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ANALYTICAL EXPLANATION OF A PHASE TRANSITION IN THE MULTIFRACTAL MEASURE CONNECTED WITH A ONE-DIMENSIONAL RANDOM FIELD ISING MODEL

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In certain one-dimensional stochastic mappings a sharp drop of the D_q-spectrum of fractal dimensions for negative values of q is observed at a special value of the noise strength. This transition is connected to the vanishing of deep valleys in the measure and can be understood by analyzing the contribution of periodic orbits. A special example is given by the one-dimensional Ising model in a bimodal random field.

1 Introduction

We consider the one-dimensional random field Ising model [1-12]. Its Hamiltonian is given by

$$H = -\sum_{i=1}^{N}(Js_i s_{i+1} + h_i s_i), \tag{1}$$

where s_i denotes the classical spin at site i which takes values 1 or -1, J is the exchange energy of adjacent spins and the local magnetic fields h_i at the sites i are independent and identically distributed random variables. We restrict ourself to bimodal distributions

$$\rho(h_i) = p_+\delta(h_i - h_+) + p_-\delta(h_i - h_-). \tag{2}$$

By successively summing up over the spins one arrives at the partition function of one spin in an effective field, which depends on the realizations of the fields on the other sites via an iterative map [2],

$$x_n = h_n + A(\beta, J, x_{n-1}), \tag{3}$$

$$A(x) = \frac{1}{2\beta}\ln\left(\frac{\cosh\beta(x+J)}{\cosh\beta(x-J)}\right), \tag{4}$$

where β denotes the inverse temperature.

The mapping defines a stochastic trajectory which in turn yields an invariant probability measure $dP_\infty(x)$ giving the probability of the trajectory to visit the small interval dx centered around x. The iteration has an attractor which is a subset of the interval $I = [x_-, x_+]$, bounded by the fixed points of the functions $f_\pm(x) = A(x) \pm h$. The map is contractive (the slope of the functions $f_\sigma(x)$ is almost everywhere smaller than one). Therefore the first image of the interval I consist of two smaller 'bands' I_+ and I_- which may or may not overlap, depending on the

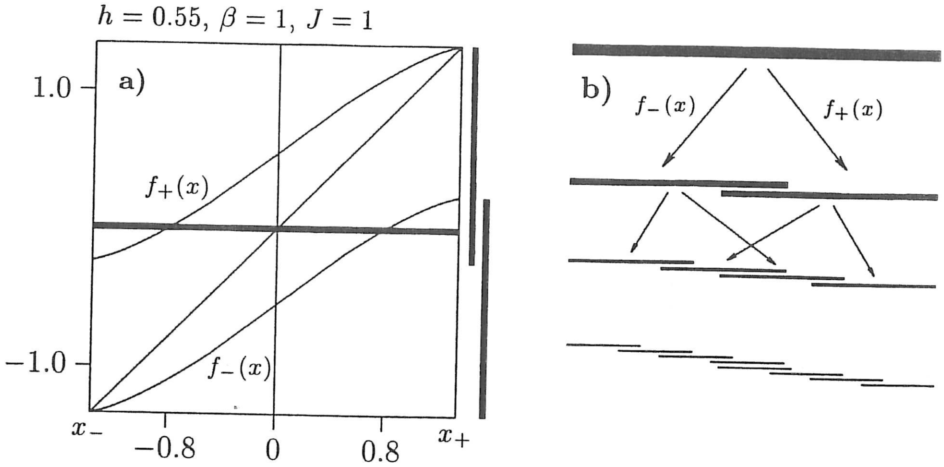

Figure 1: (a) Mapping for the case of overlapping bands f_+ and f_-. In (b) the first few images of the interval I are shown. The increasing complexity of the band structure is obvious.

value of the control parameters h_+ and h_-. In the following we restrict ourselves to a symmetric distribution with $p_+ = p_- = 0.5$, $h_+ = -h_- = h$ (cf. Figure 1). For the case of a non-symmetric distribution see ref. [5].

We consider the integrated probability distribution (the measure) $P(x)$ of the interval $(-\infty, x)$. It can be iterated starting with an initial measure $P_0(x)$ and is determined by the Frobenius-Perron equation

$$\begin{aligned} P_n(x) &= \int dh \rho(h) P_{n-1}(A^{-1}(x-h)) \\ &= \sum_{\sigma=\pm} p_\sigma P_{n-1}(f_\sigma^{-1}(x)), \end{aligned} \tag{5}$$

The fixed point of (5) gives the natural invariant measure $P_\infty(x)$ for almost all initial measures.

