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sir-Algebras and Galois Connections between Operations and Relations*

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Abstract

This paper contributes to the investigation of general relation algebras in connection with first order definable operations: sir-algebras can be characterized as algebras of relations closed with respect to set-theoretical operations definable by first order formulas without equality. As in the case of relational clones and Krasner algebras, sir-algebras are Galois closed sets with respect to a Galois connection – the strong invariance – between operations (here unary operations) and relations. They can internally be described also as Θ-extensions of Krasner algebras. The Galois closed sets of unary mappings are characterized as so-called Krasner extended monoids or, equivalently, as Θ-extensions of permutation groups.

Contents

Introduction

Systems of relations can be considered under various algebraic aspects. The algebraic structure of the set of binary relations (on a fixed set A) led to the classical notion of relation algebra (in the sense of Tarski [Tar 41], which goes even back to E. Schröder [Sch 10]).
In connection with algebras $\underline{A} = \langle A; (f_i)_{i\in I}\rangle$ also relations of other arities came into play, e.g. as subalgebras of direct powers of $\underline{A}$ or, equivalently, as so-called invariant relations $\varrho \in \mathit{Inv}\, F$ of the set $F = \{f_i \mid i \in I\}$ of fundamental operations. These invariant relations have an algebraic structure which depends on the kind of operations in F and can be expressed by closure properties with respect to certain formulas of the first order logic. In Table 1 below the first three cases list different notions of such general relational structures, indicate to which kind of operations they correspond, mention which kind of first order formulas are involved (here $\mathrm{Lop}_A(\exists, \wedge, \ldots)$ denotes the so-called logical operations corresponding to first order formulas in which only the indicated signs $\exists, \wedge, \ldots$ appear) and indicate the corresponding Galois connection. For details we refer to [Pös-K 79], [Bod-K-K-R 68] (and to the references given there; the cases (2) and (3) go back to M. Krasner; the base set A is assumed to be finite, for infinite A some modifications are neccessary).
In each case the corresponding operational (functional) systems were the starting point and the relation/Krasner algebras are characterizable as Galois closed sets with respect to some Galois connection (based on the property "function preserves relation"). Moreover, there are some other Galois connections not mentioned in Table 1 which restrict or extend the operations and/or relations under consideration (e.g. partial mappings [Gan-K-S 93], [Sus-K 94], correspondences [Bör 88], operations on relations [Bör-P 91], for more examples see [Pös 80]).

Once all this was established the way was open to investigate the situation from the logical (relational) point of view: Starting with a natural class of first order formulas investigate the algebraic structure of relational systems closed w.r.t. the corresponding logical operations and look for a possible Galois theory.
This will be done in the present paper for first order formulas without equality (case (4) in Table 1).
In *section 1* we introduce the notion of a sir-algebra[1] by closure properties with respect to some few set-theoretical operations on relations, and show that these are just the relation algebras closed under $\mathrm{Lop}_A(\exists, \wedge, \vee, \neg)$ (without equality sign). Structure properties of sir-algebras are described in *section 2*, in particular we show that every sir-algebra is a Θ-extension of some Krasner algebra. Finally, in *section 3*, we present a Galois connection such that the sir-algebras are just the Galois closed elements. Moreover, we characterize the corresponding Galois

[1] The name "sir-algebra" is derived from "algebra of **s**trongly **i**nvariant **r**elations". This will be explained in section 3.

	notion of the relation algebra	corresponding operational systems	closure w.r.t. logical operations of first order formulas	Galois connection
(1)	(general) relation algebra *relational clone*	clones of finitary functions	$\mathrm{Lop}_A(\exists, \wedge, =)$ (primitive positive formulas)	*Inv* – *Pol*
(2)	*weak Krasner algebra* (Krasner algebra of first kind)	monoids of unary functions	$\mathrm{Lop}_A(\exists, \wedge, \vee, =)$ (positive $\exists$-formulas)	*Inv* – *End*
(3)	*Krasner algebra* (Krasner algebra of second kind)	groups of permutations	$\mathrm{Lop}_A(\exists, \wedge, \vee, \neg, =)$ (all formulas)	*sInv* – *Aut*
This paper:				
(4)	*sir-algebra*	Krasner extended monoids of unary functions	$\mathrm{Lop}_A(\exists, \wedge, \vee, \neg)$ (all formulas without equality)	*sInv* – *sEnd*

Table 1: Several notions of relation algebras

closed monoids of unary mappings.
In this paper sir-algebras are introduced as heterogeneous (many-based) algebras in order to avoid difficulties in distinguishing different arities.

1 sir-algebras

1.1. Let $\mathbf{N} = \{0, 1, 2, \ldots\}$ denote the set of all natural numbers, and let $\mathbf{N}^+ := \mathbf{N} \setminus \{0\}$. For $m \in \mathbf{N}^+$ we put $\underline{m} := \{1, \ldots, m\}$. Throughout let A be a non-empty finite set. The set of all permutations on $\underline{m}$ is denoted by $\mathrm{Sym}(\underline{m})$. The set of all *$m$-ary relations on A* ($m \in \mathbf{N}^+$) is

$$R_A^{(m)} \ := \ \{\varrho \mid \varrho \subseteq A^m\} \ ,$$

and the set of all finitary relations on A is

$$R_A \ := \ \bigcup_{m \in \mathbf{N}^+} R_A^{(m)} \quad .$$

We distinguish between empty subsets of A^m for different $m \in N^+$. So we have different empty relations $\emptyset^{(m)} \subseteq A^m$ for every m. With this additional convention, every relation $\varrho \in R_A$ has a unique *arity* $\mathrm{ar}(\varrho) \in N^+$ with $\varrho \subseteq A^{\mathrm{ar}(\varrho)}$. For $Q \subseteq R_A$ we put $Q^{(m)} := Q \cap R_A^{(m)}$.

In this paper we consider *operations on relations* of the following form :

$$F \;:\; R_A^{(m_1)} \times \ldots \times R_A^{(m_k)} \;\to\; R_A^{(m)} \;,$$

where $k \in \mathbf{N}, m, m_1, \ldots m_k \in \mathbf{N}^+$. The number k is called the *arity of the operation* F. A set $Q \subseteq R_A$ of relations is said to be *closed under the operation* F , if $F(\varrho_1, \ldots, \varrho_k) \in Q$ whenever $\varrho_i \in Q^{(m_i)}$ for all $i = 1 \ldots k$.

1.2. We consider the following operations on relations.

(1) The *Boolean operations on* $R_A^{(m)}$:
$\cup^{(m)}$ (union), $\cap^{(m)}$ (intersection), $\complement^{(m)}$ (complement $\complement^{(m)}\varrho := A^m \setminus \varrho$). [2]

(2) *Permuting coordinates,* according to a permutation π on the set $\underline{m} = \{1, \ldots, m\}$:

$$\begin{aligned} \mathrm{W}_\pi \;:\; R_A^{(m)} &\to R_A^{(m)} \\ \varrho &\mapsto \{(a_1, \ldots, a_m) \mid (a_{\pi(1)}, \ldots, a_{\pi(m)}) \in \varrho\}. \end{aligned}$$

(3) *Deleting the last coordinate* (= projection onto the first m coordinates):

$$\begin{aligned} \mathrm{pr}^{(m)} : R_A^{(m+1)} &\to R_A^{(m)} \\ \varrho &\mapsto \{(a_1, \ldots, a_m) \mid \exists a_{m+1} \in A : (a_1, \ldots, a_m, a_{m+1}) \in \varrho\}. \end{aligned}$$

(4) *Adjoining a fictive coordinate*:

$$\begin{aligned} \nabla^{(m)} \;:\; R_A^{(m)} &\to R_A^{(m+1)} \\ \varrho &\mapsto \varrho \times A \\ &\quad = \{(a_1, \ldots, a_m, a_{m+1}) \in A^{m+1} \mid (a_1, \ldots, a_m) \in \varrho\}. \end{aligned}$$

1.3. Definition. The many-sorted algebra

$$\mathcal{R}_A^\diamond := \langle (R_A^{(m)})_{m \in N^+} \;;\; (\cup^{(m)}, \cap^{(m)}, \complement^{(m)},\; \mathrm{pr}^{(m)}, \nabla^{(m)},\, \mathrm{W}_\pi)_{m \in \mathbf{N}^+, \pi \in \mathrm{Sym}(\underline{m})} \rangle$$

is called the *full sir-algebra*[3] *on the set* A. Every non-empty subalgebra $\underline{R}$ of $\mathcal{R}_A^\diamond$ (notation: $\underline{R} \leq \mathcal{R}_A^\diamond$) is called a *sir-algebra on* A. For $Q \subseteq R_A$, we write $[Q]_{\mathrm{sir}}$ for the *sir-algebra, generated by* Q (more exactly: by the family $(Q^{(m)})_{m \in \mathbf{N}^+}$).

