# THE ADJOINT PROBLEM IN THE PRESENCE OF A DEFORMED SURFACE: THE EXAMPLE OF THE ROSENSWEIG INSTABILITY ON MAGNETIC FLUIDS

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The Rosensweig instability is the phenomenon that above a certain threshold of a vertical magnetic field peaks appear on the free surface of a horizontal layer of magnetic fluid. In contrast to almost all classical hydrodynamical systems, the nonlinearities of the Rosensweig instability are entirely triggered by the properties of a deformed and *a priori* unknown surface. The resulting problems in defining an adjoint operator for such nonlinearities are illustrated. The implications concerning amplitude equations for pattern forming systems with a deformed surface are discussed.

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## 1. Introduction

Magnetic fluids (MFs) are stable colloidal suspensions of ferromagnetic nanoparticles dispersed in a non-magnetic carrier liquid. The nanoparticles are coated with a layer of chemically adsorbed surfactants to avoid agglomeration. The behavior of MFs is characterized by the intricate interaction of their hydrodynamic and magnetic properties with external forces. This complex interaction causes many fascinating phenomena, as the "negative viscosity" effect and the Weissenberg effect (for a review see Ref. 1) or as the labyrinthine instability and the Rosensweig instability.[2] The latter instability occurs when a horizontal layer of MF with a free surface is subjected to a uniform and vertically oriented magnetic field. Above a certain threshold of the magnetic field that surface becomes unstable, giving rise to a hexagonal pattern of peaks.[2,3] Despite the fact that the Rosensweig instability has been known for many decades, some aspects have been addressed only recently: the inclusion of the fluid viscosity[4] and of a finite layer thickness,[5] the hexagon-square transition[6] or the wave number selection problem.[6,7]

The *quantitative* comparison of theoretical and experimental results is presently limited to linear aspects of the Rosensweig instability. Convincing quantitative agreement is found for the wave number of maximal growth,[7] for the parametric stabilization of the Rosensweig instability,[8] and for the wave resistance in magnetic

fluids.[9] For the nonlinear aspects, comparisons are restricted to the detection of the same *qualitative* features. Stability regions for hexagons and squares are found in the theory[10−16] and in the experiment.[6] The experimentally observed increase in the wavelength of the emerging hexagonal pattern[6] is detectable as well in a theoretical analysis.[16] The main reason for lacking *quantitative* comparisons is the difference between the susceptibility of the experimentally used fluids and the susceptibility up to which the nonlinear analyses are valid. Finite amplitudes for all three patterns of hexagons, squares, and rolls are given for susceptibilities $\chi$ smaller then 0.41 for an infinitely thick layer (see Figs. 3 and 6 in Ref. 16). But the experiments were performed with a magnetic fluid of $\chi_{\mathrm{exp}} = 1.4$.[6] Pattern selection studies with fluids of different susceptibilities have not yet been carried out.

Weakly nonlinear analyses of the Rosensweig instability were done by means of an energy minimization principle,[10,11,16] by methods of functional analysis,[13−15] by a generalized Swift–Hohenberg equation,[17,18] and by a multiple-scale analysis.[19,20] The first two approaches are suited only for static problems and the Swift–Hohenberg equation lacks coefficients containing the fluid properties and the geometry of the system. Consequently, amplitude equations stemming from a multiple-scale analysis based on the fundamental hydrodynamic equations are customized for the study of static *and* dynamical problems as well as for the quantitative comparison with the experiment. The standard route of the multiple-scale analysis is modified in Refs. 19 and 20 to circumvent Fredholms theorem, i.e. the definition of an adjoint operator. In the present paper a multiple-scale analysis is presented which involves the expansion of all physical quantities at the deformed surface. The resulting problem in defining an adjoint operator and the subsequent consequences are the main purpose of this paper. It is organized as follows: the system and the relevant equations of the problem are displayed in the next section. Based on the governing equations and the boundary conditions the different character of the nonlinearities of the Rosensweig instability is emphasised (Sec. 2.3). Whereas the linear problem is shortly recapitulated in Sec. 3, the adjoint problem is addressed in detail in Sec. 4. In the final Sec. 5, the problems concerning amplitude equations for pattern forming systems with a deformed surface are discussed as well as open questions and further prospects are outlined.

## 2.  System and Equations of the Problem

A horizontally unbounded layer of an incompressible, nonconducting, and viscous magnetic fluid of finite thickness $d$ and constant density $\rho$ is considered. The MF is bounded from below ($z = -d$) by the bottom of a container made of a magnetically impermeable material and has a free surface described by $z_{\mathrm{s}} = \zeta(x, y, t)$ with air above. The electrically insulating fluid justifies the stationary form of the Maxwell equations, which reduce to the Laplace equation for the magnetic potentials $\Phi^{(i)}$ in each of the three different regions. Upper indices denote the considered media: (1) air, (2) magnetic fluid, and (3) container (Fig. 1).
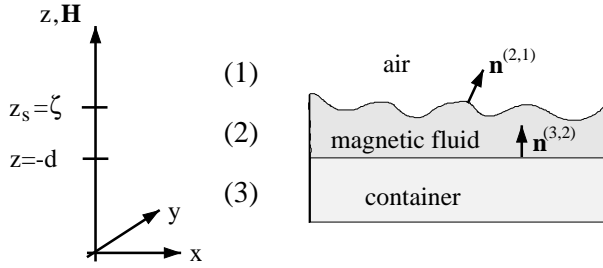
Fig. 1.   Sketch of the system.

It is assumed that the magnetization $\mathbf{M}^{(2)}$ of the magnetic fluid depends linearly on the magnetic field $\mathbf{H}^{(2)}$, $\mathbf{M}^{(2)} = (\mu_r - 1)\mathbf{H}^{(2)}$, where $\mu_r$ is the relative permeability of the fluid. Additionally, the magnetization is considered to be a linear function of the density $\rho^{(2)}$ which results in the usual form of the Kelvin force $\mathbf{F}_{\mathrm{K}} = \mu_0(\mathbf{M}^{(2)}\,\mathrm{grad})\mathbf{H}^{(2)}$.[2] The system is governed by the equation of continuity, the Navier–Stokes equations for the magnetic fluid,

$$\mathrm{div}\,\mathbf{v}^{(2)} = 0\,, \tag{1}$$

$$\rho^{(2)}\partial_t\mathbf{v}^{(2)} + \rho^{(2)}(\mathbf{v}^{(2)}\,\mathrm{grad})\mathbf{v}^{(2)} = -\mathrm{grad}\,p^{(2)} + \mu^{(2)}\Delta\mathbf{v}^{(2)}$$
$$+ \mu_0(\mathbf{M}^{(2)}\,\mathrm{grad})\mathbf{H}^{(2)} + \rho^{(2)}\mathbf{g}\,, \tag{2}$$

