

12th International Conference on Magnetic Fluids

Thermomagnetic convection in magnetic fluids subjected to spatially modulated magnetic fields

Adrian Lange*, Stefan Odenbach

TU Dresden, Institute of Fluid Mechanics, Chair of Magnetofluidynamics, 01062 Dresden, Germany

Abstract

A horizontal layer of magnetic fluid subjected to a vertical temperature gradient and a spatially modulated magnetic field is considered. For the case of symmetric modulation the initial state characterised by a nonzero flow field is identified. By conducting a linear stability analysis the area of stability of the initial state against small perturbations is determined.

© 2010 Published by Elsevier Ltd

Keywords: Magnetic fluid; thermomagnetic convection; spatially modulated magnetic field

1. Introduction

Thermal convection in a plane horizontal fluid layer heated from below develops if the vertical temperature gradient surpasses a certain threshold. Beyond this threshold the destabilizing buoyancy force, caused by the temperature driven gradient in the fluid density, prevails the stabilising effect of friction and heat conduction: the motionless initial state becomes unstable. By using a magnetic fluid (MF) as working substance subjected to an external vertical magnetic field, an additional convection driving mechanism appears. Now the gradient in the temperature causes also a gradient in the magnetisation of the MF which together with the gradient of the internal magnetic field leads to a magnetic force. This force can generate beyond a certain threshold a convection, too, called thermomagnetic convection.

Whereas this type of convection in static magnetic fields has been studied in detail theoretically [1] as well as experimentally [2, 3, 4], analyses for modulated magnetic fields are just starting to appear. A first work on parametric modulation of the thermomagnetic convection was done by Engler and Odenbach [5]. The case of a spatially modulated magnetic field has not been studied yet and it is therefore the aim of this work. A spatially modulated magnetic field can be easily realised by placing spatially modulated iron bars inside a spatially constant external field. In such a way many different types of modulation can be accomplished by varying the number of bars, their wavelengths or their phase shift. Therefore the spatial modulation of magnetic fields overcomes some of the restrictions of temporal modulation.

* Corresponding author. Tel.: 0044-351-46339867; fax: 0044-351-46333384.
E-mail address: adrian.lange@tu-dresden.de.

2. System and equations

A horizontally infinite layer of a viscous, nonconducting magnetic fluid of thickness \hat{d} is considered, whose lower and upper boundary is at $\hat{z} = \mp \hat{d}/2$ and is held at the constant temperature \hat{T}_b and $\hat{T}_t < \hat{T}_b$, respectively. Under the assumption of translational symmetry with respect to the depth of the layer, the analysis is restricted to two dimensions, i.e., to the $\hat{x}-\hat{z}$ -plane. The system is governed by the equation of continuity, $\text{div}\hat{v} = 0$, the Navier-Stokes equation in the Boussinesq approximation,

$$\hat{\rho}_0 \left[\frac{\partial \hat{v}}{\partial \hat{t}} + (\hat{v} \text{grad}) \hat{v} \right] = -\text{grad}\hat{p} + \hat{\rho} \hat{g} + \hat{\mu} \Delta \hat{v} + \hat{\mu}_0 \left(\hat{M} \text{grad} \right) \hat{H} \quad (1)$$

the equation of heat conduction,

$$\frac{\partial \hat{T}}{\partial \hat{t}} + (\hat{v} \text{grad}) \hat{T} = \hat{\kappa} \Delta T \quad (2)$$

and the relevant Maxwell equations: $\text{div}\hat{B} = 0$ and $\text{rot}\hat{H} = 0$. By introducing the scalar function $\hat{\Psi}$ for the fluid velocity $\hat{v} = (\partial_z \hat{\Psi}, -\partial_x \hat{\Psi})$, the equation of continuity is automatically fulfilled. $\hat{p}, \hat{T}, \hat{H}$, and \hat{M} are the pressure, the temperature, the magnetic field inside the fluid, and the magnetization of the MF which follows a linear law, $\hat{M} = \chi \hat{H}$, with the susceptibility χ . The acceleration due to gravity is denoted by $\hat{g} = (0, -\hat{g})$, the dynamic viscosity by $\hat{\mu}$, the permeability of vacuum by $\hat{\mu}_0$, the thermal diffusivity by $\hat{\kappa}$, and the magnetic induction by $\hat{B} = \hat{\mu}_0 (\hat{M} + \hat{H})$. The density of the fluid is a linear function of the temperature, $\hat{\rho} = \hat{\rho}_0 [1 - \hat{\alpha}(\hat{T} - \hat{T}_a)]$, where $\hat{\alpha}$ is the coefficient of volume expansion and $\hat{\rho}_0$ the density of the fluid at a reference temperature \hat{T}_a .

