	Yes	Yes	Yes	Yes [**69**]
$(\mathbb{T}; E)$	Yes [**139**]	**Open [110]**	**Open**	**Open [69]**
$(\mathbb{Q}; <)$	Yes	Yes [**158**]	No [**160**]	No
$(\mathbb{P}; \leq)$	Yes [**139**]	**Open [110]**	No [**98**]	No
$(\mathbb{Q}; <_1, <_2)$	?	**Open [110]**	No	No
Example 67	Yes [**110, 139**]	No	No	No
Evans-Hewitt	No [**54**]	No	No	No

FIGURE 6.2. Some open problems in the context of this chapter; see Section 6.7. If there is no reference then the result is trivial or can be deduced from other entries in the table and/or the results of this chapter.

6.7. Open Problems

We list open problems from the literature that fall into the context of this chapter.

(1) Does the automorphism group of the countable universal homogeneous tournament $(\mathbb{T}; E)$ have the small index property (see [**110**])?
(2) Does the class of all finite tournaments have the EPPA (see [**69**])?
(3) Does the automorphism group of the countable universal homogeneous poset $(\mathbb{P}; \leq)$ have the small index property (see [**110**])?
(4) Does the automorphism group of the countable universal homogeneous permutation $(\mathbb{Q}; <_1, <_2)$ have the small index property (see [**110**])?
(5) Is there for every $n \in \mathbb{N}$ a countably infinite structure which has n-generic automorphisms, but not $(n+1)$-generic automorphisms (see [**146**])? We only know that the answer is positive for $n = 1$ (Example 86).
(6) Is the automorphism group of every homogeneous structure with a finite relational language G-finite (Macpherson [**110**])?

CHAPTER 7

Ramsey Classes and Topological Dynamics

This section is under construction. [**53**, **78**, **79**, **89**, **90**, **102**, **119**, **121**–**126**, **148**, **149**, **153**–**155**, **163**, **164**]

7.1. Ramsey Classes

CONJECTURE 7.1 (see, e.g., [**15**]). *Every homogeneous structure with a finite relational signature has a finite homogeneous Ramsey expansion.*

7.2. The Kechris-Pestov-Todorcevic Connection

In *topological dynamics* one studies continuous actions of topological groups G on compact Hausdorff spaces X. Such an action is called a *G-flow*, and X is called a *G-space* (if the reference to the underlying G-flow is clear). A point $x \in X$ such that $g \cdot x = x$ for all $g \in G$ is called a *fixed point* of the G-flow.

By the Kechris-Pestov-Todorcevic correspondence [**89**], a homogeneous structure $\underline{B}$ is Ramsey (with respect to colorings of embeddings) if and only if its automorphism group $\underline{G} := \mathrm{Aut}(\underline{B})$ is *extremely amenable*, meaning that every $\underline{G}$-flow has a fixed point.

7.2.1. Universal Minimal Flows. Let X be a G-space and suppose that $Y \subseteq X$ is preserved by the action of G on X. Then Y naturally gives rise to a G-flow by restricting the action to Y, and Y is then called a *G-subspace of X*. A G-space is called *minimal* if X and $\emptyset$ are the only closed G-subspaces of X.

LEMMA 7.2.1. *Every G-space contains a minimal G-subspace.*

PROOF. Apply Zorn's Lemma: TODO. □

A function $f\colon X \to Y$ between a G-space X and a G-space Y is called *equivariant* if $f(g \cdot x) = g \cdot f(x)$ for all $g \in G$ and $x \in X$. If f is bijective, then the inverse is also equivariant, and f is called an *isomorphism*, and the G-spaces X and Y are called *isomorphic*. A *factor* of a G-space X is a G-space Y such that there exists a continuous G-equivariant surjective map from X to Y.

DEFINITION 7.2.2. *A G-space is called* universal *if every minimal G-space X is a factor of U_G.*

The following was shown by Ellis, with new proofs by Auslander, Uspenskij, and Gutman and Li.

THEOREM 7.2.3. *For every G-space there exists a universal minimal G-space, which is unique up to isomorphism.*

PROOF. For the existence, let $\mathcal{S}$ be a set of minimal G-spaces such that every minimal G-space is isomorphic to an element of $\mathcal{S}$. Then $\prod_{X \in \mathcal{S}} X$ (with the diagonal action) is a universal G-space:

For the uniqueness, we follow a proof of Gutman and Li. They show that a universal minimal G-space is *coalescent*, i.e., every surjective □

An active field of research studies the question for which Polish groups G the universal minimal flow $M(G)$ is metrizable. If $M(G)$ is metrizable then it has a comeagre orbit (Ben Yaacov, Melleray, Tsankov 2017): TODO.

7.3. Compact Spaces for Oligomorphic Groups

DEFINITION 7.3.1. *Let $\underline{G}$ be a topological group with a continuous action on a topological space A. Let $\sim$ be the orbit equivalence relation on A where $a \sim b$ if there exists $\alpha \in G$ such that $a = \alpha b$. We write $A/\underline{G}$ for the quotient space $A/_\sim$ with the quotient topology.*

The following statement is taken from [**26**].

PROPOSITION 7.3.2. *Let A, B be countably infinite sets and let $\underline{G}$ be a permutation group on B. Equip B with the discrete topology and B^A with the product topology. Then $B^A/\underline{G}$ is compact if and only if $\underline{G}$ is oligomorphic.*

PROOF. Suppose that $A = \mathbb{N}$. We first prove that if $\underline{G}$ is oligomorphic, then $B^A/\underline{G}$ is compact. Let $\mathcal{U} := \{U_i \mid i \in I\}$ be a family of open subsets of $B^A/\underline{G}$ such that no finite subset of $\mathcal{U}$ covers $B^A/\underline{G}$. For $n \in \mathbb{N}$, let $\sim_n$ be the equivalence relation on B^A where $f \sim_n g$ if there exists an $\alpha \in \underline{G}$ such that $f(a) = \alpha g(a)$ for all $a \in \{0, \dots, n-1\}$. Note that each equivalence class of $\sim_n$ is a union of elements of $B^A/\underline{G}$, and that the oligomorphicity of $\underline{G}$ implies that $\sim_n$ has finitely many classes for each $n \in \mathbb{N}$. If each of the finitely many equivalence classes of $\sim_n$ were contained in the complement of U_i for some $i \in I$, then we would have found a finite subset of $\mathcal{U}$ that covers $B^A/\underline{G}$, contrary to our assumptions. So for each n there exists a $\sim_n$-equivalence class which is not contained in $\bigcup_{i \in I} U_i$.

Consider the following tree: the vertices of the tree are the equivalence classes of $\sim_n$, for all $n \in \mathbb{N}$, that are not contained $\bigcup_{i \in I} U_i$. Let the equivalence class of $f \colon \{0, \dots, n-1\} \to \mathbb{N}$ be adjacent to the equivalence class of $g \colon \{0, \dots, n\} \to \mathbb{N}$ if f is the restriction of g. Clearly, the resulting tree is finitely branching and by KHonigs tree lemma contains an infinite path. From this infinite path $F_1, F_2, \dots$ one can construct a function $f \in B^A$ inductively as follows. Initially, pick any function f_1 from F_1. By the definition of edges in the tree there exists an $\alpha \in G$ such that αf_1 is the restriction of some $g_2 \in F_2$. We define f_2 to be $\alpha^{-1} g_2$ which is an extension of f_1 and in F_2. We continue with f_2 instead of f_1, and iterate to obtain an infinite sequence of functions $f_1, f_2, \dots$ which converges against some $f \in B^A$. Note that $f/_\sim$ is not contained in $\bigcup_{i \in I} U_i$ which finishes the proof that $B^A/\underline{G}$ is compact.