and the Laplace equation in each medium,

$$\Delta\Phi^{(i)} = 0\,. \tag{3}$$

The scalar magnetic potentials are defined by $\mathbf{H}^{(i)} = -\mathrm{grad}\,\Phi^{(i)}$. The velocity field in the MF is denoted by $\mathbf{v}^{(2)}$, the dynamic viscosity by $\mu^{(2)}$, the pressure field by $p^{(2)}$, and the acceleration due to gravity by $\mathbf{g} = (0, 0, -g)$. The first three terms on the right-hand side of Eq. (2) result from $\mathrm{div}\,\overset{\leftrightarrow}{T}^{(2)}$, where the components of the stress tensor $\overset{\leftrightarrow}{T}^{(2)}$ read[2]

$$T_{ij}^{(2)} = \left\{-p^{(2)} - \mu_0\frac{(H^{(2)})^2}{2}\right\}\delta_{ij} + H_i^{(2)}B_j^{(2)} + \mu^{(2)}(\partial_i v_j^{(2)} + \partial_j v_i^{(2)})\,. \tag{4}$$

$M^{(2)}$, $H^{(2)}$, and $B^{(2)}$ denote the absolute value of the magnetization, the magnetic field, and the induction $\mathbf{B}^{(2)}$ in the MF. Since the final arrangement of peaks is a static configuration, the velocity field in the MF is set to zero, $\mathbf{v}^{(2)} = 0$, and the surface depends only on the horizontal spatial coordinates, $z_{\mathrm{s}} = \zeta(x, y)$. Applying all assumptions, the static form of the Navier–Stokes equation is

$$0 = -\mathrm{grad}\,p^{(2)} + \frac{\mu_0}{2}(\mu_r - 1)\,\mathrm{grad}(H^{(2)})^2 - \rho^{(2)}g\mathbf{e}_z\,, \tag{5}$$

where $\mathbf{e}_z = (0, 0, 1)$ is the unit vector in $z$-direction. The governing equations have to be supplemented by the appropriate boundary conditions. These are the continuity

of the normal (tangential) component of the magnetic induction (magnetic field) at the top and bottom interface,

$$\mathbf{n}^{(2,1)} \cdot (\mathbf{B}^{(1)} - \mathbf{B}^{(2)}) = 0, \qquad \mathbf{n}^{(2,1)} \times (\mathbf{H}^{(1)} - \mathbf{H}^{(2)}) = 0 \quad \text{at } z = z_{\mathrm{s}}, \qquad (6)$$

$$\mathbf{n}^{(3,2)} \cdot (\mathbf{B}^{(2)} - \mathbf{B}^{(3)}) = 0, \qquad \mathbf{n}^{(3,2)} \times (\mathbf{H}^{(2)} - \mathbf{H}^{(3)}) = 0 \quad \text{at } z = -d, \qquad (7)$$

and the continuity of the normal component of the stress tensor across the free surface

$$(\overset{\leftrightarrow}{T}^{(1)} - \overset{\leftrightarrow}{T}^{(2)})\mathbf{n}^{(2,1)} - \sigma K \mathbf{n}^{(2,1)} = 0, \qquad \text{at } z = z_{\mathrm{s}}. \qquad (8)$$

In (8) the surface tension between the magnetic fluid and air is denoted by $\sigma$, the curvature of the surface by $K = \operatorname{div} \mathbf{n}^{(2,1)}$, and the unit vector normal to the MF surface by

$$\mathbf{n}^{(2,1)} = \frac{\operatorname{grad}[z - \zeta(x,y)]}{|\operatorname{grad}[z - \zeta(x,y)]|} = \frac{(-\partial_x \zeta, -\partial_y \zeta, 1)}{\sqrt{1 + (\partial_x \zeta)^2 + (\partial_y \zeta)^2}}. \qquad (9)$$

The upper index $(2,1)$ at the unit vector indicates that $\mathbf{n}^{(2,1)}$ points from medium 2 towards medium 1; analogous for the normal vector $\mathbf{n}^{(3,2)} = (0,0,1)$ (see Fig. 1). The difference of the tangential components of the stress tensor is identically zero because of the continuity of the magnetic fields and inductions in (6). Neglecting the influence of the air pressure with respect to the fluid pressure, $p^{(1)} \simeq 0$, and using $\mathbf{M}^{(1)} \equiv 0$, one finally gets from Eq. (8)

$$\frac{\mu_0}{2}[\mathbf{M}^{(2)}\mathbf{n}^{(2,1)}]^2 + p^{(2)} - \sigma K = 0, \qquad \text{at } z = z_{\mathrm{s}}, \qquad (10)$$

where $p^{(2)}$ is the solution of Eq. (5).

## 2.1. *Basic state*

As long as the applied spatially homogeneous magnetic field perpendicular to the surface,

$$\mathbf{H}^{(i)} = \mathbf{H}_{\mathrm{G}}^{(i)} = (0,0,H_{\mathrm{G}}^{(i)}), \qquad \mathbf{B}^{(i)} = \mathbf{B}_{\mathrm{G}}^{(i)} = (0,0,B_{\mathrm{G}}^{(i)}),$$

$$\mathbf{M}^{(2)} = \mathbf{M}_{\mathrm{G}}^{(2)} = (0,0,M_{\mathrm{G}}^{(2)}), \qquad (11)$$

is below a certain strength, the system is in its basic or ground state. This state is given by the plane surface $z_{\mathrm{s}} = 0$. The corresponding solution for the fluid pressure is

$$p_{\mathrm{G}}^{(2)} = -\rho^{(2)}gz - \frac{\mu_0}{2}(M_{\mathrm{G}}^{(2)})^2, \qquad (12)$$

where the constant resulting from (5) was determined by (10) and $H_{\mathrm{G}}^{(2)} = H_{\mathrm{G}}^{(2)}|_0$ was used. Here $|_0$ denotes the evaluation at the plane interface $z_{\mathrm{s}} = 0$. The boundary conditions (6) and (7) are fulfilled by $B_{\mathrm{G}}^{(1)} = B_{\mathrm{G}}^{(2)}$ and $B_{\mathrm{G}}^{(2)} = B_{\mathrm{G}}^{(3)}$, respectively.

