SOLID PSEUDOVARIETIES

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Dedicated to Prof. Hans-Jürgen Hoehnke

Abstract

Departing from the well known notion of a solid variety, we introduce the notion of a solid pseudovariety and give equational characterizations of solid pseudovarieties using both identities and pseudoidentities. We also establish a connection between lattices of solid varieties and pseudovarieties.

Introduction

Recall that a *pseudovariety* is a class of finite algebras that is closed under the formation of homomorphic images, subalgebras, and finitary direct products. The notion of a pseudovariety is rather natural per se but it has got a strong additional motivation coming from recent developments in finite automata and formal languages [11, 20, 2].

It is clear from the very definition that the notion of a pseudovariety is somehow related to the classical notion of a variety. Therefore it appears to be reasonable to extend to the pseudovariety setting the approaches that have been proved to be successful for studying varieties. One of the most interesting concepts recently arisen in the theory of varieties is the notion of a hyperidentity and the related concept of a solid variety. Recall that an identity $u \approx v$ is said to be a *hyperidentity* in a variety $\mathcal{V}$ if $\mathcal{V}$ *hypersatisfies* $u \approx v$ in the sense that whenever the operation symbols occurring in the terms u and v are replaced by any term (of the same type) of the appropriate arity, $\mathcal{V}$ satisfies the resulting identity [26]. A variety is called *solid* if it hypersatisfies all its identities [14]. A detailed survey on hyperidentities and solid varieties supplied with an extensive bibliography may be found in [24].

The aim of the present paper is to find a reasonable pseudovariety analogue of the concept of a solid variety. The main problem here is to understand what might be the right "pseudo"-version of the crucial notion of

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hypersatisfaction. Recall that, generally speaking, pseudovarieties are not exhausted by the "finite parts" of varieties, that is, not every pseudovariety can be directly defined by identities. There exist, however, two (different but, in a sense, equivalent, see [15]) ways to characterize pseudovarieties by means of certain equations. The first approach, found by Eilenberg and Schützenberger [12], deals with sequences (or, more generally, filters) of usual identities but uses a slightly different from the classical definition of what it means for such a sequence to be satisfied in a finite algebra. The second way, due to Reiterman [23] (for algebras of finite type) and Banashewski [5] (for the general case), is based on an extension of the notion of an identity to that of a pseudoidentity while the meaning of being satisfied is closer to the usual one. Of course, "right" solid pseudovarieties should admit natural characterizations in both of these languages. It is also natural to require that the "finite part" of any solid variety should constitute a solid pseudovariety.

To meet the latter requirement, we adopt Schweigert's approach to solidity via the concept of a derived algebra. Recall that a *derived algebra* of an algebra $\mathfrak{A} = (A;\ f_i)$ is any algebra of the kind $\mathfrak{A}' = (A;\ t_i)$ on the same carrier set A with each of the fundamental operations f_i replaced by a term operation t_i of the same arity. It is known ([8, Theorem 2.3], [14], see also [24, Theorem 1.3.4]) that a variety $\mathcal{V}$ is solid if and only if it is closed under taking derived algebras. Since every derived algebra of a finite algebra is finite, we may define a solid pseudovariety in the same way: a *solid pseudovariety* is a pseudovariety closed under taking derived algebras. Then, of course, the set of all finite members of any solid variety is a solid pseudovariety.

In Section 1 we give an equational characterization of solid pseudovarieties à la Eilenberg and Schützenberger and in Section 2 we characterize them in the language of pseudoidentities. In Section 3 we establish a connection between lattices of solid varieties and pseudovarieties. Section 4 contains some examples; in particular we show that all solid pseudovarieties of semigroups are equational, that is, they are exactly the finite parts of solid semigroup varieties, and we exhibit also a non-equational solid pseudovariety (of unary algebras).

1 Filters and sequences of identities

1.1. First we recall how arbitrary pseudovarieties may be described by filters of identities. Let us fix a type τ and let $T(X)$ denote the set of all terms of this type over a countable set $X = \{x_1, \ldots, x_n, \ldots\}$. Identities of type τ are pairs of terms, that is, elements of the cartesian square $T(X) \times T(X)$.

The set $T(X)$ is the carrier set of the free algebra $\mathfrak{T}(X)$ of type τ. An algebra $\mathfrak{A}$ of type τ is said *to satisfy an identity system* $\Psi \subset T(X) \times T(X)$ (in symbols $\mathfrak{A} \models \Psi$) if, for any homomorphism $\varphi : \mathfrak{T}(X) \longrightarrow \mathfrak{A}$ and for any identity $(u \approx v) \in \Psi$, we have $\varphi(u) = \varphi(v)$.