For the other direction, assume that $\underline{G}$ is not oligomorphic. Pick an $n \geq 1$ such that the componentwise action of $\underline{G}$ on B^n has infinitely orbits, and enumerate these orbits by $(O_i)_{i \in \omega}$. For each $i \in \omega$ let U_i consist of all classes $f/_\sim$ in $B^A/\underline{G}$ with the property that $f|_{\{1,\dots,n\}}$ belongs to O_i; this is well defined since for all $f, g \in B^A$ with $f \sim g$ we have that $f|_{\{1,\dots,n\}}$ belongs to O_i if and only if $g|_{\{1,\dots,n\}}$ belongs to O_i. Then $B^A/\underline{G}$ is the disjoint union of the U_i. But each U_i is open, and hence $B^A/\underline{G}$ is not compact. □

COROLLARY 7.3.3. *Let $\underline{B}$ be an ω-categorical structure. Then* $\mathrm{End}(\underline{B})/\mathrm{Aut}(\underline{B})$ *is compact.*

PROOF. $\mathrm{End}(\underline{B})$ is a closed subset of B^B which is preserved by $\mathrm{Aut}(\underline{B})$. Since $\underline{B}$ is ω-categorical, $\mathrm{Aut}(\underline{B})$ is an oligomorphic permutation group by the theorem of Engeler, Svenonius, and Ryll-Nardzewski (Theorem 3.2.3). Proposition 7.3.2 implies that $B^B/\mathrm{Aut}(\underline{B})$ is compact. Note that $\mathrm{End}(\underline{B})/\underline{G}$ is a closed subspace of $B^B/\mathrm{Aut}(\underline{B})$, so the statement follows from Proposition 4.1.13 □

If $\underline{G}$ is an oligomorphic permutation group on a countable set B, then the space $B^B/\underline{G}$ is not Hausdorff, as the following example shows.

REMARK 7.3.4. *Consider any function f in B^B which lies in the closure of $\underline{G}$ but not in $\underline{G}$; Exercise 141 shows that if $\underline{G}$ is oligomorphic, such functions must exist. Then f is inequivalent to every element of $\underline{G}$, but $f/_{\sim}$ cannot be separated from $\mathrm{id}_B/_{\sim}$ by open sets: if U is an open subset of $B^B/\underline{G}$ that contains $f/_{\sim}$, then $\bigcup U$ is open in B^B and hence must contain a basic open set $T_{a,b}$ where $a, b \in B^n$ for some $n \in \mathbb{N}$ and $f(a) = b$. Since f is in the closure of $\underline{G}$ there also exists an $\alpha \in \underline{G}$ with $\alpha a = b$, and $\alpha \sim \mathrm{id}_B$. So every open set that contains $f/_{\sim}$ also contains $\mathrm{id}_Y/_{\sim}$.*

We will work with a certain compact Hausdorff space.

DEFINITION 7.3.5. *Let $\underline{G} \curvearrowright B$ be a permutation group, and A be a set. On B^A, define an equivalence relation $\approx$ by setting $f \approx g$ if $f \in \overline{G\,g}$. We also write $B^A /\!\!/ \underline{G}$ instead of $B^A/_{\approx}$.*

Note that in Definition 7.3.5, transitivity and symmetry follow from the fact that $\underline{G}$ is a group. Also note that $f \approx g$ if and only if $f = g$ holds locally modulo $\underline{G}$.

LEMMA 7.3.6. *If $\underline{G} \curvearrowright B$ is oligomorphic, then the space $B^A /\!\!/ \underline{G}$ is a compact Hausdorff space.*

PROOF. Since $B^A /\!\!/ \underline{G}$ is a quotient of $B^A/\mathscr{G}$, and since $B^A/\underline{G}$ is compact (Proposition 7.3.2), the compactness of $B^A /\!\!/ \underline{G}$ follows from Proposition 4.1.13. To prove that $B^A /\!\!/ \underline{G}$ is Hausdorff, let $s_1/{\approx}$ and $s_2/{\approx}$ be elements of $B^A /\!\!/ \underline{G}$. If these two elements are distinct, there exists $t \in A^n$ such that $s_1(t), s_2(t) \in B^n$ lie in different orbits of n-tuples under $\underline{G}$. Then $s_1 \in U_1 := \{u \in B^A \mid u(t) = s_1(t)\}$ and $s_2 \in U_2 := \{u \in B^A \mid u(t) = s_2(t)\}$, and U_1 and U_2 are open and disjoint. $\square$

Exercises.

(141) Show that if $\underline{G}$ is an oligomorphic permutation group on a set A, then the closure of G in A^A contains some non-surjective maps (this is related to Exercise 106).

7.4. Canonical Functions

The material in this section stems from [**27**]. If $f\colon \mathbb{Q} \to \mathbb{Q}$ is any function from the order of the rational numbers to itself, then there are arbitrarily large finite subsets of $\mathbb{Q}$ on which f "behaves regularly"; that is, it is either strictly increasing, strictly decreasing, or constant. A direct (although arguably unnecessarily elaborate) way to see this is by applying Ramsey's theorem (see Section 2.2.3): two-element subsets of $\mathbb{Q}$ are coloured with three colours according to the local behaviour of f on them. In particular, it follows that

$$\overline{\{\beta\, f\, \alpha \mid \alpha, \beta \in \mathrm{Aut}(\mathbb{Q}; <)\}} \subseteq \mathbb{Q}^{\mathbb{Q}}$$

equipped with the pointwise convergence topology, contains a function which behaves regularly everywhere. This function of regular behavior is called canonical.

More generally, a function $f\colon \underline{A} \to \underline{B}$ between two structures $\underline{A}, \underline{B}$ is called *canonical* when it behaves regularly in an analogous way, that is, when it sends orbits of n-tuples of $\mathrm{Aut}(\underline{A})$ to orbits of n-tuples of $\mathrm{Aut}(\underline{B})$. Similarly as in the example above, canonical functions can be obtained from f, in the fashion stated above, when $\underline{A}$ has sufficient Ramsey-theoretic properties (for example, the Ramsey property) and when $\mathrm{Aut}(\underline{B})$ is sufficiently rich (for example ω-categorical) [**23, 24, 30**].

REMARK 7.4.1. *The concept of canonical functions has turned out useful in numerous applications: for classifying first-order reducts they are used in [**1, 2, 19, 28, 107, 128, 133**], for complexity classification for constraint satisfaction problems (CSPs) in [**20, 21, 25, 31, 96**], for decidability of meta-problems in the context of the CSPs in [**30**], for lifting algorithmic results from finite-domain CSPs to CSPs over infinite domains in [**22**], for lifting algorithmic results from finite-domain CSPs to homomorphism problems from definable infinite structures to finite structures [**94**], and for decidability questions in computations with atoms in [**95**].*

As indicated above, the technique is available for a function $f\colon A \to B$ whenever $\underline{A}$ is a Ramsey structure and $\underline{B}$ is ω-categorical, and the existence of canonical functions in the set

$$\overline{\{\beta\, f\, \alpha \mid \alpha \in \operatorname{Aut}(\underline{A}), \beta \in \operatorname{Aut}(\underline{B})\}} \subseteq B^A$$

was originally shown under these conditions by a combinatorial argument [**23**,**24**,**30**]. It is natural to ask for a perhaps more elegant proof of the existence of canonical functions via topological dynamics, reminiscent of the numerous proofs of combinatorial statements obtained in this fashion (cf. the survey [**11**] for ergodic Ramsey theory; [**91**] mentions some applications of extreme amenability). In this section we present such a proof, taken from [**27**].

7.4.1. Canonicity. Let $\underline{G} \leq \operatorname{Sym}(A)$ and $\underline{H} \leq \operatorname{Sym}(B)$. A function $f\colon A \to B$ is called *canonical with respect to* $(\underline{G}, \underline{H})$ if for every $k \geq 1$, $t \in A^k$, and $\alpha \in G$ there exists $\beta \in H$ such that $f\,\alpha(t) = \beta\, f(t)$. Hence, functions that are canonical with respect to $(\underline{G}, \underline{H})$ induce for each integer $k \geq 1$ a function from the orbits of the componentwise action of $\underline{G}$ of A^k to the orbits of the componentwise action of $\underline{H}$ on B^k. For oligomorphic permutation groups we have the following equivalent characterisations of canonicity.

PROPOSITION 7.4.2. *Let $\underline{G} \curvearrowright A$ and $\underline{H} \curvearrowright B$ be permutation groups, where $\underline{H} \curvearrowright B$ is oligomorphic. Then for any function $f\colon A \to B$ the following are equivalent.*

(1) *f is canonical with respect to $(\underline{G}, \underline{H})$;*
(2) *for every $\alpha \in G$ we have $f\alpha \in \overline{Hf} := \overline{\{\beta f \mid \beta \in H\}}$;*
(3) *for every $\alpha \in G$ there are $e_1, e_2 \in \overline{H}$ such that $e_1 f \alpha = e_2 f$.*

A stronger condition is to require that for all $\alpha \in G$ there is an $e \in H$ such that $f\alpha = ef$. To illustrate that this is strictly stronger, already if $\underline{G} = \underline{H}$, we give an explicit example.