[2] For different arities m we have different Boolean operations. Thus we cannot form e.g. the union of a binary and a unary relation. But if the arity is fixed and clear from the context, then we simply write $\cup, \cap$ and $\complement$ to denote the Boolean operations.

[3] see footnote 1 on page 1

The more well-known notion of a Krasner algebra is very close to sir-algebras, namely by adding a single constant d_A:

1.4. Definition. Let

$$d_A := \{(a,a) \;\mid a \in A\}$$

be the equality relation. The many-sorted algebra

$$\mathcal{K}_A^{\diamond} := \langle (R_A^{(m)})_{m\in N^+} \;;\; (\cup^{(m)}, \cap^{(m)}, \complement^{(m)},\; \mathrm{pr}^{(m)}, \nabla^{(m)},\; \mathrm{W}_\pi)_{m\in\mathbf{N}^+,\pi\in\mathrm{Sym}(\underline{m})}, d_A\rangle$$

is called the *full Krasner algebra on the set* A, and a nonempty subalgebra $\underline{Q}$ of $\mathcal{K}_A^{\diamond}$ is called a *Krasner algebra on* A (notation: $\underline{Q} \leq \mathcal{K}_A^{\diamond}$). For $Q \subseteq R_A$ let $[\overline{Q}]_{\mathrm{KA}}$ denote the *Krasner algebra, generated by* Q.

1.5. Remark. To every nonempty set $Q \subseteq R_A$ of relations we can assign the family $\underline{Q} := (Q^{(m)})_{m\in\mathbf{N}^+}$. Then $\underline{Q}$ is (the carrier family of) a sir-algebra, if and only if the set Q is closed under all Boolean operations, the operations W_π for all $\pi \in \mathrm{Sym}(\underline{m})$, and under the operations $\mathrm{pr}^{(m)}, \nabla^{(m)}$ for all $m \in \mathbf{N}^+$. And if, in addition, $d_A \in Q^{(2)}$, then $\underline{Q}$ is (the carrier of) a Krasner algebra.

There is a bijective connection between sets Q of relations and families $\underline{Q}$. Therefore we do not distinguish sharply between them. So we call a nonempty set $Q \subseteq R_A$ a sir-algebra and write $Q \leq \mathcal{R}_A^{\diamond}$, if $\underline{Q}$ is a sir-algebra. (Similar for Krasner algebras.)

This, in principle, shows that sir-algebras and Krasner algebras also could be defined via algebras with a single base set R_A (homogeneous approach). Concerning Krasner algebras (called there “Krasner algebras of second kind”) we refer to [Pös-K 79, p. 43].

1.6. Examples.

a) The set $E_A := \{\emptyset^{(m)} \mid m \in \mathbf{N}^+\} \cup \{A^m \mid m \in \mathbf{N}^+\}$ is closed under all the operations in 1.2. Consequently it is a sir-algebra.

   Obviously, E_A is contained in every sir-algebra, i.e. E_A is the least sir-algebra, and $E_A = [\emptyset]_{\mathrm{sir}}$. We call E_A the *trivial sir-algebra on* A.

b) The sir-algebra $D_A := [\{d_A\}]_{\mathrm{sir}}$ generated by the equality relation d_A contains all relations of the form

$$\delta_\varepsilon = \{(a_1,\ldots,a_m) \mid \forall\, i,j \in \underline{m}:\; (i,j)\in\varepsilon \Leftrightarrow a_i = a_j\}\ ,$$

   where ε is an equivalence relation on $\underline{m} := \{1,\ldots,m\}$, $m \in \mathbf{N}^+$. Every relation in D_A is empty or a union of relations of this form. Moreover D_A is a Krasner algebra, the least Krasner algebra on A.

1.7. The above operations 1.2 (1) - (4) generate (by composition) many other operations and, vice versa, can be obtained from other ones by composition. Therefore sir-algebras can be defined in various ways. We mention here some further operations and shall investigate the interdependence of all these operations in the next Lemmata[4].

(5) *Identification of (the last two) coordinates*:

$$\begin{aligned} \Delta^{(m)} : R_A^{(m+1)} &\to R_A^{(m)} \\ \varrho &\mapsto \{(a_1,\dots,a_m)\in A^m \mid (a_1,\dots,a_{m-1},a_m,a_m)\in\varrho\}\ . \end{aligned}$$

(6) *Transformation* W_s *of coordinates* with respect to an arbitrary function $s:\{1,\dots,n\}\to\{1,\dots,m\}$ $(n,m\in\mathbf{N}^+)$:

$$\begin{aligned} \mathrm{W}_s\ :\ R_A^{(n)} &\to R_A^{(m)} \\ \varrho &\mapsto \{(a_1,\dots,a_m)\ \mid (a_{s(1)},\dots,a_{s(n)})\in\varrho\}\ . \end{aligned}$$

Note: If $s=\pi$ is a permutation, then obviously $\mathrm{W}_s=\mathrm{W}_\pi$.

1.8. Lemma. *The operation* $\Delta^{(m)}$ *is a composition of the operations in* (1)–(4). *More exactly, let* $m\in\mathbf{N}^+$ *, and let* $\pi_1,\pi_2\ :\underline{m+2}\to\underline{m+2}$ *be the permutations with* $\pi_i(j)=j$ *for all* $j<m$ $(i=1,2)$ *and* $\pi_1(m)=m+2$, $\pi_1(m+1)=m+1$, $\pi_1(m+2)=m$, $\pi_2(m)=m+2$, $\pi_2(m+1)=m$, $\pi_2(m+2)=m+1$. *Then for all* $\varrho\in R_A^{(m+1)}$ *we have:*

$$\Delta^{(m)}(\varrho)=\mathrm{pr}^{(m)}(\varrho\ \cap\ \complement\,\mathrm{pr}^{(m+1)}(\mathrm{W}_{\pi_1}\nabla^{(m+1)}(\varrho)\ \cap\ \complement\,\mathrm{W}_{\pi_2}\nabla^{(m+1)}(\varrho)))\,.$$

PROOF. Let us abbreviate $\vec{a}:=a_1,\dots,a_{m-1}$. Then an application of our definitions in (1)–(4) shows:

$$\begin{aligned} &\mathrm{pr}^{(m)}(\varrho\ \cap\ \complement\,\mathrm{pr}^{(m+1)}(\mathrm{W}_{\pi_1}\nabla^{(m+1)}(\varrho)\ \cap\ \complement\,\mathrm{W}_{\pi_2}\nabla^{(m+1)}(\varrho))) \\ &=\ \{(\vec{a},a_m)\ \mid(\exists a_{m+1})((\vec{a},a_m,a_{m+1})\in\varrho\wedge \\ &\qquad(\forall a_{m+2})((\vec{a},a_{m+2},a_{m+1})\in\varrho\to(\vec{a},a_{m+2},a_m)\in\varrho))\} \end{aligned}$$

If $(\vec{a},a_m)\in\Delta^{(m)}(\varrho)$, i.e. if $(\vec{a},a_m,a_m)\in\varrho$, then $(\vec{a},a_m)$ fulfills this condition (with $a_{m+1}=a_m$), i.e. belongs to this set.
Vice versa, if $(\vec{a},a_m)$ fulfills the condition on the right side, then there is an $a_{m+1}\in A$ such that $(\vec{a},a_m,a_{m+1})\in\varrho$. Moreover, for $a_{m+2}=a_m$ the second condition yields $(\vec{a},a_m,a_{m+1})\in\varrho\to(\vec{a},a_m,a_m)\in\varrho$, therefore $(\vec{a},a_m,a_m)\in\varrho$.
Consequently our formula is equivalent with $(\vec{a},a_m,a_m)\in\varrho$, and this is the defining formula for $\Delta^{(m)}$. □

[4]Lemma 1.8 was unexpected and a surprise for the authors. For a relatively long time we tried to prove the contrary, namely that identification of variables (5) is independent of the operations (1) - (4).

The operations $\nabla^{(m)}, \Delta^{(m)}$ and W_π are special cases of W_s. We have:

1.9. Lemma.

(i) *Let* $u : \{1, \ldots, m\} \to \{1, \ldots, m+1\}$ *be the function with* $u(j) = j$ *for all* $j \leq m$. *Then* $\mathrm{W}_u = \nabla^{(m)}$.

(ii) *Let* $v : \{1, \ldots, m+1\} \to \{1, \ldots, m\}$ *be the function with* $v(j) = j$ *for all* $j \leq m$ *and* $v(m+1) = m$. *Then* $\mathrm{W}_v = \Delta^{(m)}$.