By an *identity filter* we mean any filter of subsets of $T(X) \times T(X)$, that is, any collection of subsets of $T(X) \times T(X)$ closed under the formation of oversets and finitary intersections. We say that an algebra $\mathfrak{A}$ *ultimately satisfies an identity filter* $\mathbf{F}$ if there exists a set $\Psi \in \mathbf{F}$ such that $\mathfrak{A} \models \Psi$.

1.2. Proposition ([2, Section 3.2]). *For each identity filter* $\mathbf{F}$*, the class of all finite algebras which ultimately satisfy* $\mathbf{F}$ *is a pseudovariety. Conversely, for any pseudovariety* $\mathcal{P}$*, there exists an identity filter* $\mathbf{F}$ *such that* $\mathcal{P}$ *consists precisely of all finite algebras which ultimately satisfy* $\mathbf{F}$. ∎

1.3. For the case of τ being a finite type, we may additionally choose $\mathbf{F}$ to have a countable basis, that is, an identity sequence Ψ such that $\mathbf{F}$ consists precisely of all subsets of $T(X) \times T(X)$ having a cofinite intersection with Ψ. Then an algebra $\mathfrak{A}$ ultimately satisfies $\mathbf{F}$ if and only if it satisfies all but finitely many identities of Ψ, and, in this situation, we shall say that $\mathfrak{A}$ *ultimately satisfies* Ψ. Thus we arrive to the initial Eilenberg-Schützenberger's characterization of pseudovarieties of a finite type via sequences of identities:

1.4. Proposition ([12],[2, Theorem 3.2.5]). *For each identity sequence* Ψ*, the class of all finite algebras which ultimately satisfy* Ψ *is a pseudovariety. Conversely, for any pseudovariety* $\mathcal{P}$ *of finite type, there exists an identity sequence* Ψ *such that* $\mathcal{P}$ *consists precisely of all finite algebras which ultimately satisfy* Ψ. ∎

1.5. Now we recall the notion of a hypersubstitution. A *hypersubstitution* σ is a mapping which assigns a term $\sigma(\mathbf{f}_i) \in T(X)$ of appropriate arity to each fundamental operation symbol $\mathbf{f}_i$. Every hypersubstitution defines a mapping $\hat{\sigma} : T(X) \longrightarrow T(X)$ by letting $\hat{\sigma}(x) = x$ for all $x \in X$ and $\hat{\sigma}(\mathbf{f}_i(t_1, \ldots, t_k)) = \sigma(\mathbf{f}_i)(\hat{\sigma}(t_1), \ldots, \hat{\sigma}(t_k))$ for all terms $t_1, \ldots, t_k \in T(X)$. Hypersubstitutions may also be applied to algebras in the following sense. If $\mathfrak{A} = (A;\ f_i)$ is an algebra and σ is a hypersubstitution, then we denote by $\mathfrak{A}^\sigma$ the algebra $(A;\ \sigma(\mathbf{f}_i))$. (Here and below we denote by f_i the operation corresponding to the fundamental operation symbol $\mathbf{f}_i$. The domain of this operation will be always clear from the context.) Obviously, $\mathfrak{A}^\sigma$ is a derived algebra of the algebra $\mathfrak{A}$ as defined in the introduction, and every derived algebra of $\mathfrak{A}$ has the form $\mathfrak{A}^\sigma$ for a suitable hypersubstitution σ. We need the following result which interrelates the actions of a hypersubstitution on algebras and identities.

1.6. Lemma ([8, Proposition 1.7a]). *Let* $\mathfrak{A}$ *be an algebra and* σ *a hypersubstitution. The algebra* $\mathfrak{A}^\sigma$ *satisfies an identity* $u \approx v$ *if and only if the algebra* $\mathfrak{A}$ *satisfies the identity* $\hat{\sigma}(u) \approx \hat{\sigma}(v)$. ∎

1.7. For Ψ being a system of identities and a hypersubstitution σ, we denote by $\sigma(\Psi)$ the set of all identities of the form $\hat{\sigma}(u) \approx \hat{\sigma}(v)$ where $u \approx v$ runs over Ψ. Now we say that an algebra $\mathfrak{A}$ *ultimately hypersatisfies an identity filter* $\mathbf{F}$ if, for every hypersubstitution σ, there exists a set $\Psi \in \mathbf{F}$ such that $\mathfrak{A}$ satisfies $\sigma(\Psi)$. (Warning: we do not assume that $\sigma(\Psi)$ belongs to $\mathbf{F}$!) A class of algebras $\mathcal{P}$ is said to *ultimately [hyper]satisfy* an identity filter $\mathbf{F}$ if each algebra of $\mathcal{P}$ ultimately [hyper]satisfies $\mathbf{F}$. We are ready to give our first equational characterization of solid pseudovarieties:

1.8. Theorem. *For any class $\mathcal{P}$ of finite algebras, the following conditions are equivalent:*

(i) *$\mathcal{P}$ is a solid pseudovariety;*

(ii) *$\mathcal{P}$ is a pseudovariety and, for each identity filter $\mathbf{F}$, $\mathcal{P}$ ultimately hypersatisfies $\mathbf{F}$ whenever $\mathcal{P}$ ultimately satisfies $\mathbf{F}$;*

(iii) *there exists an identity filter $\mathbf{F}$ such that $\mathcal{P}$ consists precisely of all finite algebras which ultimately hypersatisfy $\mathbf{F}$.*

Proof. (i) $\Rightarrow$ (ii). Let $\mathbf{F}$ be any filter of identities such that $\mathcal{P}$ ultimately satisfies $\mathbf{F}$. Take any algebra $\mathfrak{A} \in \mathcal{P}$ and any hypersubstitution σ. Since $\mathcal{P}$ is solid, the derived algebra $\mathfrak{A}^\sigma$ belongs to $\mathcal{P}$; this implies that $\mathfrak{A}^\sigma$ ultimately satisfies $\mathbf{F}$, that is, $\mathfrak{A}^\sigma \models \Psi$ for some $\Psi \in \mathbf{F}$. Then by Lemma 1.6 the algebra $\mathfrak{A}$ satisfies $\sigma(\Psi)$. Thus, $\mathfrak{A}$ ultimately hypersatisfies $\mathbf{F}$.

(ii) $\Rightarrow$ (iii). By Proposition 1.2 there is an identity filter $\mathbf{F}$ such that $\mathcal{P}$ consists precisely of all finite algebras which ultimately satisfy $\mathbf{F}$. In view of our condition (ii), every algebra of $\mathcal{P}$ in fact ultimately hypersatisfies $\mathbf{F}$. Thus, $\mathcal{P}$ consists precisely of all finite algebras which ultimately hypersatisfy $\mathbf{F}$.

(iii) $\Rightarrow$ (i). Let $\mathbf{F}$ be a filter of identities and $\mathcal{P}$ the class of all finite algebras which ultimately hypersatisfy $\mathbf{F}$. Clearly, if an algebra ultimately hypersatisfies $\mathbf{F}$, then so do its subalgebras and its homomorphic images. Moreover, if the algebras $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ ultimately hypersatisfy $\mathbf{F}$, then so does their direct product $\mathfrak{A} = \mathfrak{A}_1 \times \cdots \times \mathfrak{A}_n$. Indeed, for each hypersubstitution σ and for every $j = 1, \ldots, n$, there exist sets $\Psi_j \in \mathbf{F}$ such that $\mathfrak{A}_j \models \sigma(\Psi_j)$. Then the set $\Psi = \bigcap_{j=1}^n \Psi_j$ belongs to $\mathbf{F}$, and $\mathfrak{A}$ satisfies $\sigma(\Psi) = \bigcap_{j=1}^n \sigma(\Psi_j)$. We have checked that $\mathcal{P}$ is a pseudovariety. To prove that the pseudovariety $\mathcal{P}$ is solid, we have to show that it is closed under the formation of derived algebras. Let $\mathfrak{A}$ be an arbitrary algebra of $\mathcal{P}$ and $\mathfrak{A}'$ any of its derived algebras. Then $\mathfrak{A}'$ has the form $\mathfrak{A}^\sigma$ for a suitable hypersubstitution σ. Now, for an arbitrary hypersubstitution ξ, we define a hypersubstitution $\sigma \circ \xi$ by letting $(\sigma \circ \xi)(\mathbf{f}_i) = \hat{\sigma}(\xi(\mathbf{f}_i))$ for each fundamental operation symbol $\mathbf{f}_i$. There exists a set $\Psi \in \mathbf{F}$ such that $\mathfrak{A}$ satisfies $(\sigma \circ \xi)(\Psi) = \sigma(\xi(\Psi))$. By Lemma

1.6 this means that $\mathfrak{A}^\sigma$ satisfies $\xi(\Psi)$, and therefore, the algebra $\mathfrak{A}^\sigma$ belongs to the pseudovariety $\mathcal{P}$. ∎

1.9. For algebras of finite type, we can obtain a simplified equational description of solid pseudovarieties. First of all, repeating mutatis mutandi the argument in the proof of Theorem 1.8, we can get a straightforward analogue of Theorem 1.8 to characterize solid pseudovarieties via sequences of identities rather than filters. It is possible, however, to go a step further in using the finiteness of the type in order to show that solid pseudovarieties of finite type hypersatisfy identity sequences in a much stronger sense than general solid pseudovarieties do with respect to filters.