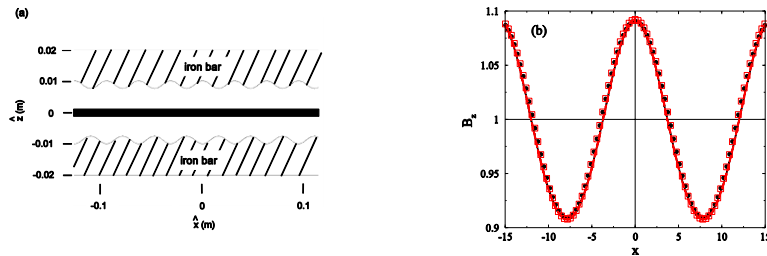


Figure 1: (a) Sketch of the symmetric modulation of a magnetic field by two iron bars. (b) (Color online) Numerical solution (solid lines) of scaled B_z versus scaled x at the heights $\hat{z} = 0$ (black) and $\hat{z} = 0.8$ mm (red) inside the MF. The symbols indicate the analytical approximation (see text).

By using $\hat{\Psi}$ and combining the $x(z)$ -component of the Navier-Stokes equation differentiated with respect to $z(x)$, one yields

$$0 = \Delta \Psi^2 - Ra \frac{\partial T}{\partial x} + Ra_m \left[M_x \frac{\partial}{\partial x} \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) + M_z \frac{\partial}{\partial z} \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) \right] \quad (3)$$

for real MFs whose Prandtl numbers $Pr = \hat{\mu}/(\hat{\kappa}\hat{\rho}_0)$ are much larger than 10 [6]. For the dimensionless quantities in Eq. (3) the lengths were scaled with \hat{d} , the time with $\hat{d}^2/\hat{\kappa}$, the velocities with $\hat{\kappa}/\hat{d}$, the temperature with $(\hat{T}_b - \hat{T}_t)$, and the magnetic field by $\hat{K}(\hat{T}_b - \hat{T}_t)$. With these scalings two dimensionless parameters appear, the Rayleigh

number, $Ra = \left[\hat{\rho}_0 \hat{g} \hat{\alpha} (\hat{T}_b - \hat{T}_t) \hat{d}^3 \right] / (\hat{\mu} \hat{\kappa})$, and the magnetic Rayleigh number, $Ra_m = \left[\hat{\mu}_0 \hat{K}^2 (\hat{T}_b - \hat{T}_t)^2 \hat{d}^2 \right] / (\hat{\mu} \hat{\kappa})$,

where \hat{K} is the pyromagnetic coefficient of the MF. In order to determine analytically the stationary initial state of the system, one has to know the magnetic field inside the MF for the given spatial modulation of the external magnetic field.

3. Magnetic field and initial state

A simple and symmetrical modulation is considered, where two iron bars are placed symmetrically above and beneath a layer of MF ($\hat{d} = 2$ mm) in an vertical external homogenous field, see Fig. 1(a). One horizontal boundary of the bar is sinusoidal ($\hat{\lambda} \approx 0.03$ m) shaped. The numerical solution of the corresponding magnetic problem results in a sinusoidal modulation of $B_z(x)$, Fig. 1(b), and $|B_x(x)/B_z(x)| \approx 10^{-2}$. These numerical results can be well fitted by $B_x = B_z \sin(kx)$ and $B_z = 1 + (E + Gz^2) \cos(kx)$ with $k = 2\pi\hat{d}/\hat{\lambda}$, $B = -kE$, and $G = k^2E/2$. With the remaining fit parameter, here $E = 0.091$, an excellent agreement with the numerical data can be achieved, see symbols in Fig. 1(b). By using this analytical approximation, the solution of

$$0 = \Delta^2 \Psi - Ra \frac{\partial T}{\partial x} + \frac{Ra_m \chi k^3 E z \sin(kx) [1 + \cos(kx)E]}{(\chi + 1)^2} \quad (4)$$

and

$$\frac{\partial \Psi}{\partial z} \frac{\partial T}{\partial x} - \frac{\partial \Psi}{\partial x} \frac{\partial T}{\partial z} = \Delta T \quad (5)$$