(iii) $\mathrm{W}_{\mathrm{id}_{\underline{m}}} = \mathrm{id}_{R_A^{(m)}}$ *for all* $m \in \mathbf{N}^+$.

(iv) *Let* $s : \underline{n} \to \underline{m}$ *and* $t : \underline{m} \to \underline{k}$. *Then* $\mathrm{W}_{t \circ s} = \mathrm{W}_t \circ \mathrm{W}_s$.

(v) *For arbitrary* s*, the operation* W_s *is a composition of operations of the form* $\nabla^{(m)}, \Delta^{(m)}$ *and* W_π *(*π *a permutation).*

PROOF. (i)-(iv) are easy consequences of the definitions. Concerning (v), every operation $s : \underline{n} \to \underline{k}$ is a composition of some permutations and of some functions of the form as u and v in (i) and (ii). Thus it is a consequence of (iv), that W_s is a composition of the corresponding operations $\mathrm{W}_\pi, \nabla^{(m)}, \Delta^{(m)}$. □

1.10. Logical Operations. First order formulas (briefly formulas) are convenient tools for the description of operations on relations. Moreover, to every formula φ we can assign an operation F_φ on relations.
So let $m \in \mathbf{N}^+, k \in \mathbf{N}$, and let $\varphi = \varphi(P_1, \ldots, P_k; x_1, \ldots, x_m)$ be a first order formula, where all free variables are among $x_1, \ldots, x_m$ and where all occuring predicate symbols are among $P_1, \ldots, P_k$. Furthermore let us assume, that the arity of P_i is $m_i \in \mathbf{N}^+$ for $i = 1 \ldots k$. Then we assign to φ the *logical operation* F_φ on R_A as follows:

$$\begin{array}{rcl} F_\varphi \;:\; R_A^{(m_1)} \times \ldots \times R_A^{(m_k)} & \to & R_A^{(m)} \\ (\varrho_1, \ldots, \varrho_k) & \mapsto & \{(a_1, \ldots, a_m) \in A^m \;\mid \varphi(\varrho_1, \ldots, \varrho_k; a_1, \ldots, a_m)\} \end{array}$$

(Here $\varphi(\varrho_1, \ldots, \varrho_k; a_1, \ldots, a_m)$ means that the formula φ with evaluation $x_i := a_i$ holds in the model $\langle A\,;\, \varrho_1, \ldots, \varrho_k\rangle$. For more information we refer to [Pös-K 79, 2.1].)
The set of all logical operations F_φ on R_A with first order formulas φ *without* identity is denoted with the symbol $\mathrm{Lop}_A(\exists, \wedge, \vee, \neg)$, while the set of all operations F_φ with arbitrary first order formulas φ is denoted by $\mathrm{Lop}_A(\exists, \wedge, \vee, \neg, =)$. Compositions of logical operations from $\mathrm{Lop}_A(\exists, \wedge, \vee, \neg)$ are again in the set $\mathrm{Lop}_A(\exists, \wedge, \vee, \neg)$, and the same holds for $\mathrm{Lop}_A(\exists, \wedge, \vee, \neg, =)$. Furthermore, if the formulas $\varphi = \varphi(P_1, \ldots, P_k; x_1, \ldots, x_m)$ and $\psi = \psi(P_1, \ldots, P_k; x_1, \ldots, x_m)$ are equivalent, then $F_\varphi = F_\psi$.

1.11. It can be easily seen that the operations (1) - (6) (cf. 1.2,1.7) are logical operations from the set $\mathrm{Lop}_A(\exists, \wedge, \vee, \neg)$. Therefore arbitrary compositions of these operations are again in $\mathrm{Lop}_A(\exists, \wedge, \vee, \neg)$. We want to show that, conversely, all operations in $\mathrm{Lop}_A(\exists, \wedge, \vee, \neg)$ can be obtained as compositions of the Boolean operations and the operations $\mathrm{pr}^{(m)}, \nabla^{(m)}, \mathrm{W}_\pi$.
The Lemmata 1.8 and 1.9 show that the operations $\Delta^{(m)}$ and W_s are compositions of $\mathrm{pr}^{(m)}, \nabla^{(m)}, \mathrm{W}_\pi$ and the Boolean operations. Using these operations, we can give a proof by induction on the structure of the formulas of the predicate calculus (without identity).
Every formula of the predicate calculus without identity is equivalent to a formula φ which contains only the logical connectives $\exists, \wedge, \vee$ and $\neg$. We start with formulas $\varphi(x_1, \ldots, x_m)$ of the form $\varphi = P(x_{s(1)}, \ldots, x_{s(n)})$, where P is an n–ary predicate symbol and $s : \underline{n} \to \underline{m}$ is a function. Obviously we have $F_\varphi = \mathrm{W}_s$.
For every logical construction with $\wedge, \vee, \neg, \exists$ we can find a corresponding composition of the operations $\cap, \cup, \complement, \mathrm{pr}$. We do not go into details here, we only mention that we need the additional help of the operations ∇ and W_π to bring our relations in a suitable form.
Consequently all operations in $\mathrm{Lop}_A(\exists, \wedge, \vee, \neg)$ can be obtained from $\mathrm{pr}, \nabla, \mathrm{W}_\pi$ and the Boolean operations. To receive the operations in the set $\mathrm{Lop}_A(\exists, \wedge, \vee, \neg, =)$ it is only necessary to add a special constant relation – the binary *equality relation* d_A on A (cf. 1.4). This relation corresponds to the equality sign.

We summarize our results in the next Lemma.

1.12. Lemma. *All the operations defined in* (1) - (6) *are logical operations in* $\mathrm{Lop}_A(\exists, \wedge, \vee, \neg)$. *Vice versa, the operations* W_π *(π a permutation),* $\nabla^{(m)}$, $\mathrm{pr}^{(m)}$, $\cap^{(m)}$, $\cup^{(m)}$, $\complement^{(m)}$ *(with $m \in \mathbf{N}^+$) generate all the operations in* $\mathrm{Lop}_A(\exists, \wedge, \vee, \neg)$. *Together with the constant binary relation d_A they generate all operations in* $\mathrm{Lop}_A(\exists, \wedge, \vee, \neg, =)$.

As a consequence of the Definition 1.3 and Lemma 1.12 we have:

1.13. Corollary. *Let $Q \subseteq R_A$ be a nonempty set. Then Q is a sir-algebra iff Q is closed under all operations in* $\mathrm{Lop}_A(\exists, \wedge, \vee, \neg)$. *$Q$ is a Krasner algebra iff it is closed under all operations in* $\mathrm{Lop}_A(\exists, \wedge, \vee, \neg, =)$.

2 Structure properties of sir-algebras

2.1. If Q is a sir-algebra on A , then the set $Q^{(m)}$ (for $m \in \mathbf{N}^+$) is nonempty and closed under the Boolean operations. Therefore it contains the constants $\emptyset^{(m)}$ and A^m , and

$$\underline{Q}^{(m)} := \langle Q^{(m)}\,;\, \emptyset^{(m)}, A^m, \cap, \cup, \complement\,\rangle$$

is a Boolean algebra (BA).

The set A is assumed to be finite, therefore $Q^{(m)}$ is finite, and the Boolean algebra $\underline{Q}^{(m)}$ is atomic. With $\mathrm{At}\underline{Q}^{(m)}$ we denote the *set of all of its atoms.* Obviously this set is a partition of A^m . If $\mathrm{At}\underline{Q}^{(m)} = \{\beta_1, \ldots, \beta_q\}$, then for every relation $\varrho \in Q^{(m)}$ there is a unique partition $\varrho = \bigcup\limits_{i\in I} \beta_i$ into some atoms. Consequently $|Q^{(m)}| = 2^q$.

The set $Q^{(m)}$ is also closed under the operations W_π with $\pi \in \mathrm{Sym}(\underline{m})$. So we can consider every sir-algebra $\underline{Q}$ as a chain of Boolean algebras $\underline{Q}^{(m)}$, which are closed under W_π and connected by the operations $\mathrm{pr}^{(m)}, \nabla^{(m)}$. In the following we want to describe some easy properties of these connections.

2.2. Lemma. *Let $Q \leq \mathcal{R}_A^\diamond$, and let $s : \underline{m} \to \underline{n}$ be a function. Then $\mathrm{W}_s : Q^{(m)} \to Q^{(n)}$ is a BA–Homomorphism. If s is injective, then W_s is an embedding. If s is surjective, then W_s is also surjective.*

The operations $\Delta^{(m)}, \nabla^{(m)}$ and W_π ($\pi \in \mathrm{Sym}(\underline{m})$) are BA–Homomorphisms. W_π is bijective, $\nabla^{(m)}$ is injective, and $\Delta^{(m)}$ is surjective.