EXAMPLE 93 (Trung Van Pham). Let $\underline{G} := \operatorname{Aut}(\mathbb{Q};<)$. Note that $(\mathbb{Q};<)$ and $(\mathbb{Q} \setminus \{0\};<)$ are isomorphic, and let f be such an isomorphism. Then f, viewed as a function from $\mathbb{Q} \to \mathbb{Q}$, is clearly canonical with respect to $(\underline{G}, \underline{G})$. But f does not satisfy the stronger condition above: there is no $e \in \overline{\underline{G}}$ such that $f\alpha = ef$. To see this, choose $b, c \in \mathbb{Q}$ such that $f(b) < 0 < f(c)$. By transitivity there exists an $\alpha \in \underline{G}$ such that $\alpha(b) = c$. Note that $0 < f\alpha(b) < f\alpha(c)$. Morever, the image of $f\alpha$ equals the image of f, and hence any $e \in \overline{\underline{G}}$ such that $f\alpha = ef$ must fix 0. Since e must also preserve $<$, it cannot map $f(b) < 0$ to $f\alpha(b) > 0$. Hence, there is no $e \in \overline{\underline{G}}$ such that $f\alpha = ef$. $\triangle$

In Proposition 7.4.2, the implications from (1) to (2) and from (3) to (1) follow straightforwardly from the definitions. For the implication from (2) to (3) we need a lift lemma, which is in essence from [**29**]. This lemma has been applied frequently lately [**8**,**20**,**22**], in various slightly different forms.

Let $\mathrm{H} \curvearrowright B$ be a permutation group, and let $f, g \in B^A$, for some A. We say that $f = g$ *holds locally modulo H* if for all finite $F \subseteq A$ there exist $\beta_1, \beta_2 \in \mathrm{H}$ such that

$\beta_1 f|_F = \beta_2 g|_F$. We say that $f = g$ *holds globally modulo H (modulo $\overline{H}$)* if there exist $e_1, e_2 \in$ H ($e_1, e_2 \in \overline{H}$, respectively) such that $e_1 f = e_2 g$. Of course, if $f = g$ holds globally modulo $\overline{H}$, then it holds locally modulo H. On the other hand, if $f = g$ holds locally modulo $\underline{H}$, then it need not hold globally modulo H. To see this, let $f(x, y)\colon \omega^2 \to \omega$ be an injection, set $g := f(y, x)$, and let $\underline{H}$ be the group of all permutations of ω. Then $f = g$ holds locally modulo H, but not globally. However, there exist injections $e_1, e_2 \in \omega^\omega$ such that $e_1 f = e_2 g$, so $f = g$ holds globally modulo $\overline{H}$. This is true in general, as we see in the following lift lemma.

LEMMA 7.4.3. *Let $\underline{H} \curvearrowright B$ be an oligomorphic permutation group, let I be an index set, and let A_i be a set for every $i \in I$. Let f_i, g_i be functions in B^{A_i} such that $f_i = g_i$ holds locally modulo $\underline{H}$ for all $i \in I$. Then there exist $e, e_i \in \overline{\underline{H}}$ such that $e\, f_i = e_i\, g_i$ holds globally for all $i \in I$.*

PROOF. For simplicity of notation, assume that the A_i are countable; then B^{A_i} is a metric space (otherwise, we would have to work with more general topological notions than sequences). We have $f_i \in \overline{H\, g_i}$; so let $(\beta_i^j\, g_i)_{j \in \omega}$ be a sequence converging to f_i for all $i \in I$. Setting $A := B$ we see that $A^A \mathbin{/\!\!/} \underline{G}$ is compact by Lemma 7.3.6. Therefore, the set

$$\{([\delta]_\approx, ([\delta\, \beta_i^j]_\approx)_{i \in I}) \mid j \in \omega, \delta \in H\}$$

is a subset of a compact space, $(A^A \mathbin{/\!\!/} \underline{G}) \times (A^A \mathbin{/\!\!/} \underline{G})^I$. Hence, it has an accumulation point $([e]_\approx, ([e_i]_\approx)_{i \in I})$. Clearly, $e, e_i \in H$ for all $i \in I$, and the functions e_i prove the lemma. $\square$

The implication from (2) to (3) in Proposition 7.4.2 now is a direct consequence of Lemma 7.4.3.

7.4.2. Canonisation. The following is the *canonisation theorem*, first proved combinatorially in [**30**] in a slightly more specialized context.

THEOREM 7.4.4. *Let $\underline{G} \curvearrowright A$, $\underline{H} \curvearrowright B$ be permutation groups where $\mathcal{G}$ is extremely amenable and H is oligomorphic, and let $f\colon A \to B$. Then*

$$\overline{H\, f\, G} := \overline{\{\beta\, f\, \alpha \mid \alpha \in G, \beta \in H\}}$$

contains a canonical function with respect to $(\underline{G}, \underline{H})$.

PROOF. The space $\overline{H\, f\, G}/_\sim$ is a closed subspace of the compact Hausdorff space $Y^X \mathbin{/\!\!/} \underline{G}$ from Lemma 7.3.6, and hence is a compact Hausdorff space as well. We define a continuous action of $\underline{G}$ on this space by

$$(\alpha, [g]_\approx) \mapsto [g\, \alpha^{-1}]_\approx\ .$$

Clearly, this assignment is a function, it is a group action, and it is continuous. Since $\underline{G}$ is extremely amenable, the action has a fixed point $[g]_\approx$. Any member g of this fixed point is canonical: whenever $\alpha \in \underline{G}$, then $[g\, \alpha]_\approx = [g]_\approx$, which is the definition of canonicity. $\square$

7.5. Model-Complete Cores of Ramsey Structures

Model companions and model-complete cores are a powerful method to construct new structures from known ones. They inherit quite a number of important properties from the structures we start from. We will see that for first-order reducts of homogeneous Ramsey structures the theory of model companions and model complete cores is particularly well behaved (Section 7.5.3). We present the theory for model complete cores; the theory for model companions can be seen as a special case, as we will see Remark 7.5.3.

7.5.1. Model Complete Cores. The results from this section are from [**14**]; we follow the presentation in [**16**]. An ω-categorical structure $\underline{C}$ is called

- *model complete* if every embedding from $\underline{C}$ into $\underline{C}$ preserves all first-order formulas.
- a *core* if every endomorphism of $\underline{C}$ is an embedding.

PROPOSITION 7.5.1 (see [**16**]). *Let $\underline{C}$ be an ω-categorical structure. Then the following are equivalent.*

(1) *$\underline{C}$ is a model-complete core.*
(2) *Every endomorphism of $\underline{C}$ preserves all first-order formulas.*
(3) *For every $n \in \mathbb{N}$, the orbits of n-tuples of $\mathrm{Aut}(\underline{C})$ are primitively positively definable in $\underline{C}$.*
(4) *For every $e \in \mathrm{End}(\underline{B})$, $n \in \mathbb{N}$, and $a \in C^n$ there exists $i \in \mathrm{End}(\underline{C})$ such that $i(e(a)) = a$.*

THEOREM 7.5.2 (from [**14**]; see [**16**]). *For every ω-categorical structure $\underline{B}$ there exists an model-complete core structure $\underline{C}$ which is homomorphically equivalent to $\underline{B}$; the structure $\underline{C}$ is unique up to isomorphism, and ω-categorical. We may assume that $\underline{C}$ is an induced substructure of $\underline{B}$.*

REMARK 7.5.3. *Saracino's theorem.*

7.5.2. Range Rigid Functions. The results in this section are from [**120**]. Let $\underline{G}$ be a permutation group on a set X. A function $g\colon X \to X$ is called *range-rigid with respect to $\underline{G}$* if for all $\beta \in G$ we have

$$g \in \overline{\{\alpha \circ g \circ \beta \circ g \mid \alpha \in G\}}.$$

In particular, the identity map is range-rigid. Note that a function $g\colon X \to X$ is range-rigid with respect to $\underline{G}$ if and only if for all $t \in X^n$, $n \in \mathbb{N}$, if there exists $s \in X^n$ and $\alpha \in G$ such that $\alpha g(s) = t$, then t and $g(t)$ lie in the same orbit. In other words, g preserves the orbits of n-tuples that have non-empty intersection with $g(X)^n$.