To investigate the structure of the measure we consider a finite iteration of order n of an arbitrary smooth initial measure P_0 and take the limit of n going to infinity at the end.

The iteration is realized by the 2^n composite functions $f_{\{\sigma\}_n}$

$$f_{\{\sigma\}_n} = f_{\sigma_n} \circ f_{\sigma_{n-1}} \circ \ldots \circ f_{\sigma_1}. \tag{6}$$

$\{\sigma\}_n$ denotes the symbolic sequence of n plus and minus signs $\{\sigma\}_n = \sigma_n \sigma_{n-1} \ldots \sigma_1$. Every function maps the invariant interval onto a small band $I_{\{\sigma\}_n}$ around the fixed point $x_{\{\sigma\}_n}$ of this function. Every band carries a total weight of $(\frac{1}{2})^n$ and the measure is a superposition of these bands. In taking the limit $n \to \infty$ the bands shrink to the corresponding fixed points.

This is most clearly seen by iterating the Frobenius-Perron equation [5]. In the first step $P_1(x)$ is a sum of two terms that involve $P_0(x)$. $P_2(x)$ becomes a sum of four terms. Finally we arrive at an expression for $P_n(x)$ as a sum of 2^n terms, involving P_0 at the predecessors $y_{\{\sigma\}_n}$ of x

$$P_n(x) = \sum_{\{\sigma\}_n} P_{\{\sigma\}_n}(y_{\{\sigma\}_n}), \tag{7}$$

$$P_{\{\sigma\}_n}(y_{\{\sigma\}_n}) = \frac{1}{2^n} P_0(y_{\{\sigma\}_n}), \tag{8}$$

$$y_{\{\sigma\}_n} = f^{-1}_{\{\sigma\}_n}(x). \tag{9}$$

For fixed x this is a path integral in the space of the symbolic dynamics.

In general, most of the terms vanish and only a small fraction of overlapping bands do contribute to the measure at a specific point. For certain values of the control parameter h and for certain points x however, the sum has only one term and the behavior of the measure around this point can be analysed.

As we will see, the map (3) generates a multifractal measure [5]. A quantity to characterize these type of measures is given by the so called D_q−spectrum [16]. One divides the invariant interval into N boxes of size ϵ and calculates the partition function $Z(q)$. This is found to scale as

$$Z_\epsilon(q) = \sum_i^N P_i^q \propto \epsilon^{(q-1)D_q}, \tag{10}$$

which defines the function D_q in the limit $\epsilon(N) \to 0$. P_i is the measure inside box i and only boxes with a non vanishing value of P_i are taken into account. For values of q greater than zero, boxes with a higher concentration of the measure contribute most whereas for negative values the most rarefied parts of the measure dominate.

In this article we are going to investigate the dependence of the measure on the parameter h. For large values of h the support of the measure is topologically equivalent to a multi-scale Cantor set [5]. The first bands I_+ and I_- are well separated, leaving a gap between them. By iteration, this gap has multiple images of ever decreasing width. The bands carrying the measure shrink to a set of Lebesgue-measure zero as it is typical for the Cantor construction. If h decreases beyond the value $h_c^{(1)}$ however, the bands start overlapping each other. The gaps close in every order of the iteration procedure and the attractor becomes a closed set, the whole interval $[x_-, x_+]$.

Beside this very drastic change there are other transitions that occur if we decrease h still further. As long as h is greater than $h_c^{(2)}$ the coarse grained density has deep valleys on every scale as is shown in Figure (3). These valleys correspond to large values in the D_q−spectrum for negative q. If h crosses $h_c^{(2)}$ the valleys vanish all of a sudden and the D_q−spectrum shows a sharp drop as is seen in Figure (2) where $D(q)$ for certain fixed q is plotted as a function of h.

The plotted spectrum was obtained by using the method of the new natural partition [12,13] since the method using an equipartition of the interval converges too slowly. The same remarkable behaviour has been previously found [14] in a model of learning in neural networks [15] and is in fact a very general feature of bimodal stochastic maps. The main objective of this article will be the explanation of this type of transition.