Proof. The definition of W_s in 1.7 shows that W_s is compatible with all Boolean operations. If s is injective, then there is a function $t : \underline{n} \to \underline{m}$ with $t \circ s = \mathrm{id}_{\underline{m}}$. Consequently by 1.9(iii) and (iv) we have

$$\mathrm{id}_{R_A^{(m)}} = \mathrm{W}_{\mathrm{id}_{\underline{m}}} = \mathrm{W}_{t\circ s} = \mathrm{W}_t \circ \mathrm{W}_s \ .$$

Therefore W_s has to be injective.
If s is surjective, then there is an t with $s \circ t = \mathrm{id}_{\underline{n}}$, consequently $\mathrm{W}_s \circ \mathrm{W}_t = \mathrm{id}_{R_A^{(n)}}$ (this follows from 1.9(iii) and (iv)), i.e. W_s has to be surjective. The last claim is a corollary of 1.9(i) and (ii). $\square$

2.3. Lemma. *Let $Q \leq \mathcal{R}_A^\diamond$. For all $m \in \mathbf{N}^+$, $\alpha, \beta \in R_A^{(m+1)}, \varrho, \sigma \in R_A^{(m)}$ the following hold:*

$$\begin{array}{rcll} \mathrm{pr}^{(m)}\emptyset^{(m+1)} &=& \emptyset^{(m)} & \textit{and} \quad \mathrm{pr}^{(m)}A^{m+1} \;=\; A^m \\ \mathrm{pr}^{(m)}(\alpha \cup \beta) &=& \mathrm{pr}^{(m)}\alpha \cup \mathrm{pr}^{(m)}\beta & \textit{and} \quad \mathrm{pr}^{(m)}\complement\,\alpha \;\geq\; \complement\,\mathrm{pr}^{(m)}\alpha \\ \mathrm{pr}^{(m)}\nabla^{(m)}\varrho &=& \varrho & \textit{and} \quad \alpha \;\leq\; \nabla^{(m)}\mathrm{pr}^{(m)}\alpha \end{array}$$

$\mathrm{pr}^{(m)} : Q^{(m+1)} \to Q^{(m)}$ is surjective. Furthermore $\mathrm{pr}^{(m)}$ maps $\mathrm{At}\underline{Q}^{(m+1)}$ onto $\mathrm{At}\underline{Q}^{(m)}$.

PROOF. All the identities are direct consequences of the definitions of $\mathrm{pr}^{(m)}$ and $\nabla^{(m)}$ in 1.2. The equation $\mathrm{pr}^{(m)}\nabla^{(m)}\varrho = \varrho$ shows, that $\mathrm{pr}^{(m)}$ is surjective. And the connection $\alpha \leq \nabla^{(m)}\mathrm{pr}^{(m)}\alpha$ shows that $\alpha \neq \emptyset$ always implies $\mathrm{pr}^{(m)}\alpha \neq \emptyset$.
If ϱ is an atom in $Q^{(m)}$, then $\mathrm{pr}^{(m)}\nabla^{(m)}\varrho = \varrho$. This implies $\varrho = \mathrm{pr}^{(m)}\beta$ for every atom $\beta \leq \nabla^{(m)}\varrho$.
Now let $\beta \in \mathrm{At}Q^{(m+1)}$. Then $\mathrm{pr}^{(m)}\beta \neq \emptyset$. Lets assume that $\mathrm{pr}^{(m)}\beta$ is *not* an atom. Then there are $\sigma, \varrho \in Q^{(m)}$ with $\sigma, \varrho \neq \emptyset$, $\sigma \cap \varrho = \emptyset$ and $\mathrm{pr}^{(m)}\beta = \sigma \cup \varrho$. This implies $\nabla^{(m)}\sigma \cap \nabla^{(m)}\varrho = \emptyset$ and $\nabla^{(m)}\sigma \cup \nabla^{(m)}\varrho = \nabla^{(m)}\mathrm{pr}^{(m)}\beta \geq \beta$. But β is an atom, so either $\beta \subseteq \nabla^{(m)}\sigma$ or $\beta \subseteq \nabla^{(m)}\varrho$. W.l.o.g. lets assume $\beta \subseteq \nabla^{(m)}\sigma$. Then $\mathrm{pr}^{(m)}\beta \subseteq \mathrm{pr}^{(m)}\nabla^{(m)}\sigma = \sigma$, a contradiction. □

2.4. By 2.3, the projection $\mathrm{pr}^{(m)}$ maps $\mathrm{At}\underline{Q}^{(m+1)}$ onto $\mathrm{At}\underline{Q}^{(m)}$. Consequently the Boolean Algebra $\underline{Q}^{(m+1)}$ determines the algebra $\underline{Q}^{(m)}$. Furthermore by 2.2 the operation W_π ($\pi \in \mathrm{Sym}(\underline{m})$) is a BA-isomorphism. Consequently W_π permutes the elements of $\mathrm{At}\underline{Q}^{(m)}$. It turns out (see 2.6 below), that these properties are enough to characterize the atom systems of sir-algebras.

2.5. Definition. A system $(\Sigma_m)_{m\in\mathbf{N}^+}$ of partitions Σ_m of A^m is called *compatible* (on A) if it satisfies the following properties:

(i) Every Σ_m is $\mathrm{Sym}(\underline{m})$–stable ($\mathrm{W}_\pi[\Sigma_m] = \Sigma_m$),
i.e. $\forall \pi \in \mathrm{Sym}(\underline{m})\ \forall \varrho \in \Sigma_m\ :\ \mathrm{W}_\pi(\varrho) \in \Sigma_m$

(ii) $\forall\, m \in \mathbf{N}^+\ \forall \varrho \in \Sigma_{m+1}\ :\ pr^{(m)}\varrho \in \Sigma_m$ (i.e. $\mathrm{pr}^{(m)}$ maps Σ_{m+1} into Σ_m)

(iii) $\forall m \in \mathbf{N}^+\ \forall \varrho \in \Sigma_m$: $\nabla^{(m)}\varrho$ is a union of elements of Σ_{m+1}.

2.6. Theorem. *There is a one-to-one correspondence between sir-algebras on A and compatible systems $(\Sigma_m)_{m\in\mathbf{N}^+}$ on A:*

a) *For a sir-algebra $\underline{Q} \leq \mathcal{R}_A^\diamond$, the system $(At\,\underline{Q}^{(m)})_{m\in\mathbf{N}^+}$ is compatible.*

b) *For a compatible system $(\Sigma_m)_{m\in\mathbf{N}^+}$, the algebra $\underline{Q} := (\underline{Q}^{(m)})_{m\in\mathbf{N}^+} \leq \mathcal{R}_A^\diamond$ is a sir-algebra, where $\underline{Q}^{(m)}$ is the Boolean algebra generated by Σ_m, and we have $At\,\underline{Q}^{(m)} = \Sigma_m$.*

PROOF. a) follows from 2.1 – 2.4.
b): since Σ_m is a partition, $Q^{(m)}$ consists of all unions of elements of Σ_m : $Q^{(m)} = \bigcup\limits_{\mathcal{B}\in\Sigma_m} \mathcal{B}$. All properties of a sir-algebra can easily be checked. □

2.7. There is a simple construction of large sir-algebras from smaller ones. Let Θ be an equivalence relation on A and $L := A/\Theta = \{[a]_\Theta \mid a \in L\}$ the corresponding set of all equivalence classes of A . Now, every $\varrho \in R_L^{(m)}$ can be extended to a relation $\varrho^\Theta \in R_A^{(m)}$ as follows:

$$\varrho^\Theta := \{(a_1, \ldots, a_m) \in A^m \ \mid\ ([a_1]_\Theta, \ldots, [a_m]_\Theta) \in \varrho\}$$

For $Q \subseteq R_L$ we put

$$Q^\Theta := \{\varrho^\Theta \mid \varrho \in Q\}$$

and call it the Θ-*extension of* Q .

2.8. Lemma.

(a) *If* $Q \leq \mathcal{R}_L^\diamond$ *then* Q^Θ *is again a sir-algebra, i.e.* $Q^\Theta \leq \mathcal{R}_A^\diamond$.

(b) *If* $Q \leq \mathcal{R}_A^\diamond$, *then* $\widehat{Q} \leq \mathcal{R}_L^\diamond$ *where* $\widehat{Q} := \{\varrho \in R_L \mid \varrho^\Theta \in Q\}$.