## 2.2. *Small disturbances and their expansion*

In order to study the stability of the basic state and the pattern selection problem in the weakly nonlinear regime, small deviations from the basic state are considered

$$z_{\mathrm{s}} = 0 + \zeta, \qquad \mathbf{H}^{(i)} = \mathbf{H}_{\mathrm{G}}^{(i)} + \mathbf{h}^{(i)}, \qquad \mathbf{B}^{(i)} = \mathbf{B}_{\mathrm{G}}^{(i)} + \mathbf{b}^{(i)},$$

$$\mathbf{M}^{(2)} = \mathbf{M}_{\mathrm{G}}^{(2)} + \mathbf{m}^{(2)}, \qquad \Phi^{(i)} = \Phi_{\mathrm{G}}^{(i)} + \phi^{(i)}, \qquad p^{(2)} = p_{\mathrm{G}}^{(2)} + \pi^{(2)}. \qquad (13)$$

Dimensionless quantities are introduced, where physical quantities are denoted by a hat in the rest of the paper. All quantities associated with the magnetic field are scaled by the critical magnetic induction in the limit of an infinite thickness of the MF layer, $\hat{B}_{\mathrm{c},\infty}$. This preference is based on the fact that magnetic inductions can be directly measured by Hall probes. Furthermore, it is assumed that the deviations are proportional to the applied external induction. Thus the following scaling is used

$$\hat{\mathbf{B}}_{\mathrm{G}}^{(i)} = \mathbf{B}_{\mathrm{ext}} \hat{B}_{\mathrm{c},\infty}, \qquad\qquad\qquad \hat{\mathbf{b}}^{(i)} = \mathbf{b}^{(i)} B_{\mathrm{ext}} \hat{B}_{\mathrm{c},\infty}, \qquad (14)$$

$$\hat{\mathbf{M}}_{\mathrm{G}}^{(i)} = \mathbf{B}_{\mathrm{ext}} \frac{(\mu_r - 1)}{\hat{\mu}_0 \mu_r^{(i)}} \hat{B}_{\mathrm{c},\infty}, \qquad\qquad \hat{\mathbf{m}}^{(i)} = \mathbf{m}^{(i)} B_{\mathrm{ext}} \frac{(\mu_r - 1)}{\hat{\mu}_0 \mu_r^{(i)}} \hat{B}_{\mathrm{c},\infty}, \qquad (15)$$

$$\hat{\mathbf{H}}_{\mathrm{G}}^{(i)} = \mathbf{B}_{\mathrm{ext}} \frac{\hat{B}_{\mathrm{c},\infty}}{\hat{\mu}_0 \mu_r^{(i)}}, \qquad\qquad\qquad \hat{\mathbf{h}}^{(i)} = \mathbf{h}^{(i)} B_{\mathrm{ext}} \frac{\hat{B}_{\mathrm{c},\infty}}{\hat{\mu}_0 \mu_r^{(i)}}, \qquad (16)$$

$$\hat{\pi}^{(2)} = \pi^{(2)} B_{\mathrm{ext}}^2 \frac{2(\mu_r + 1)}{\mu_r (\mu_r - 1)^2} \sqrt{\hat{\rho}^{(2)} \hat{g} \hat{\sigma}}, \qquad \hat{l} = \frac{l}{\hat{k}_{\mathrm{c},\infty}}. \qquad (17)$$

For an infinite thickness of the layer, the critical induction and critical wave number, respectively, are[3]

$$\hat{B}_{\mathrm{c},\infty}^2 = \frac{2\hat{\mu}_0 \mu_r (\mu_r + 1) \sqrt{\hat{\rho}^{(2)} \hat{\sigma} \hat{g}}}{(\mu_r - 1)^2}, \qquad \hat{k}_{\mathrm{c},\infty} = \sqrt{\frac{\hat{\rho}^{(2)} \hat{g}}{\hat{\sigma}}}. \qquad (18)$$

The dimensionless quantity $B_{\mathrm{ext}}$ in $\mathbf{B}_{\mathrm{ext}} = (0, 0, B_{\mathrm{ext}})$ measures the strength of the applied external induction in units of $\hat{B}_{\mathrm{c},\infty}$.

With this scaling the solution of the Navier–Stokes equation (5) becomes

$$\pi^{(2)} = \frac{\eta}{1 - \eta} [2h_z^{(2)} + (\mathbf{h}^{(2)})^2] + c, \qquad (19)$$

with a yet unknown constant $c$. The Laplace equation (3) in each medium is

$$\Delta \phi^{(i)} = 0, \qquad (20)$$

and the boundary conditions (6), (7), and (10) are

$$\text{at } z = -d, \qquad 0 = -\phi^{(2)} + \phi^{(3)}, \qquad (21)$$

$$\text{at } z = -d, \qquad 0 = -\frac{1 + \eta}{1 - \eta} \partial_z \phi^{(2)} + \partial_z \phi^{(3)}, \qquad (22)$$

$$\text{at } z = \zeta, \qquad 0 = -\frac{2\eta}{1 - \eta} \zeta + \phi^{(1)} - \phi^{(2)}, \qquad (23)$$

at $z = \zeta$,    $0 = -\partial_x \zeta \left( -\partial_x \phi^{(1)} + \frac{1+\eta}{1-\eta} \partial_x \phi^{(2)} \right)$

$$- \partial_y \zeta \left( -\partial_y \phi^{(1)} + \frac{1+\eta}{1-\eta} \partial_y \phi^{(2)} \right)$$

$$- \partial_z \phi^{(1)} + \frac{1+\eta}{1-\eta} \partial_z \phi^{(2)} , \qquad (24)$$

at $z = \zeta$,    $0 = -\frac{2B_{\text{ext}}^2}{(1+\eta)} \frac{[(\partial_x \zeta)^2 + (\partial_y \zeta)^2]}{[1 + (\partial_x \zeta)^2 + (\partial_y \zeta)^2]} + \frac{2B_{\text{ext}}^2(1-\eta)}{\eta(1+\eta)}$

$$\times \left( \frac{2\eta}{(1-\eta)} \frac{\mathbf{h}^{(2)} \mathbf{n}^{(2,1)}}{\sqrt{1 + (\partial_x \zeta)^2 + (\partial_y \zeta)^2}} + \mathbf{h}^{(2)} \mathbf{e}_z \right)$$

$$+ \frac{B_{\text{ext}}^2(1-\eta)}{\eta(1+\eta)} \left[ \frac{2\eta(\mathbf{h}^{(2)} \mathbf{n}^{(2,1)})^2}{(1-\eta)} + (\mathbf{h}^{(2)})^2 \right]$$

$$- z - K + \frac{B_{\text{ext}}^2(1-\eta)^2}{\eta^2(1+\eta)} c . \qquad (25)$$

The curvature of the surface is

$$K = \frac{-\partial_{xx}\zeta - \partial_{yy}\zeta}{\sqrt{1 + (\partial_x \zeta)^2 + (\partial_y \zeta)^2}} + \frac{(\partial_x \zeta)^2 \partial_{xx}\zeta + 2\partial_x \zeta \partial_y \zeta \partial_{xy}\zeta + (\partial_y \zeta)^2 \partial_{yy}\zeta}{[1 + (\partial_x \zeta)^2 + (\partial_y \zeta)^2]^{3/2}} , \quad (26)$$

and the widely used quantity

$$\eta = \frac{\mu_r - 1}{\mu_r + 1} \qquad (27)$$

was introduced ($\gamma = (3/4)\eta$ in Refs. 11, 16 and 21).