fulfilling the boundary conditions

$$\Psi = \partial_z \Psi = 0 \quad \text{at } z = \pm 1/2 \quad (6)$$

$$T = T_t \quad \text{at } z = 1/2 \quad (7)$$

$$T = T_b \quad \text{at } z = -1/2 \quad (8)$$

gives the stationary initial state. By the fact of long wavelength modulations, $k < 1$, the perturbational ansatz

$$\Psi_{init} = \sin(kx) [\Phi_{init,k}^0(z) + k\Phi_{init,k}^1(z) + k^2\Phi_{init,k}^2(z) + \dots] + \sin(2kx) [\Phi_{init,2k}^0(z) + k\Phi_{init,2k}^1(z) + k^2\Phi_{init,2k}^2(z) + \dots] \quad (9)$$

$$T_{init} = T_{init}^0(z) + kT_{init}^1(z) + k^2T_{init}^2(z) + \dots + \cos(kx) [T_{init,k}^0(z) + kT_{init,k}^1(z) + k^2T_{init,k}^2(z) + \dots] + \cos(2kx) [T_{init,2k}^0(z) + kT_{init,2k}^1(z) + k^2T_{init,2k}^2(z) + \dots] \quad (10)$$

is employed. Solving the hierarchy of equations with respect to powers of k , the initial state up to k^3 is given by

$$T_{init} = T_a - z \quad T_a = \frac{\hat{T}_b + \hat{T}_t}{2(\hat{T}_b - \hat{T}_t)} \quad (11)$$

$$\Psi_{init} = -\frac{Ra_m \chi E z (4z^2 - 1)^2 k^3}{1920(\chi + 1)^2} \left[\sin(kx) + \frac{E}{2} \sin(2kx) \right] \quad (12)$$

Whereas the temperature profile (11) is the classical vertical one found in conventional Rayleigh-Bénard setup [7], the initial flow field is not any longer the quiescent state. Instead, as Fig. 2 shows, the initial state is characterized by a double vortex structure. An initial state in motion is unknown in classical Rayleigh-Bénard configuration and shows the potential of spatial modulations of the external driving to open new horizons in the field of pattern formation with soft magnetic matter.

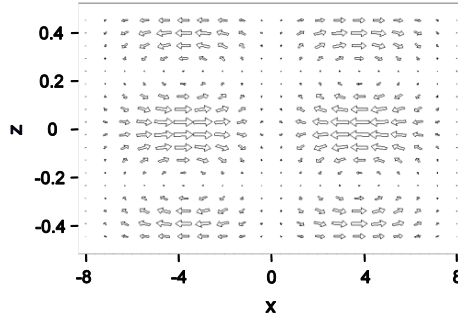


Figure 2: Flow field $\vec{v}_{init} = (\partial_z \Psi_{init}, -\partial_x \Psi_{init})$ according to Eq (12) for the fluid parameter $\hat{K} = 35 A/(mK)$, $(\hat{T}_b - \hat{T}_t) = 10K$, $\hat{\mu} = 0.15 Pa \cdot s$, $\hat{\kappa} = 5 \cdot 10^{-8} m^2/s$, and $\chi = 0.2$.

4. Stability analysis and results

To examine the stability of the initial state, a linear stability analysis against small perturbations, $\Theta(x, z, t)$ and $\Phi(x, z, t)$, is conducted,

$$T = T_{init}(z) + \Theta(x, z, t) \quad (13)$$

$$\Psi = \Psi_{init}(x, z) + \Phi(x, z, t) \quad (14)$$

which leads to the following two differential equations for the perturbations (restricted to terms up to k^3)

$$\Delta \Theta = \frac{\partial \Theta}{\partial t} + \frac{\partial \Phi}{\partial x} + \frac{\partial \Psi_{init}}{\partial z} \frac{\partial \Theta}{\partial x}, \quad (15)$$

$$\Delta^2 \Phi = Ra \frac{\partial \Theta}{\partial x}. \quad (16)$$

By differentiating Eq. (15) with respect to x and using the expression for $\partial_x \Theta$ from Eq. (16), the temperature field Θ can be eliminated. The resulting linear equation for Φ may be solved by a separation ansatz, $\Phi(x, z, t) = \Phi_1(x) \Phi_2(z) e^{\sigma t}$, where σ is the complex-valued growth rate. To simplify the analysis considerably, homogeneous stress-free boundary conditions are assumed and only the first member of the appropriate series, $\cos(\pi z)$, is used. Following the procedure outlined in [8], one ends up with a homogenous system of equations, where the condition $\sigma = 0$ gives the so-called neutral stability curve. It is an implicit relation for the determination of the Rayleigh number in dependence of the wave number q (stemming from the Floquet ansatz for $\Phi_1(x)$, see [8]) and the parameter for the strength of the magnetic excitation Ra_m . The lowest value of Ra presents the critical Rayleigh number Ra_c for the onset of instability of the initial state.