EXAMPLE 94. Let U, V be unary relation symbols, and let $\underline{B}$ be a countably infinite $\{U, V\}$-structure where $U = \{u\}$, $V = \{v\}$, and $u \neq v$. Let $w \in B \setminus \{u, v\}$. Then the map $g\colon B \to B$ with $g(w) = g(u) = u$ and $g(x) = v$ for all $x \in B \setminus \{u, w\}$ is range-rigid with respect to $\mathrm{Aut}(\underline{B})$, but not canonical with respect to $\underline{B}$. The identity map is canonical with respect to $\mathrm{Aut}(\underline{B})$ but not range-rigid with respect to $\mathrm{Aut}(\underline{B})$. △

We can use canonisation to find range-rigid functions in sufficiently rich sets of operations.

THEOREM 7.5.4. *Let $\underline{G}$ be a closed oligomorphic extremely amenable permutation group on a countable set X and let $\underline{M}$ be a non-empty closed transformation semigroup on X such that $G \circ M \circ G \subseteq M$. Then M contains a function which is range-rigid with respect to $\underline{G}$ and canonical with respect to $\underline{G}$.*

PROOF. Pick any $f \in M$. Applying Proposition 7.4.2 to f, we obtain a function $f' \in \overline{\{\alpha f \beta \mid \alpha, \beta \in G\}} \subseteq M$ which is canonical with respect to $\underline{G}$. Since $\underline{G}$ is oligomorphic, for every $n \in \mathbb{N}$ there are finitely many orbits of n-tuples for every n, so we may compose f' with itself sufficiently many times to obtain a function such that for all $t \in X^n$ whose orbit contain a tuple of the form $g(s)$ for $s \in X^n$ we have that $g(t)$ lies in the same orbit as t. A standard compactness argument shows that there is one function that does it for all n; note that the resulting map is range-rigid with respect to $\underline{G}$ and canonical with respect to $\underline{G}$. □

LEMMA 7.5.5. *Let $\underline{A}$ be a homogeneous structure and let $g\colon A \to A$ be range-rigid with respect to* $\mathrm{Aut}(\underline{A})$. *Then the age $\mathcal{C}$ of the substructure induced by the image of g in $\underline{A}$ has the amalgamation property.*

PROOF. Let $\underline{B}_1, \underline{B}_2 \in \mathcal{C}$. We claim that the restriction g_i of g to B_i, for $i \in \{1,2\}$, shows that $\underline{A}[g(B_1 \cup B_2)] \in \mathcal{C}$ is an amalgam of $\underline{B}_1$ and $\underline{B}_2$. It suffices to show that the restriction g_i of g to B_i is an embedding. By the definition of $\mathcal{C}$ there exists a finite set $B_i' \subseteq A$ such that $B_i = g(B_i')$. Since g is range-rigid, we have in particular that $g \in \overline{\{\alpha \circ g \circ g \mid \alpha \in \mathrm{Aut}(\underline{A})\}}$, and hence there exists an $\alpha_i \in \mathrm{Aut}(\underline{A})$ such that $g \circ g(a) = \alpha_i \circ g(a)$ for all $a \in B_i'$. Hence, for all $b \in B_i$ we have $g_i(b) = \alpha_i(b)$, which shows that g_i is an embedding of $\underline{B}_i$ into $\underline{C}$. $\square$

DEFINITION 7.5.6. *Let $\underline{A}$ be a homogeneous structure and let $g\colon A \to A$ be range-rigid with respect to* $\mathrm{Aut}(\underline{A})$. *We denote by $\underline{A}_g$ the Fraïssé-limit of the age of the substructure induced by $g(A)$ in $\underline{A}$ (which has the amalgamation property by Theorem 3.3.5). By the homogeneity of $\underline{A}$, we may assume that $\underline{A}_g$ is an induced substructure of $\underline{A}$.*

LEMMA 7.5.7. *Let $\underline{A}$ be a homogeneous structure and let $g\colon A \to A$ be range-rigid with respect to* $\mathrm{Aut}(\underline{A})$. *If $\underline{A}$ is ω-categorical, then $\underline{A}_g$ is ω-categorical as well.*

PROOF. If $\underline{A}$ is ω-categorical, then $\underline{A}$ has for every n only finitely many inequivalent atomic formulas with n variables. Since the age of $\underline{A}_g$ is contained in the age of $\underline{A}$, the same is true for $\underline{A}_g$, and the statement thus follows from the homogeneity of $\underline{A}_g$. $\square$

LEMMA 7.5.8. *Let $\underline{A}$ be a homogeneous τ-structure and let $g\colon A \to A$ be range-rigid with respect to* $\mathrm{Aut}(\underline{A})$. *If $\underline{A}$ is finitely bounded, then so is $\underline{A}_g$.*

PROOF. Suppose that $\mathcal{F}$ is a finite set of finite τ-structures such that $\mathrm{Forb}(\mathcal{F}) = \mathrm{Age}(\underline{A})$. Let $m \geq 1$ be the maximum arity of the relations of $\underline{B}$. Let $\mathcal{F}'$ be the union of $\mathcal{F}$ with the set of all structures on the set $\{1,\dots,m\}$ that do not embed into $\underline{A}_g$. Clearly, $\mathrm{Age}(\underline{A}_g) \subseteq \mathrm{Forb}(\mathcal{F}')$. Let $\underline{D} \in \mathrm{Forb}(\mathcal{F}')$; we have to show that $\underline{D} \in \mathrm{Age}(\underline{A}_g)$. Since $\mathrm{Forb}(\mathcal{F}') \subseteq \mathrm{Forb}(\mathcal{F}) = \mathrm{Age}(\underline{A})$ we may assume that $\underline{D}$ is a substructure of $\underline{A}$ and claim that the restriction of g to $\underline{D}$ is an embedding. Let $t \in D^m$; since $\underline{D} \subseteq \mathrm{Forb}(\mathcal{F}')$ we have that the substructure induced by $\{t_1,\dots,t_m\}$ in $\underline{D}$ is in $\mathrm{Age}(\underline{A}_g)$. By the homogeneity of $\underline{A}$ there exists $\beta \in \mathrm{Aut}(\underline{A})$ and $s \in A^m$ such that $t = \beta g(s)$. Since g is range rigid, there exists $\alpha \in \mathrm{Aut}(\underline{A})$ such that $\alpha g \beta g(s) = g(s)$, and hence $t = \beta g(s)$ and $g(t) = g(\beta g(s)) = \alpha^{-1} g(s) = \alpha^{-1}\beta^{-1} t$ lie in the same orbit of $\mathrm{Aut}(\underline{A})$, which proves the claim. $\square$

LEMMA 7.5.9. *Let $\underline{A}$ be a homogeneous structure and let $g\colon A \to A$ be range-rigid with respect to* $\mathrm{Aut}(\underline{A})$. *If $\underline{A}$ is Ramsey, then so is $\underline{A}_g$.*

PROOF. Let $\underline{S}$ and $\underline{M}$ be finite substructures of $\underline{A}_g$ and let $\chi\colon \binom{\underline{A}_g}{\underline{S}} \to \{0,1\}$ be a colouring of the copies of $\underline{S}$ in $\underline{A}_g$ with the colours 0 and 1. Let $\underline{B}$ be the substructure of $\underline{A}$ induced by $g(A)$. By the homogeneity of $\underline{A}_b$ and since $\mathrm{Age}(\underline{B}) = \mathrm{Age}(\underline{A}_b)$ there exists an embedding $f\colon \underline{B} \to \underline{A}_b$. We then define a colouring χ' of $\binom{\underline{A}}{\underline{S}} \to \{0,1\}$ by $\chi'(\underline{S}') := \chi(\underline{A}_g[f \circ g(S')])$. Since $\underline{A}$ is Ramsey, there exists $\underline{M}' \in \binom{\underline{A}}{\underline{M}}$ such that χ' is constant $c \in \{0,1\}$ on $\binom{\underline{M}'}{\underline{S}}$.

Claim 1. $\underline{M}'' := \underline{B}_g[f \circ g(M')]$ is isomorphic to $\underline{M}$. First note that $\underline{M}$ embeds into $\underline{B}$ because $\underline{A}_g$ and $\underline{B}$ have the same age; since $\underline{A}$ is homogeneous, there exists $\alpha \in \mathrm{Aut}(\underline{A})$ which maps this copy of $\underline{M}$ in $\underline{B} = \underline{A}[g(B)]$ to $\underline{M}'$. By the range rigidity of g, there exists $\beta \in \mathrm{Aut}(\underline{A})$ such that $\beta \circ g(M') = \alpha(M')$; hence, the restriction of g to M' is an embedding, and the claim follows.