Since the disturbances of the magnetic field are located in the vicinity of the magnetic fluid, they have to disappear as $z$ goes to infinity, i.e. the magnetic field disturbances have to fulfill two more conditions:

at $z = -\infty$,    $0 = \phi^{(3)} , \qquad (28)$

at $z = \infty$,    $0 = \phi^{(1)} . \qquad (29)$

In summary, the system is governed by three differential equations of second order (20) and the corresponding six boundary conditions (21)–(24), (28), (29) for the magnetic potentials. The remaining equation (25) is a differential equation for $\zeta(x, y)$ whose solution determines the shape of the surface.

To derive solutions with a finite but small amplitude, the Eqs. (20)–(25) are solved perturbatively. The external control parameter $B_{\text{ext}}$ and the various physical quantities are expanded

$$B_{\text{ext}}^2 = B_c^2 + \varepsilon B_1 + \varepsilon^2 B_2 + \cdots , \qquad (30)$$

$$X = \varepsilon X_0 + \varepsilon^2 X_1 + \varepsilon^3 X_2 + \cdots , \qquad (31)$$

where $X$ can be any of the quantities $\phi^{(i)}$, $\zeta$, and $c$. $\varepsilon$ is an expansion parameter with respect to the magnitude of the surface deformation ($0 < \varepsilon \ll 1$). The boundary conditions (23)–(25) have to be evaluated at an *a priori* unknown surface $\zeta(x, y)$. Therefore it is convenient to expand all potentials $\phi^{(i)}$ on the surface in terms of $\phi^{(i)}$ at $z = 0$,

$$\phi^{(i)}(x, y, z = \zeta) = \phi^{(i)}(x, y, 0) + \partial_z \phi^{(i)}(x, y, z)|_0 \cdot \zeta + \frac{1}{2}\partial_{zz}\phi^{(i)}(x, y, z)|_0 \cdot \zeta^2 + \cdots,$$
(32)

and similarly for all space derivatives of $\phi^{(i)}$. The chosen expansion of the external parameter follows the way as it is known from thermal convection problems of ordinary fluids [22,23] or magnetic fluids[24] and from the Bénard–Marangoni problem.[25−27] The expansion (30) and (31) describes a situation slightly above the threshold of the linear instability and differs from those used in Refs. 19 and 20, where the magnetic field $H$, acting as external parameter, is not expanded. The consequences of this difference will be discussed in Sec. 5.

## 2.3. *Different nonlinearity*

Before going into the details of the expansion procedure, it is illuminating to analyze the set of Eqs. (20)–(25). The differential equations (20) and the boundary conditions at the bottom of the container (21), (22) are *linear* equations. Only the three remaining boundary equations (23)–(25) contain *nonlinear* terms. In contrast to almost all classical hydrodynamic systems, the nonlinearity stems from the boundary conditions and not from the differential equations which describe phenomena in the bulk of the system. For example in convection systems the nonlinear contributions are caused by the advective term, i.e. by a bulk term, in the Navier–Stokes equation and in the equation of head conduction. For the Rosensweig instability the nonlinearities are caused by the *deformed* and *unknown* surface $\zeta$ at which the potentials have to be calculated. Thus each linear term in $\phi^{(1)}$ and $\phi^{(2)}$ in Eqs. (23)–(25) generates nonlinear terms via Eq. (32). A further nonlinear contribution comes from the Kelvin force density in which a term quadratic in the potential $\phi^{(2)}$ appears, to determine again at the deformed and unknown surface. This qualitatively different character of the Rosensweig instability makes it fascinating as well as hard to treat the instability with the classical methods. One indicator of the essential role of the boundaries is the fact that the external parameter $B_{\text{ext}}$ appears only in the boundary condition (25).

Performing the expansion according to (30)–(32) in (20)–(25) generates a hierarchy of linear equations for $U_0$, $U_1$, and $U_2$

$$L_0 U_0 = 0,$$
(33)

$$L_0 U_1 = -L_1 U_0 + N_1(U_0, U_0),$$
(34)

$$L_0 U_2 = -L_1 U_1 - L_2 U_0 + N(U_0, U_1) + N(U_1, U_0)$$
$$+ N(U_0, U_0, U_0) + N_2(U_0, U_0),$$
(35)

by matching the three lowest powers of $\varepsilon$. The vector

$$U = (\phi^{(3)}, \phi^{(2)}, \phi^{(1)}, \phi^{(1)}|_0, \phi^{(2)}|_0, \zeta)^T = \varepsilon U_0 + \varepsilon^2 U_1 + \varepsilon^3 U_2 + \cdots \qquad (36)$$

is the augmented state vector, $L_0$, $L_1$, and $L_2$ are linear operators, and the vectors $N(\cdot)$, $N_1(\cdot)$, and $N_2(\cdot)$ contain the nonlinear contributions. With the above mentioned, it is clear that in each order of $\varepsilon$ different nonhomogeneous boundary problems have to be solved. This again indicates the different quality of the problem considered here in comparison to classical convection problems. For the boundary conditions, terms involving $\phi^{(1)}$, $\phi^{(2)}$, and their derivatives have to be evaluated at $z = 0$ wherefore the vector $U$ contains the potentials $\phi^{(1)}$ and $\phi^{(2)}$ at $z = 0$ additionally to the potentials itself. Similar state vector were used for the Bénard–Marangoni system[26,27] or for the electroconvection problem in a thin film.[28]

## 3. Linear Problem

The linear stability problem and the corresponding adjoint problem play a fundamental role in the nonlinear analysis. Since the linear problem has been considered in details elsewhere[4,5,7,29–31] only some of its more important aspects are recapitulated here.