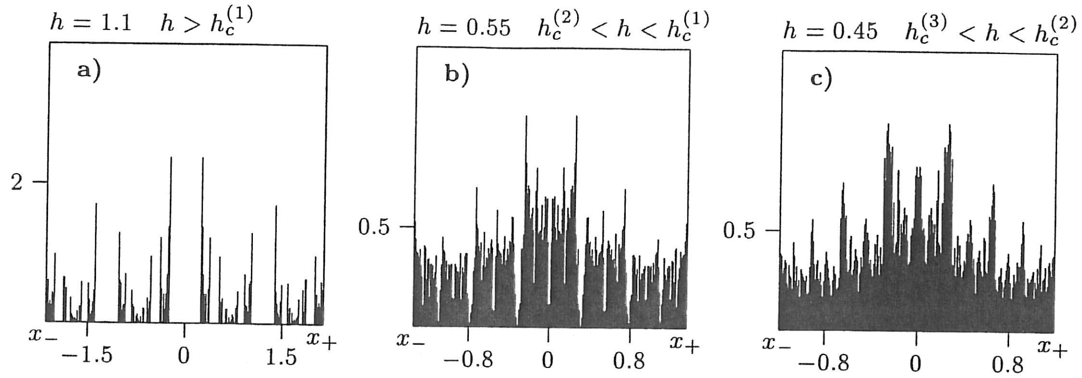

Figure 2: Qualitative changes in the shape of the measure (7). P_∞ is obtained by stochastic iteration of the trajectory for different values of h. (a) thin fractal, (b) fat fractal with deep valleys, (c) without deep valleys. ($\beta = 1$, $J = 1$ for all graphs).

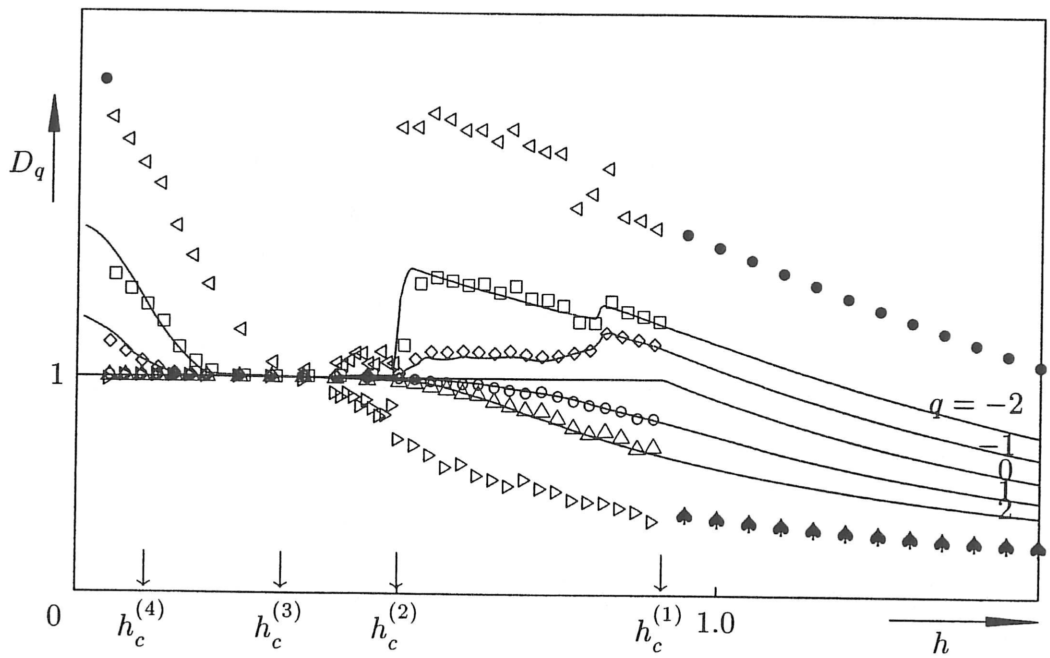

Figure 3: Generalized dimensions D_q ($q = -2, -1, 0, 1, 2$) versus the strength of the local magnetic field h for the random field ising model with $J = 1$, $\beta = 1$. The open symbols indicate the results from the digital simulation with 10^{10} iterations. The solid lines and filled symbols (• for $q = -35$ and ♠ for $q = 35$) show the results obtained from the thermodynamic formalism. The meaning of the critical values $h_c^{(n)}$, $n = 1, \ldots, 4$, is explained in the text.