Claim 2. χ is constant on $\binom{M''}{\underline{S}}$. If $\underline{S}'' \in \binom{M''}{\underline{S}}$, then $\chi(\underline{S}'') = \chi'(\underline{B}[g^{-1} \circ f^{-1}(S'')]) = c$ since $g^{-1} \circ f^{-1}(S'')$ is a copy of $\underline{S}$ in $\underline{M}'$. $\square$

7.5.3. Range-Rigidity and Model-Complete Cores. A subset S of a monoid $\underline{M}$ is called a *left ideal of* $\underline{M}$ if $MS = S$. Note that for every $f \in M$, then set $T := \overline{Mf}$ is a (closed) left ideal, because $MT = M\overline{Mf} \subseteq \overline{Mf} = T$.

Lemma 7.5.10. *Let $\underline{M}$ be a monoid and $f \in M$. Then the following are equivalent.*

(1) *f lies in an inclusion-wise minimal closed left-ideal of $\underline{M}$;*
(2) *$\overline{Mf}$ is an inclusion-wise minimal closed left-ideal;*
(3) *$f \in \overline{Mef}$ for every $e \in M$.*

Proof. (1) implies (2). Suppose that $f \in S$ for some inclusion-wise minimal closed left-ideal S of $\underline{M}$. Then $T := \overline{Mf} \subseteq \overline{MS} = \overline{S} = S$, so $T = S$ by the minimality of S, and hence T is an inclusion-wise minimal closed left-ideal S of $\underline{M}$.

(2) implies (1). We have $f \in \overline{Mf}$ since M is a monoid and thus contains 1.

(2) implies (3). Let $e \in M$. Then $\overline{Mef} \subseteq \overline{Mf}$ is closed, non-empty, and a left-ideal since $M\overline{Mef} = \overline{Mef}$. By the minimality of $\overline{Mf}$ we have that $\overline{Mef} = \overline{Mf}$, so $f \in \overline{Mef}$ since $1 \in M$.

(3) implies (2). It suffices to show minimality of $T := \overline{Mf}$. Suppose that there exists a non-empty closed left-ideal I of $\underline{M}$ contained in $\overline{Mf}$. Let $t \in I$. Then $t \in \overline{Mf}$ so there are $r_1, r_2, \dots \in M$ such that $\lim_{i\in\mathbb{N}} r_i f = t$. By assumption, for every $i \in \mathbb{N}$ we have that $f \in \overline{Mr_i f}$, so there exist $s_{i,1}, s_{i,2}, \dots \in M$ such that $\lim_{j\in\mathbb{N}} s_{i,j} r_i f = f$. We claim that $\lim_{j\in\mathbb{N}} s_{ij} t = f$. Indeed,

$$\begin{aligned}\lim_{j\in\mathbb{N}} s_{ij} t &= \lim_{j\in\mathbb{N}} s_{ij} \lim_{i\in\mathbb{N}} r_i f \\ &= \lim_{i\in\mathbb{N}} \lim_{j\in\mathbb{N}} s_{ij} r_i f = f.\end{aligned}$$

Hence, $f \in \overline{MI} = I$. Therefore, $I = \overline{MI} \supseteq \overline{Mf} \supseteq I$, showing that $\overline{Mf}$ is a minimal closed left-ideal of $\underline{M}$. $\square$

The following lemma has been shown in [**7**] (Lemma 5.4); the proof uses the equivalence relation $\approx$ from Definition 7.3.5.

Lemma 7.5.11. *Let X be countable and let $\underline{M}$ be a closed transformation monoid on X that contains a closed oligomorphic permutation group $\underline{G}$. Then $\underline{M}$ has a inclusion-wise minimal non-empty closed left-ideal S.*

Proof. Let $\mathcal{T}$ be the set of all non-empty topologically closed subsets T of $M /\!\!/ \underline{G}$ (Definition 7.3.5) with the property that whenever $[f]_\approx \in T$ and $m \in M$, then $[m \circ f]_\approx \in T$. Since $M /\!\!/ \underline{G}$ is compact (Lemma 7.3.6), arbitrary descending chains in $\mathcal{T}$ have a non-empty intersection in $\mathcal{T}$ (the proof of the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) for Theorem 4.1.19 works in arbitrary topological spaces). Hence, $\mathcal{T}$ contains a minimal element T_0 by Zorn's Lemma. Then $S_0 := \{f \in M \mid [f]_\approx \in T_0\}$ is closed and non-empty, and $M \circ S_0 = S_0$, and minimal with these properties. $\square$

Lemma 7.5.12. *Let $\underline{B}$ be a first-order reduct of a homogeneous structure $\underline{A}$. Suppose that*

- *S is a minimal non-empty closed left-ideal of* $\mathrm{End}(\underline{B})$*, and*
- *there exists $g \in S$ which is range-rigid with respect to* $\mathrm{Aut}(\underline{A})$*.*

Then the substructure $\underline{C}$ of $\underline{B}$ with domain $\underline{A}_g$ is the model-complete core of $\underline{B}$.

PROOF. Since $\underline{A}[g(A)]$ and $\underline{A}_g$ have the same age and $\underline{A}_g$ is homogeneous, there exists an embedding $f\colon \underline{A}[g(A)] \to \underline{A}_g$. Then $f \circ g$ is a homomorphism of $\underline{B}$ into $\underline{C}$, because every relation of $\underline{B}$ has a quantifier-free definition in $\underline{A}$, and the same formula defines the corresponding relations of $\underline{C}$ in $\underline{A}_g$. Since $\underline{C}$ is even a substructure of $\underline{B}$, they are homomorphically equivalent. So it suffices to prove that $\underline{C}$ is a model-complete core. Let $e \in \mathrm{End}(\underline{C})$, $n \in \mathbb{N}$, and $t \in (A_g)^n$.

Claim 1. We may choose f so that there exists $s \in A^n$ with $(f \circ g^2)(s) = t$. Indeed, there exists $s \in A^n$ such that $g(s)$ satisfies the same atomic formulas as t in $\underline{A}$, because $\underline{A}[g(A)]$ and $\underline{A}_g$ have the same age. Since g is range-rigid with respect to $\mathrm{Aut}(\underline{A})$, there exists an automorphism of $\underline{A}$ that maps $g(g(s))$ to $g(s)$. The homogeneity of $\underline{A}_g$ implies that there exists $\alpha \in \mathrm{Aut}(\underline{A}_g)$ which maps $(f \circ g^2)(s)$ to t. Therefore, $\alpha \circ f$ is an embedding of $\underline{A}[g(A)]$ into $\underline{A}_g$ which has the required property.

Claim 2. There exists $h \in \mathrm{End}(\underline{B})$ such that $h \circ (e \circ f \circ g)(g(s)) = g(s)$. Otherwise, $S' := \overline{\mathrm{End}(\underline{B}) \circ \{e \circ f \circ g^2\}} \subseteq S$ does not contain g. Since $\mathrm{End}(\underline{B}) \circ S' = S'$, this contradicts the minimality of S. Then

$$\begin{aligned} t = (f \circ g^2)(s) &= (f \circ g) \circ h \circ (e \circ f \circ g)g(s) && \text{(Claim 2)} \\ &= (f \circ g \circ h) \circ e(t). \end{aligned}$$

The restriction of $f \circ g \circ g$ to A_g therefore proves condition (2) of Proposition 7.5.1 for $\underline{C}$, and hence $\underline{C}$ is a model-complete core. $\square$

LEMMA 7.5.13. *Let X be countable and let $\underline{G} \leq \mathrm{Sym}(X)$ be closed, oligomorphic, and extremely amenable. Let $\underline{M}$ be a closed transformation monoid containing $\underline{G}$. Then $\underline{M}$ contains a function which is range-rigid and canonical with respect to $\underline{G}$ and is contained in a minimal closed left-ideal S of $\underline{M}$.*

PROOF. By Lemma 7.5.11 $\underline{M}$ contains a minimal non-empty closed left-ideal S_0. Let S_1 be the smallest closed transformation semigroup which contains S_0 such that $G \circ S_1 \circ G = S_1$. By Theorem 7.5.4, S_1 contains a function g which is range-rigid and canonical with respect to $\underline{G}$.

If thus suffices to show that every element of S_1 lies in a minimal closed left-ideal of $\underline{M}$. This is clear for every element of $S_0 \subseteq S_1$. Now suppose that f lies in a minimal closed left-ideal of $\underline{M}$, and let $h \in M$. We claim that hf and fh are contained in minimal closed left-ideals of $\underline{M}$. This then implies that every element of S_1 is contained in a minimal closed left-ideal of $\underline{M}$.