Inserting the ansatz (30)–(32) into (20)–(25) gives at order $O(\varepsilon)$

$$0 = \Delta \phi_0^{(i)}, \qquad (37)$$

$$\text{at } z = -d, \qquad 0 = -\phi_0^{(2)} + \phi_0^{(3)}, \qquad (38)$$

$$\text{at } z = -d, \qquad 0 = -\frac{1+\eta}{1-\eta} \partial_z \phi_0^{(2)} + \partial_z \phi_0^{(3)}, \qquad (39)$$

$$\text{at } z = 0, \qquad 0 = -\frac{2\eta}{1-\eta} \zeta_0 + \phi_0^{(1)} - \phi_0^{(2)}, \qquad (40)$$

$$\text{at } z = 0, \qquad 0 = -\partial_z \phi_0^{(1)} + \frac{1+\eta}{1-\eta} \partial_z \phi_0^{(2)}, \qquad (41)$$

$$\text{at } z = 0, \qquad 0 = -\frac{2(1-\eta)B_c^2}{\eta(1+\eta)} \partial_z \phi_0^{(1)} - \zeta_0$$

$$+ (\partial_{xx} + \partial_{yy}) \zeta_0 + \frac{(1-\eta)^2 B_c^2}{\eta^2(1+\eta)} c_0. \qquad (42)$$

Equations (37)–(41) are solved by

$$\phi_0^{(3)} = -\frac{\eta(1+\eta) e^{k(d+z)}}{e^{kd} - \eta^2 e^{-kd}} \bar{\zeta}_0 e^{i\mathbf{k}\cdot\mathbf{r}}, \qquad (43)$$

$$\phi_0^{(2)} = -\frac{\eta[e^{k(d+z)} + \eta e^{-k(d+z)}]}{e^{kd} - \eta^2 e^{-kd}} \bar{\zeta}_0 e^{i\mathbf{k}\cdot\mathbf{r}}, \qquad (44)$$

$$\phi_0^{(1)} = \frac{\eta(1+\eta)[e^{k(d-z)} - \eta e^{-k(d+z)}]}{(1-\eta)[e^{kd} - \eta^2 e^{-kd}]} \bar{\zeta}_0 e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{45}$$

$$\zeta_0 = \bar{\zeta}_0 e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{46}$$

with $\mathbf{k} = (k_x, k_y)$, $k = \sqrt{k_x^2 + k_y^2}$, $\mathbf{r} = (x, y)$, and $\bar{\zeta}_0$ the amplitude of the surface deformation. Inserting this solution into (42) leads to the known dispersion relation for an inviscid magnetic fluid of finite thickness in dimensionless quantities[5,7,29,30]

$$1 + k^2 - 2B_c^2 k \left( \frac{e^{kd} - \eta e^{-kd}}{e^{kd} - \eta^2 e^{-kd}} \right) = 0, \tag{47}$$

providing that the unknown constant $c_0$ is set to zero. By solving Eq. (47) numerically for a given thickness $d$, one gets the height dependent critical values $B_c(d)$ and $k_c(d)$ at the onset of the instability.

The linear operator $L_0$ together with the corresponding augmented state vector $U_0$ of Eq. (33) are defined on the basis of Eqs. (37), (40)–(42):

$$L_0 U_0 = 0 = \begin{pmatrix} \Delta & 0 & 0 & 0 & 0 & 0 \\ 0 & \Delta & 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -\dfrac{2\eta}{1-\eta} \\ 0 & \dfrac{1+\eta}{1-\eta}\partial_z|_0 & -\partial_z|_0 & 0 & 0 & 0 \\ 0 & 0 & -\dfrac{2(1-\eta)}{\eta(1+\eta)}B_c^2 \partial_z|_0 & 0 & 0 & \partial_{xx} + \partial_{yy} - 1 \end{pmatrix} \begin{pmatrix} \phi_0^{(3)} \\ \phi_0^{(2)} \\ \phi_0^{(1)} \\ \phi_0^{(1)}|_0 \\ \phi_0^{(2)}|_0 \\ \zeta_0 \end{pmatrix}. \tag{48}$$

## 4. Adjoint Problem

To proceed with the derivation towards an amplitude equation, one has to solve the linear inhomogeneous equations for $U_1$ and $U_2$, Eqs. (34), (35). According to Fredholms theorem,[32] they have a solution if and only if the right hand side is orthogonal to the zero space of the linear operator $L_0$. If $\bar{U}_0$ is an eigenvector of the adjoint linear operator $L_0^\dagger$ with zero eigenvalue, then multiplying Eq. (34) from left with $\bar{U}_0$ reads (analogous for Eq. (35))

$$-\langle \bar{U}_0, L_1 U_0 \rangle + \langle \bar{U}_0, N_1(U_0, U_0) \rangle = \langle \bar{U}_0, L_0 U_1 \rangle = \langle L_0^\dagger \bar{U}_0, U_1 \rangle = 0. \tag{49}$$

The scalar product is denoted by $\langle\,,\,\rangle$, where the explicit form depends on the considered problem. Applying these solvability conditions, by which $B_1$ and $B_2$ (see Eq. (30)) can be expressed, leads to the general amplitude equation for a pattern selection problem.

In Sec. 2.3 the different character is described of the pure "surface-nonlinearities" occuring in the Rosensweig instability. The question arises whether

the established methods for "bulk-nonlinearities", as for example for the Rayleigh–Bénard convection, will work here as well. Motivated by Refs. 26–28 where nonlinearities triggered by the bulk and the plain surface appear, the following scalar product

$$\langle \bar{U}_0, U \rangle = \lim_{l \to \infty} \frac{1}{l^2} \int_{-l/2}^{l/2} dx \int_{-l/2}^{l/2} dy \left[ \int_{-\infty}^{-d} dz \bar{\phi}_0^{(3)*} \phi^{(3)} + \frac{1+\eta}{1-\eta} \int_{-d}^{0} dz \bar{\phi}_0^{(2)*} \phi^{(2)} \right.$$

$$\left. + \int_0^\infty dz \bar{\phi}_0^{(1)*} \phi^{(1)} + d\bar{u}_{0,4}^*|_0 \phi^{(1)}|_0 + e\bar{u}_{0,5}^*|_0 \phi^{(2)}|_0 + f\bar{u}_{0,6}^*|_0 \zeta \right] \quad (50)$$