Proof. We have $\varrho^\Theta \cup \sigma^\Theta = (\varrho \cup \sigma)^\Theta$ (cf. definition). Further one can check $\mathsf{C}\,(\varrho^\Theta) = (\mathsf{C}\,\varrho)^\Theta$, $\mathrm{pr}\,\varrho^\Theta = (\mathrm{pr}\,\varrho)^\Theta$, $\mathrm{W}_\pi(\varrho^\Theta) = (\mathrm{W}_\pi\varrho)^\Theta$, $\nabla(\varrho^\Theta) = (\nabla\varrho)^\Theta$, □

2.9. Remark. If Q is a Krasner algebra, i.e. if $Q \leq \mathcal{R}_L^\diamond$ contains the diagonal d_L, and $\Theta \neq d_L$, then Q^Θ is not a Krasner algebra.

The following Theorem will be crucial for the characterization of the Galois closures of the Galois connection *sInv* – *sEnd* . We use the notation of 2.7 .

2.10. Theorem. *Every sir-algebra is a* Θ*-extension of some Krasner algebra. More concrete, let* $Q \leq \mathcal{R}_A^\diamond$ *and let* Θ *be the least equivalence relation in* $Q^{(2)}$ *and* $L := A/\Theta$. *Then*

$$\widehat{Q} := \{\varrho \in R_L \mid \varrho^\Theta \in Q\}$$

is a Krasner algebra and $\widehat{Q}^\Theta = Q$.
Remark: The least Θ exists since $A^2 \in Q$ is an equivalence relation and $Q^{(2)}$ is closed under intersections.

Proof. By 2.8, $\widehat{Q}$ is an sir-algebra. Moreover, for the diagonal d_L we have

$$d_L^\Theta = \{(a,b) \in A^2 \mid ([a]_\Theta, [b]_\Theta) \in d_L\} = \Theta \in Q,$$

thus $d_L \in \widehat{Q}$ and $\widehat{Q}$ is a Krasner algebra.

It remains to prove that every $\sigma \in Q$ is of the form ϱ^Θ for some $\varrho \in \widehat{Q}$. Fix $\sigma \in Q^{(m)}$, $(m \geq 1)$ and let

$$\varrho := [\sigma]_\Theta := \{([a_1]_\Theta, \ldots, [a_m]_\theta) \ \mid \ (a_1, \ldots, a_m) \in \sigma\}$$

Obviously $\sigma \subseteq \varrho^\Theta$ (cf. 2.7). In order to prove the converse inclusion, let $(b_1, \ldots, b_m) \in \varrho^\Theta$. Thus $([b_1]_\Theta, \ldots, [b_m]_\Theta) \in \varrho$ (by definition) and there exists $(a_1, \ldots, a_m) \in \sigma$ with $a_i \Theta b_i$, $i = 1, \ldots, m$.
Now we are going to show $(b_1, \ldots, b_j, a_{j+1}, \ldots a_m) \in \sigma$ successively for $j = 0, 1, \ldots, m$. The case $j = 0$ is clear: $(a_1, \ldots, a_m) \in \sigma$.
Now we assume $(b_1, \ldots, b_j, a_{j+1}, \ldots, a_m) \in \sigma$ $(j \in \{0, 1, \ldots, m-1\})$ and define

$$\begin{aligned}\Psi := \{(a,b) \in A^2 \mid \forall b_1, \ldots, b_j, a_{j+2}, \ldots, a_m :\\ (b_1, \ldots, b_j, b, a_{j+2}, \ldots, a_m) \in \sigma \iff (b_1, \ldots, b_j, a, a_{j+2}, \ldots, a_m) \in \sigma\}.\end{aligned}$$

Obviously this is an equivalence relation on A . Furthermore it is defined by a formula of the predicate calculus without identity. Consequently (1.13) $\Psi \in Q$ and therefore $\Theta \subseteq \Psi$. In particular $(a_{j+1}, b_{j+1}) \in \Theta \subseteq \Psi$, and therefore $(b_1, \ldots, b_j, b_{j+1}, a_{j+2}, \ldots, a_m) \in \sigma$ since $(b_1, \ldots, b_j, a_{j+1}, a_{j+2}, \ldots, a_m) \in \sigma$. Thus, for $j = m$, we always have $(b_1, \ldots, b_m) \in \sigma$, i.e. $\varrho^\Theta \subseteq \sigma$. □

3 The Galois connection *sInv* – *sEnd*

3.1. As before let A be finite, and let T_A be the *full transformation monoid* of all unary mappings $h : A \to A$. The identity mapping id_A is the unit element w.r.t. composition. We write $H \leq T_A$ if H is a submonoid of T_A (i.e. $\mathrm{id}_A \in H$ and H is closed under the composition of mappings). Let $\langle H \rangle_{T_A}$ denote the monoid generated by $H \subseteq T_A$.
For $h \in T_A$, $\vec{a} = (a_1, \ldots, a_m) \in A^m$ and $\varrho \subseteq A^m$, we use the notations

$$\begin{aligned} h \circ \vec{a} &:= (h(a_1), \ldots, h(a_m)), \\ h \star \varrho &:= \{h \circ \vec{a} \mid \vec{a} \in \varrho\}. \end{aligned}$$

As usual, we call $h \in T_A$ an *endomorphism* of a relation $\varrho \in R_A^{(m)}$ (and ϱ an *invariant relation* for h) if $h \star \varrho \subseteq \varrho$. For $Q \subseteq R_A$, $H \subseteq T_A$ we denote:

$$\begin{aligned} \mathit{End}\, Q &= \mathit{End}_A\, Q &:= \{h \in T_A \mid h \text{ preserves every } \varrho \in Q\} \\ \mathit{Inv}\, H &= \mathit{Inv}_A\, H &:= \{\varrho \in R_A \mid \varrho \text{ is invariant for every } h \in H\} \end{aligned}$$

Now we say, h is a *strong endomorphism* (or h *strongly preserves* ϱ, or ϱ is a *strongly invariant relation* for h), if h preserves both, ϱ and $\complement \varrho$. This is equivalent to

$$\forall \vec{a} \in A^m : \vec{a} \in \varrho \iff h \circ \vec{a} \in \varrho \, .$$

For $Q \subseteq R_A$ let

$$\mathit{sEnd}\, Q := \mathit{sEnd}_A\, Q := \{h \in T_A \mid h \text{ strongly preserves every } \varrho \in Q\}$$

be the set of all strong endomorphisms of Q. A permutation $g \in \mathrm{Sym}(A)$ which is a strong endomorphism of a relation ϱ is called an *automorphism* of ϱ:

$$\mathit{Aut}\, Q := \mathit{Aut}_A\, Q := \mathrm{Sym}(A) \cap \mathit{sEnd}\, Q$$

For $H \subseteq T_A$ let

$$\mathit{sInv}\, H := \mathit{sInv}_A\, H := \{\varrho \in R_A \mid \forall h \in H : h \text{ strongly preserves } \varrho\}$$

denote the set of all *strongly invariant* relations for H .

3.2. Remark. For $H \subseteq T_A$, two m-tuples $\vec{a}, \vec{b} \in A^m$ are called *H-connected* (notation $\vec{a} \sim_H \vec{b}$), if there exist $n \in \mathbf{N}$, $h_1, \dots, h_n$, $g_1, \dots, g_n \in H \cup \{\mathrm{id}_A\}$ and $\vec{a}_0, \vec{b}_0, \vec{a}_1, \vec{b}_1, \dots, \vec{a}_n, \vec{b}_n \in A^m$ such that

$$\vec{a} = \vec{a}_0, \quad h_{i+1} \circ \vec{a}_i = \vec{b}_i = g_{i+1} \circ \vec{a}_{i+1} \ (i \in \{0, 1, \cdots, n-1\}), \quad \vec{b}_n = \vec{b}.$$

i.e. we have the situation

$$\vec{a} = \vec{a}_0 \xrightarrow{h_1} \vec{b}_0 \xleftarrow{g_1} \vec{a}_1 \xrightarrow{h_2} \vec{b}_1 \xleftarrow{g_2} \vec{a}_2 \longrightarrow \cdots \longleftarrow \vec{b}_{n-1} \longrightarrow \vec{a}_n \xleftarrow{g_n} \vec{b}_n = \vec{b}.$$

H-connectedness $\sim_H$ is an equivalence relation on A^m. A relation $\varrho \in R_A^{(m)}$ is called *H-connected* if it is an equivalence class of $\sim_H$.