By Lemma 7.5.10 (2), we have that $\overline{Mf}$ is a minimal closed left-ideal. Note that $\overline{Mf} = \overline{Mhf}$ and hence hf lies in a minimal closed left-ideal by Lemma 7.5.10 (2). To show that fh lies in a minimal closed left-ideal, we use Lemma 7.5.10 (3). Let $e \in M$. Lemma 7.5.10 (3) implies that $f \in \overline{Mef}$. Thus, $fh \in \overline{Mef}h = \overline{Me(fh)}$, so Lemma 7.5.10 (3) implies that fh indeed lies in a minimal closed left-ideal of $\underline{M}$. $\square$

The following theorem summarises many of the previous statements.

THEOREM 7.5.14. *Let $\underline{B}$ be a first-order reduct of an ω-categorical homogeneous Ramsey structure $\underline{A}$ and let $\underline{C}$ be the model-complete core of $\underline{B}$. Then*

(1) *$\underline{B}$ has an endomorphism g which is range-rigid with respect to $\mathrm{Aut}(\underline{A})$ and canonical with respect to $\mathrm{Aut}(\underline{A})$.*
(2) *$\underline{A}_g$ is ω-categorical and Ramsey.*
(3) *If $\underline{A}$ is finitely bounded, then $\underline{A}_g$ is finitely bounded as well.*
(4) *The substructure of $\underline{B}$ with domain A_g is isomorphic to $\underline{C}$; we identify C with A_g along this isomorphism. Then $\underline{C}$ is a first-order reduct of $\underline{A}_g$.*

Proof. By Lemma 7.5.13 applied to $\underline{M} = \mathrm{End}(\underline{B})$ and $\underline{G} = \mathrm{Aut}(\underline{A})$, we obtain a function $g \in \mathrm{End}(\underline{B})$ which is range-rigid with respect to $\mathrm{Aut}(\underline{A})$ and canonical with respect to $\mathrm{Aut}(\underline{A})$, which shows (1), and additionally contained in a minimal closed left-ideal of $\mathrm{End}(\underline{B})$. By Lemma 7.5.7, $\underline{A}_g$ is ω-categorical, and it is Ramsey by Lemma 7.5.9, which shows (2). Lemma 7.5.8 implies item (3).

By Lemma 7.5.12, the substructure of $\underline{B}$ with domain $\underline{A}_g$ is isomorphic to the model-complete core $\underline{C}$ of $\underline{B}$. Since $\underline{A}$ is homogeneous, all relations of $\underline{B}$ have a quantifier-free definition in $\underline{A}$, and hence the same formulas define the relations of $\underline{C}$ in $\underline{A}_g$; this completes the proof of item (4). $\square$

We present an immediate consequence of this theorem (whose statement does not involve the concept of range-rigidity).

Corollary 7.5.15. *Let $\underline{B}$ be a first-order reduct of an ω-categorical homogeneous Ramsey structure $\underline{A}$ and let $\underline{C}$ be the model-complete core of $\underline{B}$. Then $\underline{C}$ has an ω-categorical homogeneous Ramsey expansion $\underline{A}'$. If $\underline{A}$ has a finite signature, then so has $\underline{A}'$. If $\underline{A}$ is finitely bounded, then so is $\underline{A}'$.*

CHAPTER 8

Reducts and Closed Supergroups

[**2**, **10**, **32**, **33**, **37**, **87**, **107**, **128**, **133**, **134**, **156**, **157**]

CONJECTURE 8.1 (Thomas [**156**]). *Let $\underline{A}$ be a homogeneous structure with a finite relational signature. Then* $\mathrm{Aut}(\underline{A})$ *has only finitely many closed supergroups in* $\mathrm{Sym}(A)$. *Equivalently, $\underline{A}$ has only finitely many first-order reducts up to interdefinability.*

Cameron's theorem for highly set-transitive permutation groups on a countably infinite set, Thomas' result about the closed supergroups of the automorphism group of the random graph.

Exercises.

(142) Let $(V;T)$ be the Fraïssé-limit of the class of all finite tournaments (see Exercise 57). Show that

$$\mathrm{Aut}(V;\{(x,y,u,v) \mid T(x,y) \Leftrightarrow T(u,v)\})$$

is isomorphic to a semidirect product of $\mathbb{Z}_2$ and $\mathrm{Aut}(V;T)$.

(143) Let $(V;E)$ be the Rado graph (Example 29), and let $R \subseteq V^4$ be the relation $\{(a,b,c,d) \mid E(a,b) \Leftrightarrow E(c,d)\}$.
Show that $\underline{N} := \mathrm{Aut}(V;E)$ is a closed normal subgroup of $\underline{G} := \mathrm{Aut}(V;R)$ of index 2, but that $\underline{G}$ is *not* isomorphic to a semidirect product of $\underline{N}$ and $\mathbb{Z}_2$ (see Proposition 1.5.4).

CHAPTER 9

Restricted Orbit Growth

[**36**, **38**, **41**, **55**, **112–115**, **135**, **136**, **144**]

CHAPTER 10

Homogeneous Structures in Restricted Signatures

[**4**, **42**, **44**–**46**, **65**, **85**, **99**–**101**, **105**, **108**, **140**]

Exercises.

(144) Show that up to isomorphism, there is only one countable homogeneous linear order.

THEOREM 10.0.1 (Woodrow). *Up to isomorphism, there are only two countable homogeneous tournaments that do not embed the 4-element tournament* $\underline{D}$ *from Exercise 62.*

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APPENDIX A

Background Material

A.1. Ultrafilter

Let X be a set. A filter on X is a certain set of subsets of X; the idea is that the elements of $\mathcal{F}$ are (in some sense) 'large'; it helps thinking of the elements $F \in \mathcal{F}$ as being 'almost all' of X.

DEFINITION A.1.1. *A* filter *$\mathcal{F}$ on X is a set of subsets of X such that*
1. *$\emptyset \notin \mathcal{F}$ and $X \in \mathcal{F}$;*
2. *if $F \in \mathcal{F}$ and $G \subseteq X$ contains F, then $G \in \mathcal{F}$.*
3. *if $F_1, F_2 \in \mathcal{F}$ then $F_1 \cap F_2 \in \mathcal{F}$.*

Note that filters have the *finite intersection property*:

$$A_1, \dots, A_n \in \mathcal{F} \Rightarrow A_1 \cap \cdots \cap A_n \neq \emptyset \qquad \text{(FIP)}$$

LEMMA A.1.2. *Every subset $\mathcal{S} \subseteq \mathcal{P}(X)$ with the FIP is contained in a smallest filter that contains $\mathcal{S}$; this filter is called the filter* generated *by $\mathcal{S}$.*

PROOF. First add finite intersections, and then all supersets to $\mathcal{S}$. $\square$

EXAMPLE 95. For a non-empty subset $Y \subseteq X$, the family

$$\mathcal{F} := \{Z \subseteq X \mid Y \subseteq Z\}$$

is a filter, the filter generated by a $\{Y\}$; such filters are called *principal.* $\triangle$

EXAMPLE 96. The *Fréchet filter*: for an infinite set X this is the filter $\mathcal{F}$ that consists of all cofinite subsets of X, i.e.,

$$\mathcal{F} := \{Y \subseteq X \mid X \setminus Y \text{ is finite}\}. \qquad \triangle$$

A filter $\mathcal{F}$ is called a *ultrafilter* if $\mathcal{F}$ is maximal, that is for every filter $\mathcal{G} \supseteq \mathcal{F}$ we have $\mathcal{G} = \mathcal{F}$.

LEMMA A.1.3. *Let $\mathcal{F}$ be a filter. Then the following are equivalent.*
1. *$\mathcal{F}$ is a ultrafilter.*
2. *For all $A \subseteq X$ either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$.*
3. *For all $A_1 \cup \cdots \cup A_n \in \mathcal{F}$ there is an $i \leq n$ with $A_i \in \mathcal{F}$.*

PROOF. (1) $\Leftarrow$ (2): No $A \subseteq X$ can be added to $\mathcal{F}$. Hence, $\mathcal{F}$ is maximal.
(2) $\Leftarrow$ (3): Note that $A \cup (X \setminus A) = X \in \mathcal{F}$.
(1) $\Rightarrow$ (3): If there is an $i \leq n$ such that $\mathcal{F} \cup \{A_i\}$ has the FIP, then by Lemma A.1.2 there is a filter that contains this set, and hence $\mathcal{F}$ was not maximal. Otherwise, there are $S_1, \dots, S_n \subseteq \mathcal{F}$ with $A_i \cap S_i = \emptyset$. Then $S_i \subseteq X \setminus A_i$ and thus $S_1 \cap \cdots \cap S_n \subseteq X \setminus (A_1 \cup \cdots \cup A_n) \notin \mathcal{F}$, a contradiction. $\square$

A filter $\mathcal{F}$ is *principal* if it contains a inclusionwise minimal element. Note that this is the case if and only if $\bigcap \mathcal{F} \in \mathcal{F}$.