is used. $\bar{U}_0$ is chosen since Eqs. (34) and (35) have to be multiplied by $\bar{U}_0$. The first three terms in the square brackets in Eq. (50) are contributions from the volume, whereas the last three terms are contributions from the surface. The possible prefactors $d$, $e$, and $f$ as well as the surface components of the adjoint vector $\bar{U}_0$, $\bar{u}_{0,4}$, $\bar{u}_{0,5}$, and $\bar{u}_{0,6}$, have yet to be determined. From Refs. 26 and 27 it is known that the surface components of the adjoint state vector are not just simply the adjoint components of the state vector. Using the identity

$$\bar{\phi}_0^{(i)*} \Delta \phi^{(i)} = \partial_x [\bar{\phi}_0^{(i)*} \partial_x \phi^{(i)} - \partial_x \bar{\phi}_0^{(i)*} \phi^{(i)}] + \partial_y [\bar{\phi}_0^{(i)*} \partial_y \phi^{(i)} - \partial_y \bar{\phi}_0^{(i)*} \phi^{(i)}]$$

$$+ \partial_{xx} \bar{\phi}_0^{(i)*} \phi^{(i)} + \partial_{yy} \bar{\phi}_0^{(i)*} \phi^{(i)} + \bar{\phi}_0^{(i)*} \partial_{zz} \phi^{(i)} , \quad (51)$$

and the conditions (21), (22), (28), and (29) one has after partial integration

$$\langle \bar{U}_0, L_0 U \rangle = \lim_{l \to \infty} \frac{1}{l^2} \int_{-l/2}^{l/2} dx \int_{-l/2}^{l/2} dy \left[ \int_{-\infty}^{-d} dz \Delta \bar{\phi}_0^{(3)*} \phi^{(3)} + \frac{1+\eta}{1-\eta} \right.$$

$$\times \int_{-d}^{0} dz \Delta \bar{\phi}_0^{(2)*} \phi^{(2)} + \int_0^\infty dz \Delta \bar{\phi}_0^{(1)*} \phi^{(1)} - \bar{\phi}_0^{(3)*}|_{-\infty} \partial_z \phi^{(3)}|_{-\infty}$$

$$+ (\bar{\phi}_0^{(3)*} - \bar{\phi}_0^{(2)*})|_{-d} \partial_z \phi^{(3)}|_{-d} - \left( \partial_z \bar{\phi}_0^{(3)*} - \frac{1+\eta}{1-\eta} \partial_z \bar{\phi}_0^{(2)*} \right)\bigg|_{-d} \phi^{(3)}|_{-d}$$

$$+ \bar{\phi}_0^{(1)*}|_\infty \partial_z \phi^{(1)}|_\infty + R \bigg] . \quad (52)$$

From Eq. (52) it is concluded that the linear adjoint system is governed by three differential equations of second order for the adjoint potentials $\bar{\phi}_0^{(i)}$. According to Eq. (52) the four conditions to be fulfilled by these adjoint potentials are

$$\text{at } z = -\infty, \qquad 0 = \bar{\phi}_0^{(3)} , \quad (53)$$

$$\text{at } z = -d, \qquad 0 = -\bar{\phi}_0^{(2)} + \bar{\phi}_0^{(3)} , \quad (54)$$

$$\text{at } z = -d, \qquad 0 = -\frac{1+\eta}{1-\eta} \partial_z \bar{\phi}_0^{(2)} + \partial_z \bar{\phi}_0^{(3)}, \quad (55)$$

$$\text{at } z = \infty, \qquad 0 = \bar{\phi}_0^{(1)} , \quad (56)$$

since only terms on the surface should remain. The residual terms

$$R = \partial_z \phi^{(2)}|_0 \left( \frac{1+\eta}{1-\eta} \bar{\phi}_0^{(2)*} + \frac{1+\eta}{1-\eta} e\bar{u}_{0,5}^* \right)\bigg|_0 + \phi^{(2)}|_0 \left( -\frac{1+\eta}{1-\eta} \partial_z \bar{\phi}_0^{(2)*} - d\bar{u}_{0,4}^* \right)\bigg|_0$$

$$+ \partial_z \phi^{(1)}|_0 \left[ -\bar{\phi}_0^{(1)*} - e\bar{u}_{0,5}^* - \frac{2(1-\eta)}{\eta(1+\eta)} B_c^2 f\bar{u}_{0,6}^* \right]\bigg|_0 + \phi^{(1)}|_0 (\partial_z \bar{\phi}_0^{(1)*} + d\bar{u}_{0,4}^*)|_0$$

$$+ \zeta \left[ \frac{2\eta}{1-\eta} d\bar{u}_{0,4}^* + (\partial_{xx} + \partial_{yy} - 1)f\bar{u}_{0,6}^* \right]\bigg|_0 \tag{57}$$

are bounded to contain only the two remaining boundary conditions and one further term, all to be determined at $z = 0$, for a proper definition of $L_0^\dagger$. Comparing the terms in $R$ and the surface components of $U$ (see Eq. (36)), one realizes that the first and third term in Eq. (57) have to vanish. As a consequence of the immediate choice $e\bar{u}_{0,5} = -\bar{\phi}_0^{(2)}$, the sixth component of the adjoint vector $\bar{U}_0$, $\bar{u}_{0,6}$, is a linear combination of $\bar{\phi}_0^{(1)}$ and $\bar{\phi}_0^{(2)}$, i.e. a linear combination of $\bar{\phi}_0^{(1)}$ and $\bar{u}_{0,5}$. Certainly a result contrary to the condition that the set of variables in $\bar{U}_0$ have to be linearly independent. Testing the choice

$$d\bar{u}_{0,4} = k\bar{\phi}_0^{(1)}, \qquad e\bar{u}_{0,5} = -\bar{\phi}_0^{(2)}, \qquad f\bar{u}_{0,6} = \frac{-\eta^2(1+\eta)}{(1-\eta)^2 B_c^2} \bar{\zeta}_0 \tag{58}$$

results in

$$\langle \bar{U}_0, L_0 U \rangle = \lim_{L \to \infty} \frac{1}{l^2} \int_{-l/2}^{l/2} dx \int_{-l/2}^{l/2} dy \left[ \int_{-\infty}^{-d} dz \Delta \bar{\phi}_0^{(3)*} \phi^{(3)} \right.$$

$$+ \frac{1+\eta}{1-\eta} \int_{-d}^0 dz \Delta \bar{\phi}_0^{(2)*} \phi^{(2)} + \int_0^\infty dz \Delta \bar{\phi}_0^{(1)*} \phi^{(1)}$$