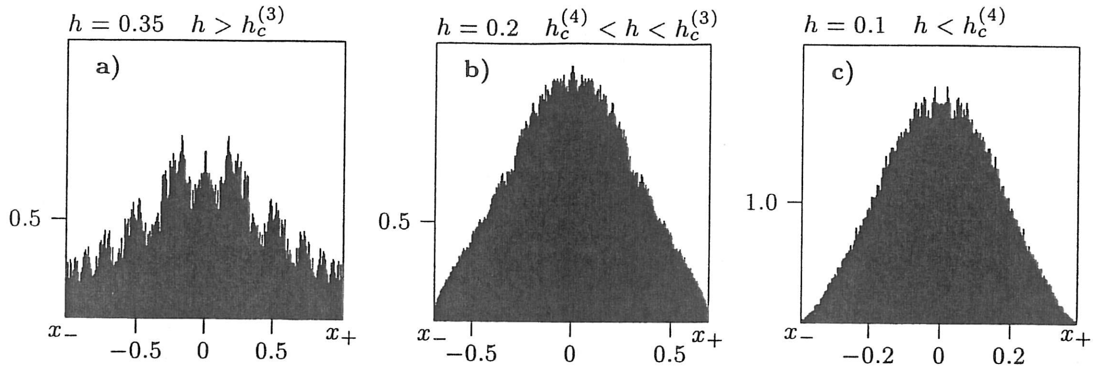

Figure 4: Qualitative changes of the behaviour of the invariant density $p(x)$ at the boundaries of the support. (a) $p(x_\pm) = \infty$, (b) $p(x_\pm) = 0$ but $p'(x_\pm) = \infty$, (c) $p(x_\pm) = 0$ and $p'(x_\pm) = 0$. ($\beta = 1$, $J = 1$ for all graphs).

The measure undergoes further changes that determine the shape of the D_q−spectrum and are explained elsewhere[12]. There the behaviour of the invariant density at the end points $x_\pm$ of the interval is analysed. Since these points, the fixed points of the functions $f_\pm(x)$, are their own predecessors under the mapping (3), the Frobenius-Perron equation for the density has in the limit of $n \to \infty$ only one term

$$p_n(x_\sigma) = \frac{1}{2f'_\sigma(x_\sigma)} p_{n-1}(x_\sigma). \tag{11}$$

Therefore, the density at $x_\pm$ is either zero or infinity, depending on the slope of the function $f_\pm$ at these points. The derivative does not depend explicitely on h. The location of the fixed points however does. If h drops below $h_c^{(3)}$ the density at the fixed points changes from infinity to zero. Since, as we will see, the density is not singular anymore inside the interval at this value of h, the behavior of the measure at these points completely determines the D_q−spectrum.

The measure is found to scale exponentially near the end points: if we iterate a point, e.g., in the vicinity of x_+ we arrive at

$$\begin{aligned} P_\infty(x_+) - P_\infty(x_+ - \epsilon) &= \frac{1}{2}(P_\infty(x_+) - P_\infty(f_+^{-1}(x_+ - \epsilon))) \\ &\simeq \frac{1}{2}(P_\infty(x_+) - P_\infty(x_+ - f_+^{-1'}(x_+)\epsilon)), \end{aligned} \tag{12}$$

which leads, supposing the scaling law $P_\infty(x_+ - \epsilon) \propto \epsilon^{\alpha_+ + 1}$ to the exponent

$$\alpha_+ = -1 - \frac{\ln(2)}{\ln(f'_+(x_+)}. \tag{13}$$

The deeper reason for this simple scaling behavior at the points $x_\pm$ lies in the fact that at these points only the rightmost respectively leftmost band contributes to the measure. How fast these bands shrinks under the mapping of $f_\pm$ determines the strength of the singularity. The total measure on this outmost bands diminishes like $(1/2)^n$. The size of the bands however scales like $\left(f'_\pm(x_\pm)\right)^n$. A small slope of the function $f_\pm$ at its fixed point therefore results in a narrow band and in a strong positive singularity. The functions $f_\pm$ have the least slope at the points $x_\pm$. Therefore the singularities at these points are the strongest.