LEMMA A.1.4. *Let $\mathcal{F}$ be a filter on a set X. Then the following are equivalent.*

(1) *$\mathcal{F}$ is a principal ultrafilter;*
(2) *$\mathcal{F}$ contains $\{a\}$ for some $a \in X$.*
(3) *$\mathcal{F}$ is of the form $\{Y \subseteq X \mid a \in Y\}$ for some $a \in X$.*
(4) *$\mathcal{F}$ is an ultrafilter and contains a finite set.*

PROOF. (1) $\Rightarrow$ (2): let $A := \bigcap \mathcal{F} \in \mathcal{F}$. If $|A| > 1$ then we can write $A = B_1 \cup B_2$ for $B_1, B_2 \subseteq X$ non-empty. But then Lemma A.1.3 (3) implies that $B_1 \in \mathcal{F}$ or $B_2 \in \mathcal{F}$, in contradiction to the definition of A. So $A = \{a\}$ for some $a \in X$.

(2) $\Rightarrow$ (3). Clearly, $\{Y \subseteq X \mid a \in Y\} \subseteq \mathcal{F}$ since $\mathcal{F}$ is closed under supersets, and $\mathcal{F} \subseteq \{Y \subseteq X \mid a \in Y\}$ since $\mathcal{F}$ does not contain the empty set.

(3) $\Rightarrow$ (4): Clearly $\mathcal{F}$ contains a finite set; use Lemma A.1.3 (2) to check that $\mathcal{F}$ is an ultrafilter.

(4) $\Rightarrow$ (1): If $A \in \mathcal{F}$ is finite, then $B := \bigcap F$ is finite, and hence B is the intersection of finitely many elements in $\mathcal{F}$, and hence in $\mathcal{F}$ since $\mathcal{F}$ is a filter. This shows that $\mathcal{F}$ is principal. $\square$

Are there non-principal ultrafilters?

LEMMA A.1.5 (Ultrafilter Lemma). *Every filter $\mathcal{F}$ is contained in a ultrafilter.*

PROOF. Let $\mathcal{M}$ be the set of all filters on X that contain $\mathcal{F}$, partially ordered by containment. Note that unions of chains of filters in this partial order are again filters. By Zorn's lemma, $\mathcal{M}$ contains a maximal filter. $\square$

Non-principal ultrafilters are also called *free ultrafilters*. In particular the Fréchet filter is contained in an ultrafilter, which must be free:

LEMMA A.1.6. *An ultrafilter is free if and only if it contains the Fréchet filter.*

PROOF. Let $\mathcal{U}$ be a free ultrafilter on X and let $x \in X$. Either $\{x\} \in \mathcal{U}$ or $X \setminus \{x\} \in \mathcal{U}$. As $\mathcal{U}$ is free, $\{x\} \notin \mathcal{U}$ (Lemma A.1.4). Hence, $X \setminus \{x\} \in \mathcal{U}$ for every $x \in X$. Let $F \subseteq X$ be finite. Then

$$X \setminus F = \bigcap_{x \in F} (X \setminus \{x\}) \in \mathcal{U}\,.$$

Now let $\mathcal{U}$ be a principal ultrafilter, i.e., there is $x \in X$ with $\{x\} \in \mathcal{U}$ (Lemma A.1.4). Then the element $X \setminus \{x\}$ of the Fréchet filters is not in $\mathcal{U}$. $\square$

Exercises.

(145) Show that a set of subsets of a set X can be extended to an ultrafilter if and only if it has the FIP.
(146) Show that a set $\mathcal{F}$ of subsets of a set X can be extended to a free ultrafilter if and only if the intersection of every finite subset of $\mathcal{F}$ is infinite.
(147) Show that every filter $\mathcal{F}$ on a set X is the intersection of all ultrafilters on X that extend $\mathcal{F}$.
(148) Show that if $\mathcal{U}$ is a free ultrafilter on X, and $S \in \mathcal{U}$ and $T \subseteq X$ are such that the symmetric difference $S \Delta T$ is finite, then $S \in \mathcal{U}$.

(149) Show that there are $2^{2^{|X|}}$ many ultrafilters on an infinite set X.
Hint: first show that there is a family $\mathcal{F}$ of $2^{|X|}$ subsets of X such that for any $A_1, \dots, A_n, B_1, \dots, B_n \in \mathcal{F}$

$$A_1 \cap \cdots \cap A_n \cap (X \setminus B_1) \cap \cdots \cap (X \setminus B_n) \neq \emptyset.$$

(150) True or false: if $\mathcal{U}$ and $\mathcal{V}$ are free ultrafilters on an infinite set X, is there is a permutation π of X such that $S \in \mathcal{U}$ if and only if $\pi(S) \in \mathcal{V}$?[1]

[1] Thanks to Lukas Juhrich for the idea for this exercise.

A.2. The Axiom of Choice and its Weaker Versions

'Das "Auswahlaxiom" (...) fordert in der gewöhnlichen (...) Fassung, daß zu jeder Menge M, deren Elemente (...) paarwise fremde und nicht-leere Mengen sind, mindestens eine "Auswahlmenge" existiere, die mit jedem Element von M genau ein Element gemeinsam hat. Die nächstliegende und mehrfach verwendete Methode, um ein schwächeres Postulat als die vorstehende Fassung zu formulieren, besteht darin, daß man entweder über die Mächtigkeit der Menge M, oder über die Mächtigkeit ihrer Elemente, oder über beides gleichzeitig, einschränkende Bedingung macht, also nur in diesen eingeschränkten Fällen die Existenz einer Auswahlmenge verlangt.'

Abraham Adolf Halevi Fraenkel [**57**]

'Let us sum things up: Topology with "choice" may be as unreal as a soap-bubble dream, but topology without "choice" is as horrible as a nightmare.'

Horst Herrlich [**66**]

The proofs of several statements in this text used the Axiom of Choice (Definition A.2.1). Sometimes these statements can also be proved using weaker forms of the Axiom of Choice, and sometimes they are even equivalent to these weaker forms.

- The Baire Category Theorem (Theorem 4.1.8);
- The Ultrafilter Lemma (Lemma A.1.5);
- The equivalence of compactness and sequential compactness (Theorem 4.1.19);
- Tychonoff's theorem (Theorem 4.1.14).

We have collected known facts about some weaker forms of the Axiom of Choice in Figure A.1. We first state these weaker forms and then provide references for the equivalences and implications that are displayed in Figure A.1. We have also collected what we know about non-implications.

Definition A.2.1 (Axiom of Choice). *For every nonempty set A there exists a function $f\colon A \to \bigcup A$ such that for every $S \in A$ we have $f(S) \in S$.*

The *Axiom of Countable Choice* is defined as the axiom of choice, but restricted to countable sets A.

Definition A.2.2 (Axiom of Dependent Choices). *Let $R \subseteq A^2$ be a relation with the property that for every $a \in A$ there exists $b \in A$ with $(a, b) \in R$. Then for every $c \in A$ there exists a function $f\colon \mathbb{N} \to A$ such that $f(0) = c$ and for every $i \in \mathbb{N}$ we have $(f(i), f(i + 1)) \in R$.*

Definition A.2.3 (Order Extension Principle). *Every partial order can be extended to a linear order.*

Definition A.2.4 (Ordering Principle). *Every set can be linearly ordered.*

The following equivalences are known to hold in Zermelo-Fraenkel set theory (ZF):

- The Axiom of Choice, Zorn's Lemma, and the Well-ordering Theorem are equivalent; see, e.g., [**84**].
- The Axiom of Choice implies Tychonoff's theorem [**161**], and Tychonoff's theorem implies the Axiom of Choice [**93**].
- The equivalence to the existence of surjections with right inverse is Exercise 153.
- The Axiom of Dependent Choices can be used to prove the Baire Category Theorem (Theorem 4.1.8) as we have seen in Section 4.1.5; the implication also holds for pseudo-metrics instead of metrics (also see the discussion in [**66**]); conversely, the Baire Category Theorem for pseudo-metric spaces implies the Axiom of Dependent Choices [**13**].