$$- \left( -\partial_z \bar{\phi}_0^{(1)*} + \frac{1+\eta}{1-\eta} \partial_z \bar{\phi}_0^{(2)*} \right)\bigg|_0 \phi^{(2)}|_0$$

$$- \frac{\eta^2(1+\eta)}{(1-\eta)^2 B_c^2} \left( -\frac{2(1-\eta)B_c^2}{\eta(1+\eta)} \partial_z \bar{\phi}_0^{(1)*} + (\partial_{xx} + \partial_{yy} - 1)\bar{\zeta}_0^* \right)\bigg|_0 \zeta$$

$$- \left( -\frac{2\eta}{(1-\eta)} \zeta_0^* + \bar{\phi}_0^{(1)*} - \bar{\phi}_0^{(2)*} \right)\bigg|_0 \partial_z \phi^{(1)}|_0 \right], \tag{59}$$

where $\partial_z \bar{\phi}_0^{(1)}|_0 = -k\bar{\phi}_0^{(1)}|_0$ was used. This identity is justified because of the Laplace equation for $\bar{\phi}_0^{(1)}$ (see last $z$-integral in Eq. (52)) and the condition (56). After inserting (58) into Eq. (50), it becomes clear that Eq. (59) has almost the proper form in order to define the adjoint operator $L_0^\dagger$ via $\langle \bar{U}_0, L_0 U \rangle = \langle L_0^\dagger \bar{U}_0, U \rangle$. The last remaining but unsolved problem is posed by the third surface term in Eq. (59). Instead of $k\phi^{(1)}|_0$, the expression $-\partial_z \phi^{(1)}|_0$ appears. Both terms are only equally in the first order of the expansion, where $\phi_0^{(1)} \sim e^{-kz}$ (see Eq. (45)). In higher orders of the expansion, the functions $\phi_1^{(1)}$ and $\phi_2^{(1)}$ containing wave vectors which are linear

combinations of two and three, respectively, wave vectors of the basic modes. As a result $z$-dependences will appear as $\phi_1^{(1)} \sim e^{-|\mathbf{k}_n \pm \mathbf{k}_m|z}$ and $\phi_2^{(1)} \sim e^{-|\mathbf{k}_n \pm \mathbf{k}_m \pm \mathbf{k}_l|z}$, respectively. The absolute value of the resulting wave vector is usually not equal to $k$. Thus the task remains that for a nonlinear analysis a linear adjoint operator should be definable which is valid for a wider set of function as $e^{-kz}$.

The generic problem of a nonzero, deformed surface is illustrated by calculating the scalar product of the augmented state $U$ vector and its adjoint one $\bar{U}$

$$\langle \bar{U}, U \rangle = \varepsilon^2 \langle \bar{U}_0, U_0 \rangle + \varepsilon^3 (\langle \bar{U}_0, U_1 \rangle + \langle \bar{U}_1, U_0 \rangle) + O(\varepsilon^4),$$

$$= \lim_{l \to \infty} \frac{1}{l^2} \int_{-l/2}^{l/2} dx \int_{-l/2}^{l/2} dy \left[ \int_{-\infty}^{-d} dz \bar{\phi}^{(3)*} \phi^{(3)} \right.$$

$$+ \frac{1+\eta}{1-\eta} \int_{-d}^{\zeta} dz \bar{\phi}^{(2)*} \phi^{(2)} + \int_{\zeta}^{\infty} dz \bar{\phi}^{(1)*} \phi^{(1)}$$

$$\left. + d\bar{u}_4^*|_\zeta \phi^{(1)}|_\zeta + e \bar{u}_5^*|_\zeta \phi^{(2)}|_\zeta + f \bar{u}_6^*|_\zeta \zeta \right]. \tag{60}$$

Since $U$ and $\bar{U}$ contain the complete information of the disturbed state, the unknown and deformed surface $\zeta$ appears as a bound at two integrals and at the surface contributions. Whereas the expansion of the three surface contributions can be performed accordingly to Eq. (32), the expansion of the two integrals involving the deformed surface $\zeta$ is accomplished as

$$\int_{-d}^{\zeta} dz \bar{\phi}^{(2)*} \phi^{(2)} = \varepsilon^2 \int_{-d}^{0} dz \phi_0^{(2)*} \phi_0^{(2)} + \varepsilon^3 \left[ \zeta_0 \bar{\phi}_0^{(2)*}|_0 \phi_0^{(2)}|_0 \right.$$

$$\left. + \int_{-d}^{0} dz (\bar{\phi}_0^{(2)*} \phi_1^{(2)} + \bar{\phi}_1^{(2)*} \phi_0^{(2)}) \right] + O(\varepsilon^4). \tag{61}$$

The analogous expansion can be applied to the second integral involving $\zeta$. Matching the two lowest powers of $\varepsilon$ in Eq. (60), one gets

$$\langle \bar{U}_0, U_0 \rangle = \lim_{l \to \infty} \frac{1}{l^2} \int_{-l/2}^{l/2} dx \int_{-l/2}^{l/2} dy \left[ \int_{-\infty}^{-d} dz \bar{\phi}_0^{(3)*} \phi_0^{(3)} + \frac{1+\eta}{1-\eta} \int_{-d}^{0} dz \bar{\phi}_0^{(2)*} \phi_0^{(2)} \right.$$

$$\left. + \int_{0}^{\infty} dz \bar{\phi}_0^{(1)*} \phi_0^{(1)} + d\bar{u}_{0,4}^*|_0 \phi_0^{(1)}|_0 + e \bar{u}_{0,5}^*|_0 \phi_0^{(2)}|_0 + f \bar{u}_{0,6}^*|_0 \zeta_0 \right] \tag{62}$$

and

$$\langle \bar{U}_0, U_1 \rangle + \langle \bar{U}_1, U_0 \rangle$$

$$= \lim_{l \to \infty} \frac{1}{l^2} \int_{-l/2}^{l/2} dx \int_{-l/2}^{l/2} dy \left\{ \int_{-\infty}^{-d} dz (\bar{\phi}_0^{(3)*} \phi_1^{(3)} + \bar{\phi}_1^{(3)*} \phi_0^{(3)}) \right.$$

$$+ \frac{1+\eta}{1-\eta} \int_{-d}^{0} dz (\bar{\phi}_0^{(2)*} \phi_1^{(2)} + \bar{\phi}_1^{(2)*} \phi_0^{(2)}) + \int_0^{\infty} dz (\bar{\phi}_0^{(1)*} \phi_1^{(1)} + \bar{\phi}_1^{(1)*} \phi_0^{(1)})$$