As long as $f'_+(x_+) < \frac{1}{2}$ $(h > h_c^{(3)})$ the density at the fixed point x_+ has a positive singularity $\alpha_+ < 0$. If $\frac{1}{2} < f'_+(x_+) < \frac{1}{\sqrt{2}}$ $(h_c^{(4)} < h < h_c^{(3)})$ the density is finite but the first derivative of the density at x_+ is singular, $0 < \alpha_+ < 1$. For $f'_+(x_+) > \frac{1}{\sqrt{2}}$, that is beyond $h_c^{(4)}$, the derivative is zero and the exponent α_+ is greater than one. As we will show in the next paragraph, analogous considerations apply to every fixed point in the case of non-overlapping bands and to a certain subset of fixed points related to periodic orbits, even in the case of overlapping bands.

2 Periodic Orbits and their Contribution to the Invariant Measure

We have seen that as long as the bands do not overlap (that is $h > h_c^{(1)}$), the attractor has the topology of a Cantor set and therefore the measure is purely singular. The bands being disjoint mean that any point x belonging to the attractor has only one predecessor and the Frobenius-Perron equation for the integrated probability density at this point has only one term. The above considerations apply in any order of the iteration to any of the 2^n fixed points. The measure has either a negative $(\alpha > 0)$ or positive $(\alpha < 0)$ singularity in the vicinity of the fixed point, depending on how fast the corresponding band shrinks, as was the case at the end points $x_\pm$ of the interval. These singularities can be calculated in principle at any order. The Frobenius-Perron equation for a point in the vicinity of a fixed point leads to an expression for the singularity

$$P_\infty(x_{\{\sigma\}_n} + \epsilon) - P_\infty(x_{\{\sigma\}_n}) \quad \propto \quad \epsilon^{\alpha_{\{\sigma\}_n}+1}, \tag{14}$$

$$\alpha_{\{\sigma\}_n} \quad = \quad -1 - \frac{\ln(2)}{\frac{1}{n}\ln(f'_{\{\sigma\}_n}(x_{\{\sigma\}_n})}. \tag{15}$$

This can be seen as follows: Since $x_{\{\sigma\}_n} = f_{\{\sigma\}_n}(x_{\{\sigma\}_n}) = f_{\sigma_n} \circ f_{\sigma_{n-1}} \circ \ldots \circ f_{\sigma_1}(x_{\{\sigma\}_n})$ is a fixed point, we can write down a sequence of predecessors $y_i = f_{\sigma_i} \circ f_{\sigma_{i-1}} \circ \ldots \circ f_{\sigma_1}(x_{\{\sigma\}_n})$

$$\begin{aligned} x_{\{\sigma\}_n} &= f_{\sigma_n}(y_{n-1}), \\ y_{n-1} &= f_{\sigma_{n-1}}(y_{n-2}), \\ &\vdots \;, \\ y_1 &= f_{\sigma_1}(x_{\{\sigma\}_n}). \end{aligned} \tag{16}$$

For every step in this sequence we find due to the Frobenius-Perron equation and the fact that all points of the sequence have only one predecessor:

$$P_\infty(y_i + \epsilon) - P_\infty(y_i) \simeq \frac{1}{2}(P_\infty(y_{i-1} + f_\sigma^{-1\prime}(y_i)\epsilon) - P_\infty(y_{i-1}))$$
$$i = 0, \ldots, n, \quad y_0 = y_n = x_{\{\sigma\}_n}, \tag{17}$$

Since the sequence is closed we obtain for the singularity at the fixed point expression (15).

If h is decreased and the bands overlap, most of this order is lost. But there are certain fixed points that are not affected immediately, the end points $x_\pm$ being an example. To find these points we need to introduce the concept of periodic orbits of the map. As before, we first consider a finite order n of the iteration procedure where we have 2^n composite functions $f_{\{\sigma\}_n}$.

We call a periodic orbit of a function $f_{\{\sigma\}_n}$ the sequence $\{y_{n-1}, \ldots, y_1, x_{\sigma_n}\}$ of the predecessors of its fixed point under consecutive iteration of the function as given in (16).

As one easily confirms, a periodic orbit consists, besides the fixed-point of the function $f_{\{\sigma\}_n}$ itself, of fixed points of all the other functions that are obtained from the function $f_{\{\sigma\}_n}$ by cyclic permutation of the symbolic sequence $\{\sigma\}_n$. This is illustrated by the example of an orbit of period three, see Figure (5).