Then we have the following
Proposition. *A relation ϱ is strongly H-invariant iff ϱ is a union of H-connected relations, i.e. $\vec{a} \in \varrho \wedge \vec{a} \sim_H \vec{b} \Longrightarrow \vec{b} \in \varrho$.*
For the PROOF notice that $h \in End\{\varrho, \complement\, \varrho\}$ iff $\forall\, \vec{a} \in A^m : \ \vec{a} \in \varrho \iff h \circ \vec{a} \in \varrho$. $\square$

3.3. Any binary relation ($\subseteq T_A \times R_A$) and therefore also the relation

" h strongly preserves ϱ "

induces a Galois connection between T_A and R_A, namely

$$\begin{aligned} sInv &: \mathcal{P}(T_A) \to \mathcal{P}(R_A) : H \mapsto sInv\, H \\ sEnd &: \mathcal{P}(R_A) \to \mathcal{P}(T_A) : Q \mapsto sEnd\, Q. \end{aligned}$$

The Galois closed elements here are the sets of the form $H = sEnd\, Q_0$ and $Q = sInv\, H_0$ ($Q_0 \subseteq R_A$, $H_0 \subseteq T_A$). They always satisfy $H = sEnd\, sInv\, H$ and $Q = sInv\, sEnd\, Q$. Our aim is to find a characterization of these sets.

3.4. Remarks. (A) Two other Galois connections are well known, namely the connection *Inv – End* between T_A and R_A and the connection *sInv – Aut* between $\mathrm{Sym}(A)$ and R_A (cf. e.g. [Pös-K 79],[Pös 80]. We mention some facts: If h is a permutation on a finite set A, then h preserves ϱ iff h strongly preserves ϱ. Thus, for $G \subseteq S_A$, we have

$$sInv\, G = Inv\, G = \{\varrho \in R_A \mid \forall g \in G : \ g \text{ preserves } \varrho\}.$$

and therefore also

$$Inv\ Aut\ Q = sInv Aut Q.$$

Sets of the form $Inv\, G$ are known to be a Krasner algebras. Conversely, if $sInv\, H$ is a Krasner algebra for some $H \subseteq T_A$, then $\complement\, d_A \in Inv\, H$ (note $\complement\, d_A = \{(x, y) \mid x \neq y\}$ is the inequality relation), and

$$H \subseteq End\ Inv\ H \subseteq End\, \{\complement\, d_A\} = \mathrm{Sym}(A)$$

shows that H must consist of permutations only. Thus, for $H \subseteq T_A$,

$$sInv\, H \text{ is a Krasner algebra} \iff H \subseteq \mathrm{Sym}(A).$$

Consequently, permutation groups G and Krasner algebras Q are just the Galois closed sets for the Galois connection $sInv - Aut$ (i.e. $G = Aut\, sInv\, G$, $Q = sInv\, Aut\, Q$) as well as for $Inv - Aut$.

(B) Note that every $\varrho \in R_A$ is strongly id_A-invariant and that for every $\varrho \in R_A \setminus E_A$ there exists an $h \in T_A$ (e.g. any constant function) which does not strongly preserve ϱ. Consequently, we get

$$sInv\, \{\mathrm{id}_A\} = R_A \quad \text{and} \quad sInv\, T_A = E_A$$

for the trivial sir-algebras R_A, E_A. Moreover, $sInv\, \{c_a\} = E_A$ for every constant $c_a : x \mapsto a$ ($a \in A$), but $sInv\, H = R_A$ implies $H = \{\mathrm{id}_A\}$ (since for every $h \neq \mathrm{id}_A$ there is some $\varrho \in R_A$ which is not strongly h-invariant).

Now we start with the investigation of $sInv - sEnd$. First we mention some simple properties:

3.5. Lemma. *Let $Q_0 \subseteq R_A$, $H_0 \subseteq T_A$.*

(a) *The set $sInv\, H_0$ contains the relations $\emptyset^{(m)}, A^m$ and is closed under all Boolean operations as well as under $\nabla^{(m)}$ and W_π (for all $m \in \mathbf{N}^+$, $\pi \in \mathrm{Sym}(\underline{m})$).*

(b) *The set $sEnd\, Q_0$ is a submonoid of T_A.*

PROOF. (a): Obviously $\emptyset^{(m)}, A^m \in sInv\, H_0$, and it is also obvious that $sInv\, H_0$ is closed under $\complement^{(m)}$. A routine checking shows that $sInv\, H_0$ is closed under $\cap$, and therefore under all Boolean operations.

Now let us assume that $\varrho \in sInv\, H$. We want to show that also $\mathrm{W}_s\varrho \in sInv\, H$ (for arbitrary $s : \underline{m} \to \underline{n}$). Let $h \in H$ and $\vec{a} \in \mathrm{W}_s\varrho$. Then $(a_{s(1)}, \dots, a_{s(m)}) \in \varrho$, therefore

$$h \circ (a_{s(1)}, \dots, a_{s(m)}) = (h(a_{s(1)}), \dots, h(a_{s(m)})) \in \varrho .$$

This implies $h \circ \vec{a} \in \mathrm{W}_s\varrho$, i.e. h preserves $\mathrm{W}_s\varrho$. Every $h \in H$ also preserves $\complement\, \varrho$, therefore $\vec{b} \in \complement\, \mathrm{W}_s\varrho = \mathrm{W}_s\complement\, \varrho$ implies $h \circ \vec{b} \in \mathrm{W}_s\complement\, \varrho$.

Thus every $h \in H$ strongly preserves $\mathrm{W}_s\varrho$ for $\varrho \in sInv\, H$. But W_π aund $\nabla^{(m)}$ are special cases of W_s, and this finishes the proof of (a).

(b): Since $End\, Q \leq T_A$ for any set Q, and because of $sEnd\, Q = \bigcap_{\varrho \in Q} End\, \{\varrho, \complement\, \varrho\}$, we have $sEnd\, Q \leq T_A$. □

We still need the following Lemma.

3.6. Lemma. *For every $h \in T_A$ (A finite) there exist numbers $k, l \in \mathbf{N}$, $l \geq 1$ such that $h^k(a) = h^{k+l}(a)$ for all $a \in A$. Moreover, one can choose the same k, l for all $h \in T_A$.* [5]

PROOF. Let $k := min\{n \in \mathbf{N} \mid \mathrm{Im}(h^n) = \mathrm{Im}(h^{n+1})\}$ [6]. Then the restriction $h|_B$ of h to the subset $B := \mathrm{Im}(h^k) \subseteq A$ is a permutation. One can choose l as the order of this permutation. To get common k, l for all $h \in T_A$, take $k := |A| - 1$ and let l be the least common multiple of all orders of the above permutations. □

3.7. Lemma. *If $H \subseteq T_A$, then $sInv\, H$ is a sir-algebra.*

PROOF. Because of Lemma 3.5(a) we only have to show that $sInv\, H$ is closed under the operations $\mathrm{pr}^{(m)}$ for all $m \in \mathbf{N}^+$. So let $\varrho \in sInv\, H$ be an $m+1$-ary relation. We have to prove that $\mathrm{pr}^{(m)}\varrho$ is strongly invariant for all $h \in H$.
The relation ϱ is strongly preserved by $h \in H$ and therefore by all functions h^n with $n \in \mathbf{N}^+$. Consequently, for all $\vec{a} \in A^m$, $a_0 \in A$ and $n \in \mathbf{N}^+$ we have

$$(\vec{a}, a_0) \in \varrho \iff (h^n \circ \vec{a}, h^n(a_0)) \in \varrho. \qquad (*)$$

If $\vec{a} \in \mathrm{pr}^{(m)}\varrho$, then there is an $a_0 \in A$ with $(\vec{a}, a_0) \in \varrho$, therefore $(h \circ \vec{a}, h(a_0)) \in \varrho$, and this implies $h \circ \vec{a} \in \mathrm{pr}^{(m)}\varrho$. So for all $\vec{a} \in A^m$ we have

$$\vec{a} \in \varrho \Longrightarrow h \circ \vec{a} \in \varrho\ .$$

Now let us assume that $h \circ \vec{a} \in \mathrm{pr}^{(m)}\varrho$. Then there is an $b_0 \in A$ with $(h \circ \vec{a}, b_0) \in \varrho$. As in Lemma 3.6, we can find $k, l \in \mathbf{N}$ $(l \geq 1)$, with $h^k(a) = h^{k+l}(a)$ for all $a \in A$. Now, applying $(*)$ several times, we get:

$$\begin{aligned} (h \circ \vec{a}, b_0) \in \varrho &\iff (h^{k+1} \circ \vec{a}, h^k(b_0)) \in \varrho \\ &\iff (h^{k+1} \circ \vec{a}, h^{k+l}(b_0)) \in \varrho \\ &\iff (\vec{a}, h^{l-1}(b_0)) \in \varrho \end{aligned}$$

Thus $h \circ \vec{a} \in \mathrm{pr}^{(m)}\varrho$ also implies $\vec{a} \in \mathrm{pr}^{(m)}\varrho$. Therefore for all $h \in H$ and $\vec{a} \in A^m$ we have:

$$\vec{a} \in \varrho \iff h \circ \vec{a} \in \varrho\ .$$

This is equivalent to "ϱ is strongly invariant for h." □

We want to show that vice versa every sir-algebra is a set of strongly invariant relations for some $H \subseteq T_A$. For this reason we use the concept of a Θ-extension (cf. 2.7). Recall (cf. 2.10) that every sir-algebra is a Θ-extension of some Krasner algebra. We can extend this concept to transformation semigroups.