$$+ d(\bar{u}_{0,4}^* \phi_1^{(1)} + \bar{u}_{1,4}^* \phi_0^{(1)})|_0 + e(\bar{u}_{0,5}^* \phi_1^{(2)} + \bar{u}_{1,5}^* \phi_0^{(2)})|_0$$

$$+ f(\bar{u}_{0,6}^*|_0 \zeta_1 + \bar{u}_{1,6}^*|_0 \zeta_0) + \zeta_0 \Big[ \bar{\phi}_0^{(2)*} \phi_0^{(2)} - \bar{\phi}_0^{(1)*} \phi_0^{(1)}$$

$$+ d(\bar{u}_{0,4}^* \partial_z \phi_0^{(1)} + \partial_z \bar{u}_{0,4}^* \phi_0^{(1)}) + e(\bar{u}_{0,5}^* \partial_z \phi_0^{(2)} + \partial_z \bar{u}_{0,5}^* \phi_0^{(2)}) \Big]\Big|_0$$

$$+ f \partial_z \bar{u}_{0,6}^*|_0 \zeta_0^2 \Big\} . \tag{63}$$

As discussed for Eq. (59), the problems are appearing beyond the first order of expansion. The form for the scalar product of $\langle \bar{U}_0, U_0 \rangle$ is the expected one (Eq. (62)). In the second order expansion, the left hand side of Eq. (63) lets await only products of functions which belong to the first and second order of expansion. But the expansion on the deformed surface generates additional terms, where each one is formed by three functions of the first order expansion. These additional terms can neither be assigned to $\langle \bar{U}_0, U_1 \rangle$ nor to $\langle \bar{U}_1, U_0 \rangle$ by simple arguments. Similar dilemmas are present for the third and any higher expansion order with many more not-assignable terms. Neglecting surface deformations is certainly not an option for the solution of this problem.

It becomes now apparent why the derivation of an amplitude equation in Refs. 26–28 with the approximation of a *plane* surface succeeded. The first order expansion on a *deformed* surface is equivalent to that approximation of a plane surface. Therefore the problems of not-assignable terms stemming from higher orders in the expansion do not occur.

## 5. Discussion

The detailed analysis of the last section showed that the definition of a linear adjoint operator for pure surface-nonlinearities has not yet been accomplished. Two attempts which led to unsatisfactory consequences were described here. They point to the fact that in the known literature no derivation can be traced in which an expansion on the deformed surface was taken into account *and* a linear adjoint operator was defined. The latter is necessary to determine higher order terms of the external parameter; for the system considered here these are $B_1$ and $B_2$, see Eq. (32). This fact is surprisingly since pattern forming systems with a deformed surface belong to the set of classical hydrodynamical systems.

For the nonlinear analysis of instabilities in MF, a very similar expansion route was followed in Refs. 19, 20 and 33. Remarkably, only very shortly the existence of an adjoint problem is mentioned in Refs. 20, 33. No details with respect to the definition of a scalar product and a linear adjoint operator were published. The

same expansion as in Eq. (31) is used, but the external parameter, the magnetic field $H$, is not expanded accordingly to Eq. (30). The latter step is a inconsistency since the multiple-scale expansion comes along with the expansion of the physical quantities *and* the external driving parameters as the Rayleigh number[22−24] or the Marangoni number.[25−27] The gain of the lacking expansion is that higher order terms of the external parameter have not to be determined. The solvability condition in second order postulated in Refs. 19 and 20 demands that the amplitude has to have a nonvanishing derivative everywhere with respect to the slow spatial variable. That means that the amplitude is a strictly monotonously increasing or decreasing function which is a very special type of solution. Selection problems between regular patterns can not be tackled with such a type of solution.

The problems caused by a deformed surface for the derivation of an amplitude equation are not unique to instabilities of magnetic fluids. The analysis of the Bénard–Marangoni convection or of parametric surface (Faraday) waves is confronted with similar difficulties. For the Bénard–Marangoni convection the surface deformation is often disregarded in fluid-gas systems[26,34] and two-fluid systems.[27] In Refs. 35 and 36 a surface deflection was imitated by a nonzero Crispation number, thus avoiding the expansion of any quantity on a deformed surface. In Ref. 25 the expansion of the layer thickness was tuned in such a way that eigenfunctions could be used which correspond to the case of an undeformed interface.

In the nonlinear analysis of Faraday waves an explicit evaluation of quantities on the free surface is rare. In Ref. 37 the driving parameter is not expanded, similar to Refs. 19, 20 and 33. No expansion on the surface is performed, but two different amplitudes are introduced for the solution of first and third order. The adjoint operator is circumvented in this way. The two different amplitudes are just the two quantities needed to attach use to the solvability conditions. Under the constrains of an inviscid fluid and an irrotational flow field, Milner[38] derived an amplitude equation for parametrically driven capillary waves. An expansion of the velocity potential on the surface was included, but an adjoint problem was not formulated.

The overall picture is that several circumventing routes were used in order to avoid the explicit evaluation of quantities on a deformed surface and the formulation of a linear adjoint operator. The used alternatives were hardly motivated which is why the real reasons behind the search for alternatives are not publicly known. Instead of using the solvability conditions to determine higher order terms of the external parameters, they were converted to constraining conditions for the amplitudes of the expanded physical quantities. This presents a rather unsatisfactory situation.

The deeper reasons for this unsatisfactory situation are disclosed in this article. The adjoint problem is presented in detail for nonlinearities purely triggered by a deformed surface as in the case of the Rosensweig instability. By expanding the physical quantities and the external parameter, the knowledge of the linear adjoint operator is essentially to proceed towards the nonlinear amplitude equation. The

main and still unsolved problem is the proper definition of the linear adjoint operator. Since this problem mounts a principal barrier and is not mentioned as a reason for the search of alternative approaches, it is presented here even if no solution can presently be offered. This situation entails one foremost question: Is it possible to apply the concept of Fredholms theorem to pure surface-nonlinearities? If yes, what is the adapted route for defining a linear adjoint operator. If not, what is a generic and mathematically proven alternative since the hitherto used routes are lacking these features. Therefore further analyses need to be done in order to solve this problem and to clarify the open questions.

If these problems for the Rosensweig instability, caused by the single external excitation of a magnetic field, are solved, other instabilities caused by different external excitations can be fruitful tackled. Phenomena as standing twin peaks[39] or domain structures[40] for parametrically excited MF under the influence of a magnetic field are designated future examples for an analysis by amplitude equations derived from the basic hydrodynamic equations.

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