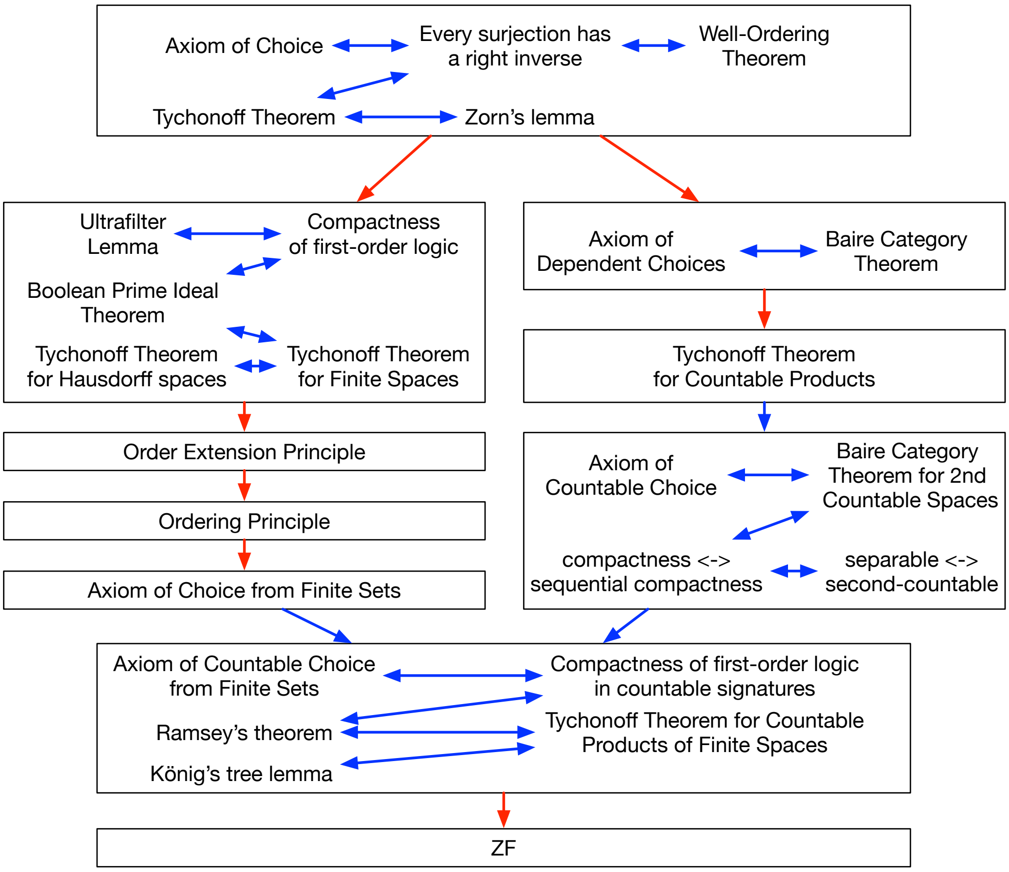

FIGURE A.1. The extensions of Zermelo-Fraenkel set theory by the Axiom of Choice or some of its weak versions, and their relationships. Blue arcs indicate implications, red arcs indicate strict implications.

- The Axiom of Countable Choice is equivalent to the Baire Category Theorem for second-countable pseudo-metric spaces [**66**, Theorem 2.4], and to Theorem 4.1.19 [**66**, Theorem 2.4].
- For the equivalence of the Boolean Prime Ideal Theorem (which we do not need here) and the Ultrafilter Lemma, see [**82**].
- The equivalence of the Ultrafilter Lemma and the compactness theorem of first-order logic is well-known; see [**17**].
- References for the equivalence between the Ultrafilter Lemma, Tychonoff's theorem for Hausdorff spaces, and Tychonoff's theorem for finite spaces can be found in [**66**, Theorem 3.4].
- The equivalence of the Axiom of Countable Choice from Finite Sets and König's Tree Lemma can be found in [**49**] (page 203), the equivalence to Tychonoff's theorem for countable products of finite spaces in [**97**], and the equivalence to Ramsey's theorem in [**109**]. A proof that countable choice from finite sets suffices to prove the compactness theorem for countable signatures can be found in [**71**], and the converse is easy to prove.

The following implications are known to hold in Zermelo-Fraenkel set theory (ZF):

- The Axiom of Choice implies the Axiom of Dependent Choices (Exercise 154).

- The Axiom of Dependent Choices implies Tychonoff's theorem for countable products of compact spaces (see the proof of Theorem 4.1.14).
- Tychonoff's theorem for countable products implies the Axiom of Countable Choice [**62**].
- The Axiom of Choice implies the Ultrafilter Lemma (Lemma A.1.5).
- The Ultrafilter Lemma implies the Order Extension Principle [**151**].
- The Order Extension Principle implies the Ordering Principle (Exercise 151).
- The Ordering Property implies the Axiom of Choice for families of non-empty finite sets (Exercise 152).
- Trivially, the Axiom of Countable Choice from Finite Sets is implied by the Axiom of Countable choice, and is implied by the Axiom of Choice from Finite Sets.

The following can be proved within ZF alone, without any additional choice axioms:

- Choice for finite families.
- The Baire category theorem for Polish spaces (Theorem 4.1.8).
- The compactness theorem for countable signature [**71**].
- König's tree lemma for countable trees (which is not the same as König's tree lemma because we cannot prove in ZF that a finitely branching tree is countable).

Independence Results. There are models of ZF that show the following.

- The Ultrafilter Lemma and the Axiom of Dependent Choices do not imply the Axiom of Choice [**131**]. In particular, the Ultrafilter Lemma does not imply the Axiom of Choice, and the Axiom of Dependent Choices does not imply the Axiom of Choice.
- The Axiom of Dependent Choices does not imply the Ultrafilter lemma; this follows from Example 82 in combination with Theorem 6.3.13. Also see Remark 2.11 (3) in [**67**].
- the Ultrafilter Lemma and the Axiom of Countable Choice do not imply the Axiom of Dependent Choices [**76**]. In particular, the Axiom of Countable Choice does not imply the Axiom of Dependent Choices [**86**], and the Ultrafilter Lemma does not apply the Axiom of Dependent Choices. It also follows that Tychonoff's theorem for countable products alone does not imply the Axiom of Dependent Choices: this is because the Ultrafilter Lemma together with Tychonoff's theorem for countable products *does* imply the Axiom of Dependent Choices (see Remark 1.7 (1) in [**67**]).
- The Boolean Prime Ideal Theorem does not follow from the Ordering Extension Property [**56**].
- The Ordering Extension Property does not follow from the Ordering Principle [**117**].
- The Ordering Principle does not follow from the Axiom of Choice for families of non-empty finite sets [**106**].
- The so-called *basic Fraenkel model* of ZF does not satisfy the Axiom of Choice from Finite Sets; there are also models of ZF that do not satisfy the Countable Axiom of Choice from two-element sets (see Section 4.3-4.5 in [**82**]).

Unkown relations. We do not know whether ZF together with the Axiom of Countable Choice implies Tychonoff's theorem for countable products of compact spaces [**66**]. It seems unlikely that the Ultrafilter Lemma implies the Axiom of Countable Choice, but I am not aware of any reference. I also don't know whether the Axiom of Dependent Choices implies the Order Extension Principle, whether the Axiom of

Countable Choice from Finite Sets is equivalent to the Axiom of Countable Choice, or whether it is equivalent to the Axiom of Choice from Finite Sets.

Exercises.

(151) Show that the Order Extension Property implies the Ordering Property.

(152) Show that the Ordering Property implies the Axiom of Choice for families of non-empty finite sets.

(153) Show that the Axiom of Choice is equivalent to the statement that every surjective function $f\colon A \to B$ has a *right inverse*, i.e., a function $g\colon B \to A$ such that $g(f(x)) = x$ for all $x \in A$.

(154) Show that the Axiom of Choice implies the Axiom of Dependent Choices.

(155) (Exercise 5.7 in [**83**]) Show that the Axiom of Dependent Choices implies the Axiom of Countable Choice (without quoting the facts stated above).
Hint. Given $(A_n)_{n\in\mathbb{N}}$, consider the set A of all choice functions on some $S_n := \{A_i \mid i \leq n\}$, ordered by extensions.