[5] Here we put $h^0 := \mathrm{id}_A$.

[6] $\mathrm{Im}\, h$ denotes the *image* $\{h(a) \mid a \in A\}$ of h

3.8. Definition. Let Θ be an equivalence relation on A, $L := A/\Theta$, and let t denote the natural map $t : A \to L,\ a \mapsto [a]_\Theta$. For $g \in \mathrm{Sym}(L)$ define:

$$[g]^\Theta := \{f \in T_A \mid\ g \circ t = t \circ f\}.^7$$

Let $G \leq \mathrm{Sym}(L)$ be a permutation group on L. The Θ-*extension* of G is defined by

$$[G]^\Theta := \bigcup_{g \in G} [g]^\Theta.$$

3.9. Remark. Notice that $[G]^\Theta$ is a semigroup with identity since $\mathrm{id}_A \in [\mathrm{id}_L]^\Theta$ and $f_1 \in [g_1]^\Theta$, $f_2 \in [g_2]^\Theta$ imply $(g_1 \circ g_2) \circ t = g_1 \circ t \circ f_2 = t \circ (f_1 \circ f_2)$ with $g_1 \circ g_2 \in G$.

3.10. Theorem. *Let Θ be an equivalence relation on A. The Θ-extension of a Krasner algebra is the set of strongly invariant relations of a suitable transformation semigroup.*
More concrete: Let $L := A/\Theta$ and let $Q \subseteq R_L$ be a Krasner algebra on L. Then we have

$$Q^\Theta = sInv_A\, [Aut_L Q]^\Theta.$$

PROOF. Let $H := [Aut_L Q]^\Theta$.
Claim 1: $Q^\Theta \subseteq sInv\, H$.
In fact, let $\varrho \in Q^{(m)}$, $f \in H$, i.e. $\exists g \in Aut\, Q : g \circ t = t \circ f$, and let $\vec{a} = (a_1, \ldots, a_m) \in A^m$. Then

$$\vec{a} \in \varrho^\Theta \iff_{(2.7)} t \circ \vec{a} \in \varrho \iff g \circ t \circ \vec{a} \in \varrho \iff t \circ f \circ \vec{a} \in \varrho \iff f \circ \vec{a} \in \varrho^\Theta,$$

i.e. $\varrho^\Theta \in sInv\, \{f\}$.
Claim 2: $sInv\ H \subseteq Q^\Theta$.
Let $s : L \to A$ be a choice function with $s([a]_\Theta]) \in [a]_\Theta$ for all $a \in A$. Note that $t \circ s = \mathrm{id}_L$. For $g \in Aut\, Q$ let $f_g := s \circ g \circ t : A \to A$. Obviously $f_g \in [g]^\Theta \subseteq H$, in particular $f_{\mathrm{id}_L} = s \circ t \in H$.
Now, let $\sigma \in sInv\, H$. Since $s \circ t \in H$ strongly preserves σ we have

$$\vec{a} \in \sigma \iff s \circ t \circ \vec{a} \in \sigma \qquad (*)$$

Let $\varrho := \{t \circ \vec{a} \mid \vec{a} \in \sigma\}$. Because of (*) we have $\sigma = \varrho^\Theta$: In fact,

$$\begin{aligned} \vec{a} \in \varrho^\Theta &\iff t \circ \vec{a} \in \varrho \iff \exists \vec{b} \in \sigma : t \circ \vec{a} = t \circ \vec{b} \\ &\iff \exists \vec{b} \in \sigma : s \circ t \circ \vec{a} = s \circ t \circ \vec{b} \iff_{(*)} \vec{a} \in \sigma. \end{aligned}$$

It remains to prove $\varrho \in Q$. We shall show $\varrho \in Inv\, Aut\, Q$ (this finishes the proof because of $Q = Inv\, Aut\, Q$, cf. 3.4).

[7] i.e. $[g]^\Theta = \{f \in T_A \mid \forall a \in A : g([a]_\Theta) = [f(a)]_\Theta\}$

Let $g \in Aut\,Q$, $\vec{b} \in \varrho$. Note $\vec{b} = t \circ s \circ \vec{b}$ (since $\vec{b} \in L^m,\ t \circ s = \mathrm{id}_L$). Then $s \circ g \circ \vec{b} = s \circ g \circ t \circ s \circ \vec{b} = f_g \circ s \circ \vec{b} \in \sigma$ (since f_g preserves σ and $s \circ \vec{b} \in \sigma$), thus $g \circ \vec{b} = t \circ s \circ g \circ \vec{b} \in t \circ \sigma = \varrho$, i.e. $\varrho \in Inv\,\{g\}$. Thus $\varrho \in Inv\,Aut\,Q = Q$. □

Now we can give the characterization for the Galois closed sets of relations.

3.11. Theorem. *Let A be finite and $Q \subseteq R_A$. Then the following are equivalent:*

(1) *$Q = sInv\,sEnd\,Q$ (i.e. Q is Galois closed).*

(2) *$Q = sInv\,H$ for some $H \subseteq T_A$.*

(3) *Q is a sir-algebra.*

(4) *Q is the Θ-extension of a Krasner algebra for some equivalence relation Θ on A.*

Furthermore for all $Q \subseteq R_A$ we have $[Q]_{\mathrm{sir}} = sInv\,sEnd\,Q$.

Proof. (1) ⇒ (2) is trivial.
(2) ⇒ (3) is the content of 3.7.
(3) ⇒ (4) is just 2.10.
(4) ⇒ (1) is a consequence of 3.10, namely Q has the form $Q = sInv\,H$ for some $H \subseteq T_A$ (3.10). By a general property of all Galois connections we have $sInv\,H = sInv\,sEnd\,sInv\,H$. Consequently:

$$Q = sInv\,H = sInv\,sEnd\,sInv\,H = sInv\,sEnd\,Q$$

To prove the last statementy for $Q \subseteq R_A$, note that $sInv\,sEnd\,Q$ is a sir-algebra which contains Q and therefore also contains the sir-algebra $[Q]_{\mathrm{sir}}$, therefore

$$[Q]_{\mathrm{sir}} \subseteq sInv\,sEnd\,Q\ .$$

On the other hand $Q \subseteq [Q]_{\mathrm{sir}}$, $[Q]_{\mathrm{sir}}$ is a sir-algebra and therefore

$$sInv\,sEnd\,Q \subseteq sInv\,sEnd\,[Q]_{\mathrm{sir}} \underset{(1)}{=} [Q]_{\mathrm{sir}}$$

Consequently $[Q]_{\mathrm{sir}} = sInv\,sEnd\,Q$. □

Now we are going to characterize the Galois closed sets of transformations. We already know that they form a submonoid of T_A (cf. 3.5 b). Every set of strongly invariant relations is a Θ-extension of some Krasner algebra. A similar result can be obtained for the sets $sEnd\,Q$.

3.12. Lemma. *Let Θ be an equivalence relation on A, let $L := A/\Theta$ and let Q be a Krasner algebra on L. Then*

$$[Aut_L\,Q]^{\Theta} = sEnd_A(Q^{\Theta})\ .$$

PROOF. By 3.10 we have

$$sEnd_A Q^{\Theta} = sEnd_A sInv_A [Aut_L Q]^{\Theta} \supseteq [Aut_L Q]^{\Theta}.$$

It remains to show the opposite inclusion. As in 3.10 let t denote the natural map $t : A \to L,\ a \mapsto [a]_{\Theta}$, and let $s : L \to A$ be a function with $t \circ s = \mathrm{id}_L$. Note that $a_1 \Theta a_2$ is equivalent to $t(a_1) = t(a_2)$.