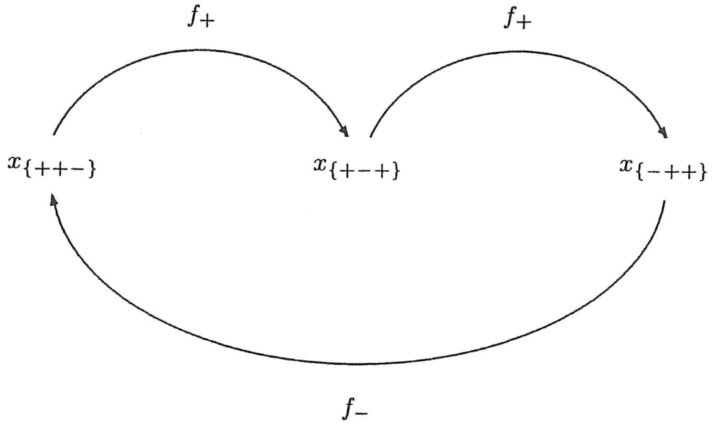

Figure 5: Period three orbit. The composite function is taken as $f_{\{-++\}}$ which is made up of f_-, f_+ and f_+. Cyclic permutation yields the functions $f_{\{+-+\}}$ and $f_{\{++-\}}$. The mapping of the fixed point $x_{\{-++\}}$ by the function f_+ leads to the point $x_{\{+-+\}}$, the repeated application of f_+ to the point $x_{\{++-\}}$ and the further mapping by f_- back to $x_{\{-++\}}$.

Now, if we are in a parameter region of h where the bands do not overlap, all points of a periodic orbit are equivalent, since eq. (15) is invariant under cyclic permutation of the functions.

What happens if the bands start overlapping each other? In the middle of the invariant interval there is now a region, bounded by $f_-(x_+)$ and $f_+(x_-)$, where the mapping is no longer one to one and onto. A point x in this overlap region has two predecessors and the Frobenius-Perron equation has two terms. If however all the points of a periodic orbit lie outside this domain, they are not affected at all (cf. Figure 6). In this case all the above considerations remain valid for this specific orbit. If the parameter h is tuned, the extent of the overlap region changes as well as the location of the orbits. If at least on point x of an orbit gets into the overlap region, this has only little influence on the D_q–spectrum as long as the singularity at the points of the orbit is positive ($\alpha_x < 0$). The second term of the Frobenius–Perron equation might contribute a singularity eventually thereby replacing the old singularity on the whole orbit if the one at the new predecessor y

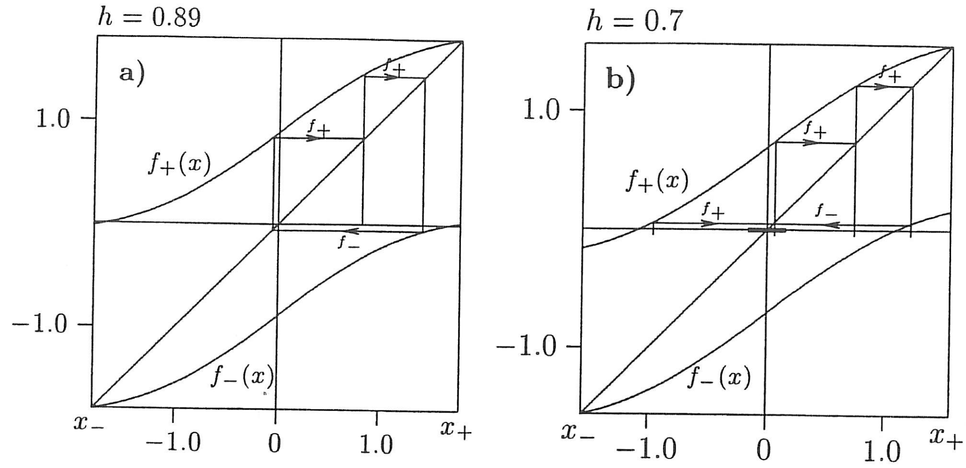

Figure 6: Orbits of period three for different values of h. (a) shows the case where no point of the orbit falls into the overlap region, whereas in (b) the point $x_{\{-++\}}$ lies in the overlap region and has therefore two predecessors. ($\beta = 1$, $J = 1$ for all graphs).

is stronger ($\alpha_y < \alpha_x$). This however changes the D_q−spectrum only slightly. But if the singularity at x is negative, it is replaced by a positive singularity whenever the density at y is greater than zero. So the number of positive singularities increases at the extent of the number of negative singularities.