Fig. 3 shows the behaviour of Ra_c as function of the magnetic Rayleigh number Ra_m , where both parameters are scaled with the critical Rayleigh number for pure thermal driving, $Ra_c(Ra_m = 0) = 27\pi^4/4$, in the case of stress-free boundary conditions [7]. It can be seen that with increasing magnetic Rayleigh number, Ra_c increases as well, but only slightly. The increase of Ra_c is accompanied by an increase of q_c (not shown here) which causes the growth of Ra_c . A larger wave number corresponds to more flow vortices per length for which more energy is necessary which means a higher thermal driving, i.e. an increase of Ra_c .

The appearance of spatially modulated external driving generates a neutral stability curve which is formed by many interwoven tongues. With increasing Ra_m some of these tongues degenerate towards shrinking islands, a

phenomenon also known from the Faraday instability [9]. The disappearance of such an island causes the jump in the stability curve, see Fig. 3, because the lowest value of Ra jumps to another tongue. That tongue will shrink to an island with further increase of Ra_m and will at its disappearance cause the next jump of Ra_c . A detailed analysis of the neutral stability curve will be presented in a forthcoming publication.

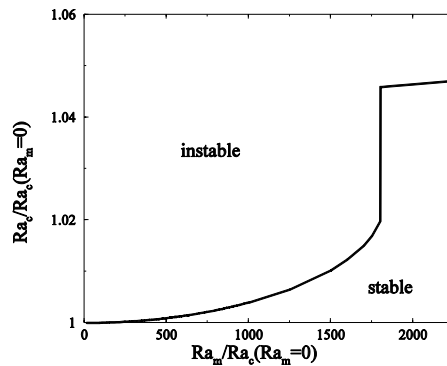


Figure 3: Stability curve for the initial state of the two vortex flow as shown in Fig 2 in dependence of the magnetic Rayleigh number Ra_m .

5. Conclusions

In summary, the initial state and its stability of the thermomagnetic convection in a horizontal fluid layer of MF subjected to a spatially modulated magnetic field was analysed. In contrast to the purely thermal driven system, the nonzero flow field of the initial state is characterised by a two vortex structure. With the possible options for the spatial modulation of magnetic fields, a new scope of research on thermomagnetic convection starts to emerge.

References

- [1] B. A. Finlayson, Convective instability of ferromagnetic fluids, *J. Fluid Mech.* 40 (1970) 753–767.
- [2] L. Schwab, Thermal convection in ferrofluids under a free surface, *J. Magn. Magn. Mat.* 85 (1990) 199–202.
- [3] L. Schwab, K. Stierstadt, Field-induced wavevector-selection by magnetic Bénard-convection, *J. Magn. Magn. Mat.* 65 (1987) 315–316.
- [4] L. Schwab, U. Hildebrandt, K. Stierstadt, Magnetic Bénard convection, *J. Magn. Magn. Mat.* 39 (1983) 113–114.
- [5] H. Engler, S. Odenbach, Parametric modulation of thermomagnetic convection in magnetic fluids, *J. Phys.: Condens. Matter* 20 (20) (2008) 204135.
- [6] S. A. Suslov, Thermomagnetic convection in a vertical layer of ferromagnetic fluid, *Phys. Fluids* 20 (8) (2008) 084101.
- [7] S. Chandrasekhar, *Hydrodynamics and hydrodynamic stability*, Dover Publ., New York, 1981.
- [8] R. Schmitz, W. Zimmermann, Spatially periodic modulated Rayleigh- Bénard convection, *Phys. Rev. E* 53 (6) (1996) 5993–6011.
- [9] M. Silber, A. C. Skeldon, Parametrically excited surface waves: two-frequency forcing, normal form symmetries, and pattern selection, *Phys. Rev. E* 59 (5) (1999) 5446–5456.