Let $f \in sEnd_A Q^{\Theta}$. Since Q is a Krasneralgebra, we have $d_L \in Q$ and thus $\Theta = d_L^{\Theta} \in Q^{\Theta}$. Therefore f strongly preserves Θ , i.e. $a_1 \Theta a_2 \iff f(a_1)\Theta f(a_2)$. Now let $g := t \circ f \circ s : L \to L$. For $b_1, b_2 \in L$ we have:

$$\begin{aligned} g(b_1) = g(b_2) &\implies t(f(s(b_1))) = t(f(s(b_2))) \implies f(s(b_1))\Theta f(s(b_2)) \\ &\implies s(b_1)\Theta s(b_2) \implies b_1 = t(s(b_1)) = t(s(b_2)) = b_2. \end{aligned}$$

This means that g is a permutation on L. Obviously $f \in [g]^{\Theta}$ (cf. 3.8). To finish the proof it is enough to show $g \in Aut_L Q$.
Let $\vec{b} \in \varrho \in Q$. Notice $\vec{b} \in \varrho \implies s \circ \vec{b} \in \varrho^{\Theta}$, and $\vec{a} \in \varrho^{\Theta} \implies t \circ \vec{a} \in \varrho$. We have

$$\vec{b} \in \varrho \implies s \circ \vec{b} \in \varrho^{\Theta} \implies f \circ s \circ \vec{b} \in \varrho^{\Theta} \implies t \circ f \circ s \circ \vec{b} \in \varrho \implies g \circ \vec{b} \in \varrho.$$

Consequently $g \in Aut_L Q$. □

3.13. Lemma. *Let $H \subseteq T_A$. Then $H = sEnd\, sInv\, H$ (i.e. H is Galois closed) if and only if $H = [G]^{\Theta}$ for some equivalence relation Θ on A and some permutation group G on A/Θ .*

PROOF. Let $H = sEnd\, Inv\, H$. Because $sInv\, H$ is a sir-algebra, it has the form Q^{Θ} for some Θ and some Krasner algebra Q on $L = A/\Theta$ (cf. 3.11). Now 3.12 implies $H = [Aut_L Q]^{\Theta}$.
Vice versa, let $H = [G]^{\Theta}$ with $G \leq \mathrm{Sym}(L)$. Then G is of the form $Aut_L Q$ for some Krasner algebra Q (cf. 3.4). But then 3.12 implies $H = sEnd_A Q^{\Theta}$. □

The equivalence relation Θ is important for the structure of sir-algebras and their endomorphism monoids. For a sir-algebra Q it is clear how to choose Θ: it is the least equivalence in $Q^{(2)}$ (cf. 2.10). We can use this equivalence to give an "inner" characterization of the Galois closed monoids.
We need some definitions.

3.14. Definition. A strongly invariant equivalence relation $\Theta \in sInv\, H$ will be called a *strong congruence* of $H \subseteq T_A$ (or of the algebra $\langle A; H\rangle$ [8]). Let $sCon\ H$ denote the set of all strong congruences.

3.15. Lemma. *Let Θ be an equvalence on the finite set A.*

[8]strong congruences Θ are congruences of the algebra $\langle A; H\rangle$ satisfying $\forall h \in H \forall a, b \in A : a\Theta b \iff h(a)\Theta h(b)$

(a) *Θ is a strong congruence of $\underline{A} = \langle A; H\rangle$ iff the operations h/Θ of the factor algebra $\underline{A}/\Theta := \langle A/\Theta; H/\Theta\rangle$ are permutations* [9].

(b) *For every strong congruence Θ of $\langle A; h\rangle$ there exist an $m \in \mathbf{N}^+$ such that $h^m(a)\Theta a$ for all $a \in A$.*

(c) *For $H \subseteq T_A$, the set $sCon\,\langle A; H\rangle$ is a filter in the congruence lattice $Con\,\langle A; H\rangle$. This filter is principal and there exists a least strong congruence.*

PROOF. (a) is straightforward (or see in the proof of 3.12).
(b) The order m of the permutation h/Θ does the job, in fact $[h^m(a)]_\Theta = (h/\Theta)^m([a]_\Theta) = [a]_\Theta$.
(c) The intersection of strong congruences is again a strong congruence since $sInv\,H$ is closed under intersections. Further, let $\Theta \in sCon\,\langle A; h\rangle$ $(h \in T_A)$ and $\Theta \subseteq \Psi$ for $\Psi \in Con\,\langle A; h\rangle$. Then obviously $a\Psi b \implies h(a)\Psi h(b)$ and, by (b), for some m we have:

$$h(a)\Psi h(b) \implies a\Theta h^m(a)\Psi h^m(b)\Theta b \implies a\Psi b$$

Thus $\Psi \in sCon\,\langle A; h\rangle$. □

3.16. Definition. Let $\psi(H)$ be the least strong congruence for $H \leq T_A$. The monoid H will be called *Krasner extended monoid* if for all $g \in T_A$ holds:

$$(\exists h \in H\, \forall a \in A:\ h(a)\psi(H)g(a)) \implies g \in H \qquad (*)$$

3.17. Lemma. *A monoid $H \leq T_A$ is Krasner extended if and only if it is the $\psi(H)$-extension of the permutation group $H/\psi(H) \leq T_{A/\psi(H)}$.*

PROOF. Let $\Theta := \psi(H)$. Then by 3.15(a) $G := H/\Theta$ is a permutation group on A/Θ.
The condition $(\forall a)h(a)\Theta g(a)$ in 3.16 is equivalent to $(h/\Theta)([a]_\Theta) = [g(a)]_\Theta$, i.e. $g \in [h/\Theta]^\Theta$. Consequently, the whole condition $(*)$ in 3.16 says nothing else than $g \in [h/\Theta]^\Theta \implies g \in H$, i.e. $[G]^\Theta \subseteq H$.
Because of $h \in [h/\Theta]^\Theta$ for $h \in H$, we can conclude: $[G]^\Theta = H$.
Vice versa, the indicated $\psi(H)$-extension is obviously Krasner extended. □

Now we can characterize the Galois closed transformation semigroups.

3.18. Theorem.

(A) *Let $H \subseteq T_A$, let $\Theta := \psi(H)$ and $L := A/\Theta$. Then we have:*

$$[\langle H/\Theta\rangle_{\mathrm{T}_L}]^\Theta = sEnd\ sInv\ H\ (= End\ sInv\ H).$$

[9] $h/\Theta : A/\Theta \to A/\Theta$ being defined by $[a]_\Theta^{h/\Theta} := [a^h]_\Theta$, $H/\Theta := \{h/\Theta \mid h \in H\}$

(B) *The following are equivalent for* $H \subseteq T_A$:

(1) $H = sEnd\, sInv\, H$, (1)$'$ $H = End\, sInv\, H$,

(2) $H = sEnd\, Q$ *for some* $Q \subseteq R_A$,

(3) H *is a Krasner extended monoid,*

(4) H *is the* Θ*-extension of a permutation group* $G \leq \mathrm{Sym}(A/\Theta)$ *for some equivalence relation* Θ *on* A,

PROOF. (A): For $H \subseteq T_A$, the set H/Θ consists of permutations $h/\Theta \in \mathrm{Sym}(L)$ (cf. 3.15(a)). For $G := \langle H/\Theta \rangle_{\mathrm{T}_L}$ there is some Krasneralgebra $Q_0 := sInv_L G$ such that $G = Aut_L Q_0$ (cf. 3.4). Now $H \subseteq [\langle H/\Theta \rangle]^\Theta$ (for notations cf. 3.8) and therefore

$$sEnd\, sInv\, H \subseteq sEnd\, sInv\, [G]^\Theta =_{(3.10)} sEnd\, Q_0^\Theta =_{(3.12)} [Aut_L Q_0]^\Theta = [G]^\Theta.$$

It remains to show $[G]^\Theta \subseteq sEnd\, sInv\, H$. By 3.11 we know that $Q := sInv\, H \leq \mathcal{R}_A^\diamond$ is an sir-algebra and therefore $Q = \widehat{Q}^\Theta$ as shown in 2.10.
Claim: $G \subseteq Aut_L \widehat{Q}$ $\quad (\widehat{Q} = \{\varrho \in R_L \mid \varrho^\Theta \in Q\})$.
In fact, let $g = h/\Theta \in H/\Theta$, $\varrho \in \widehat{Q}$ and let $([a_1]_\Theta, \ldots, [a_m]_\Theta) \in \varrho$. Then $(a_1, \ldots, a_m) \in \varrho^\Theta$, consequently $(a_1^h, \ldots, a_m^h) \in \varrho^\Theta$ (since h preserves $\varrho^\Theta \in Q = sInv\, H$), i.e. $([a_1]_\Theta, \ldots, [a_m]_\Theta)^{h/\Theta} = ([a_1^h]_\Theta, \ldots, [a_m^h]_\Theta) \in \varrho$ (cf. 3.15, 2.9); h/Θ preserves ϱ. Thus H/Θ and therefore also $G = \langle H/\Theta \rangle$ preserves ϱ.
The claim is proved and implies:

$$sEnd\, sInv\, H = sEnd\, Q = sEnd\, \widehat{Q}^\Theta =_{(3.12)} [Aut_L \widehat{Q}]^\Theta \supseteq [G]^\Theta$$

(B): We have (1) $\Longleftrightarrow$ (4) by 3.13, and (4) $\Longleftrightarrow$ (3) by 3.17. Furthermore (1) $\Longrightarrow$ (2) is trivial, and (2) $\Longrightarrow$ (1) is a general property of a Galois connection. □

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