A very special role is played by the fixed points $x_\pm$ as period-one orbits—because they are never reached by the overlap region—and by the period-two orbit $\{x_{\{+-\}}, x_{\{-+\}}\}$. Since $x_{\{+-\}}$ is mapped under application of f_- onto $x_{\{-+\}}$, all points to the right of $x_{\{+-\}}$ are mapped onto the interval $(x_{\{+-\}}, x_{\{-+\}})$ as well as are all points to the left of $x_{\{-+\}}$ under application of f_+. The interval $(x_{\{+-\}}, x_{\{-+\}})$ in turn is mapped to the right of $x_{\{+-\}}$ (to the left of $x_{\{-+\}}$) by f_+ (f_-), see Figure (7).

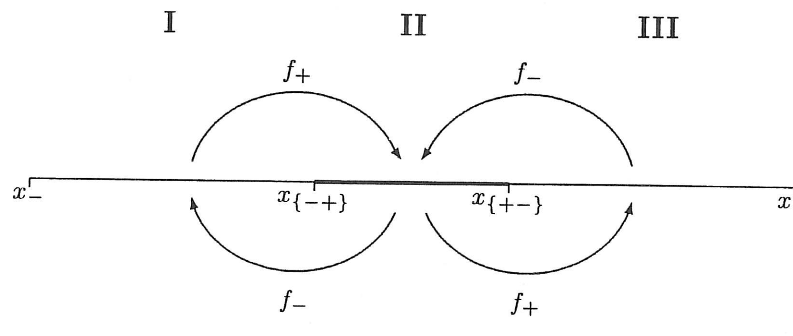

Figure 7: Mapping of the interval I elucidating the importance of the interval **II**=$[x_{-+}, x_{+-}]$. **II** is mapped onto parts of **I** and **III** under f_- (f_+) which are itself preimages of **II**.

This means that every periodic orbit of period greater than three has at least one point in the interval $(x_{\{+-\}}, x_{\{-+\}})$ and at least one point outside since the orbits are closed and the trajectory has to come back to the point where it started from.

This topology explains the transition found in the D_q−spectrum. As h is decreased, the overlap region grows whereas the location of the orbits changes only slightly. Gradually all the orbits are reached and the negative singularities are destroyed. The remaining orbits with negative singularity gather more and more weight in their influence on the D_q−spectrum. When the period-two orbit is reached, that is $f_+(x_-) = x_{-+}$, the last negative singularity vanishes because at the points $x_\pm$ the slope of the functions $f_\pm$ is a minimum and the probability density has the strongest, and for this value of h, positive singularity there. The D_q−spectrum for $q < 0$, which measures exactly the negative singularities, shows a sharp drop to one. Since all the negative singularities have vanished, the right part of the $f(\alpha)$-spectrum has now collapsed, a behaviour that has also been observed in the case of the superposition of equal-scale [17] and multi-scale [18] Cantor sets.

3 Concluding Remarks

The scenario explained in this article is found in other applications of stochastic maps [14]. In fact the only necessary conditions for this mechanism to work are:

1. A stochastic map with two smooth monotonous functions $f_\pm(x, h)$, depending on a control parameter h.
2. A nonchaotic dynamics $f'_\pm(x) < 1$ →a finite invariant interval $I = [x_+, x_-]$.
3. $f_+(x) > f_-(x) \qquad \forall x \in I$.
4. The dependence on the control parameter h need to be such that the conditions $f_+(x_-) = x_{-+}$ and $f_-(x_+) = x_{+-}$ can be reached by tuning h.
5. If the overlap reaches one of the points x_{+-}, x_{-+}, the slope at the end points $f'_\pm(x_\pm)$ need to be greater than $\frac{1}{2}$.

The last condition ensures that there are no negative singularities left since the period one orbits $x_\pm$ are not reached by the overlap and their singularity therefore remains untouched.

There are still open questions. For ones the D_q−spectrum of the 1d random field Ising model shows another slight drop for negative q at a value of h of about 0.8. This seems to be a predecessor of the explained transition and is not yet understood. Further all the above explanations are not given with full mathematical rigour. Is there a similar behaviour for discrete stochastic mappings in higher dimensions or for discrete distributions of the noise with more than two possible values? Finally there are interesting mathematical questions that should be answered: Is the measure absolutely continuous whenever the bands overlap or are there parameter values for which the measure becomes absolutely singular as is the case for the problem of the Bernoulli convolutions? Can the functional dependence of the D_q−spectrum in terms of h right at the transition point be stated?

This paper is dedicated to Professor Adolf Kühnel on the occasion of his 60th birthday.

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