Let us proceed with formal definitions. We say that an algebra $\mathfrak{A}$ *ultimately hypersatisfies an identity sequence* Ψ if, for every hypersubstitution σ, there exists a finite subset $\Psi' \subset \Psi$ such that $\mathfrak{A}$ satisfies $\sigma(\Psi \setminus \Psi')$. Clearly, this version of hypersatisfaction is nothing but a "projection" of the corresponding notion for filters: if $\mathbf{F}$ is the filter based on Ψ as explained in 1.3 then any algebra ultimately hypersatisfying Ψ ultimately hypersatisfies $\mathbf{F}$, and vice versa. Now let us permute quantifiers and say that $\mathfrak{A}$ *ultimately strongly hypersatisfies* Ψ if there exists a finite subset $\Psi' \subset \Psi$ such that $\mathfrak{A}$ satisfies $\sigma(\Psi \setminus \Psi')$ for every hypersubstitution σ (in other words, $\mathfrak{A}$ hypersatisfies $\Psi \setminus \Psi'$). In Section 4 we shall give an example showing that the latter notion is indeed stronger than the former one. However, for a finite algebra of finite type, they turn out to be equivalent:

1.10. Lemma. *Let $\mathfrak{A} = (A;\ f_i)$ be a finite algebra of finite type and Ψ a sequence of identities. If $\mathfrak{A}$ ultimately hypersatisfies Ψ, then $\mathfrak{A}$ ultimately strongly hypersatisfies Ψ.*

Proof. Since the type of $\mathfrak{A}$ is finite, there exist only finitely many non-isomorphic algebras of this type on the carrier set A. In particular, $\mathfrak{A}$ has only finitely many non-isomorphic derived algebras. Denote them by $\mathfrak{A}_1, \dots, \mathfrak{A}_n$; each of the algebras $\mathfrak{A}_j$ is of the form $\mathfrak{A}^{\sigma_j}$ for a suitable hypersubstitution σ_j. Since the algebra $\mathfrak{A}$ ultimately satisfies Ψ, for every $j = 1, \dots, n$, there exists a finite subset $\Psi_j \subset \Psi$ such that $\mathfrak{A}$ satisfies $\sigma_j(\Psi \setminus \Psi_j)$. Let $\Psi' = \bigcup_{j=1}^{n} \Psi_j$, then Ψ' is again a finite subset of Ψ, and $\mathfrak{A}$ certainly satisfies $\sigma_j(\Psi \setminus \Psi')$ for any $j = 1, \dots, n$. By Lemma 1.6, each of the algebras $\mathfrak{A}_j = \mathfrak{A}^{\sigma_j}$ satisfies the identity system $\Psi \setminus \Psi'$. Let now σ be an arbitrary hypersubstitution and $\mathfrak{A}^\sigma$ the corresponding derived algebra of the algebra $\mathfrak{A}$. Since $\mathfrak{A}^\sigma$ is isomorphic to one of the algebras $\mathfrak{A}_j$, it also satisfies $\Psi \setminus \Psi'$. Again Lemma 1.6 applies, this time showing that $\mathfrak{A} \models \sigma(\Psi \setminus \Psi')$. Thus we have found a finite subset $\Psi' \subset \Psi$ such that, for every hypersubstitution σ, the given algebra $\mathfrak{A}$ satisfies the identity system $\sigma(\Psi \setminus \Psi')$, that is, $\mathfrak{A}$ ultimately strongly hypersatisfies the sequence Ψ. ∎

1.11. As in the case of filters, a class of algebras $\mathcal{P}$ *ultimately satisfies* (*hypersatisfies, strongly hypersatisfies*) *an identity sequence* Ψ if each algebra of $\mathcal{P}$ ultimately satisfies (respectively, hypersatisfies, strongly hypersatisfies) this sequence. Using these notions, we can formulate an equational description of solid pseudovarieties of finite type as follows:

1.12. Theorem. *For any class $\mathcal{P}$ of finite algebras of finite type, the following conditions are equivalent:*

(i) *$\mathcal{P}$ is a solid pseudovariety;*

(ii) *$\mathcal{P}$ is a pseudovariety and, for each identity sequence Ψ, $\mathcal{P}$ ultimately* [*strongly*] *hypersatisfies Ψ whenever $\mathcal{P}$ ultimately satisfies Ψ;*

(iii) *there exists an identity sequence Ψ such that $\mathcal{P}$ consists precisely of all finite algebras which ultimately* [*strongly*] *hypersatisfy Ψ.*

Proof. Each of the conditions (ii) and (iii) above has a weak and a strong version depending on the word "strongly" being omitted or not. These versions, however, are equivalent in view of Lemma 1.10. On the other hand, the equivalence of (i), the weak version of (ii), and the weak version of (iii) can be obtained along the lines of the proof of Theorem 1.8 with the only difference that sequences are everywhere substituted for filters and Proposition 1.4 comes into the play instead of Proposition 1.2. ■

1.13. We conclude this section with a couple of remarks concerning solid generalized varieties. Recall that a *generalized variety* is a class of algebras that is closed under the formation of homomorphic images, subalgebras, finitary direct products, and arbitrary direct powers. Generalized varieties were proved in [4], see also [2, Section 3.2], to play the role of a bridge between varieties and pseudovarieties: on one hand, every generalized variety is a directed union of varieties while, on the other hand, every pseudovariety is the class of all finite algebras of a generalized variety. As directed unions of varieties, generalized varieties can be characterized by filters of identities: for each identity filter $\mathbf{F}$, the class of all algebras which ultimately satisfy $\mathbf{F}$ is a generalized variety, and conversely, for any generalized variety $\mathcal{G}$, there exists an identity filter $\mathbf{F}$ such that $\mathcal{G}$ consists precisely of all algebras which ultimately satisfy $\mathbf{F}$.

It is natural to define a *solid generalized variety* as a generalized variety closed under taking derived algebras. One may check that we made no real use of the fact that algebras under consideration were finite in the proof of Theorem 1.8. Thus we can apply almost the same reasoning to obtain a characterization of solid generalized varieties; the only difference in the proof is that we have to additionally take into account the obvious fact that if an algebra ultimately hypersatisfies a filter of identities, then so does any of its direct powers. Then we have:

Corollary. *For any class $\mathcal{G}$ of algebras, the following conditions are equivalent:*

(i) *$\mathcal{G}$ is a solid generalized variety;*

(ii) *$\mathcal{G}$ is a generalized variety and, for each identity filter* $\mathbf{F}$*, $\mathcal{G}$ ultimately hypersatisfies* $\mathbf{F}$ *whenever $\mathcal{G}$ ultimately satisfies* $\mathbf{F}$*;*

(iii) *there exists an identity filter* $\mathbf{F}$ *such that $\mathcal{G}$ consists precisely of all algebras which ultimately hypersatisfy* $\mathbf{F}$. ∎

On the other hand, the finiteness plays an essential role in the proof of Theorem 1.12. Therefore there seems to be no natural analogue of the latter theorem for generalized varieties.

1.14. Comparing Theorem 1.8 with Corollary 1.13, we arrive at the following characterization of solid pseudovarieties:

Corollary. *A class of finite algebras is a solid pseudovariety if and only if it consists of all finite members of a solid generalized variety.* ∎

1.15. The latter corollary is a "solidification" of one half of the aforementioned result by Ash [4]. One might expect that its other half also admits a similar solidification, that is, every solid generalized variety is a directed union of solid varieties. In Section 4 we shall exhibit an example showing that, generally speaking, this is not true. On the other hand, for algebras of finite type, we are able to apply Theorem 1.12 in its "strong" version to prove that all solid pseudovarieties come from those (obviously, solid) generalized varieties which are unions of a chain of solid varieties:

1.16. Proposition. *Every solid pseudovariety $\mathcal{P}$ of finite type coincides with the set of all finite members of the union of an ascending chain of solid varieties.*

Proof. Let $\Psi = \{\varepsilon_1, \ldots, \varepsilon_n, \ldots\}$ be a sequence of identities such that $\mathcal{P}$ consists precisely of all finite algebras which ultimately strongly hypersatisfy Ψ. For any $k = 1, 2, \ldots$, denote by Ψ_k the subsequence $\{\varepsilon_k, \ldots, \varepsilon_n, \ldots\}$ and by $\mathcal{V}_k$ the solid variety defined by Ψ_k (that is, $\mathcal{V}_k$ is the class of all algebras hypersatisfying Ψ_k). Then

$$\mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \ldots \subseteq \mathcal{V}_n \subseteq \ldots .$$

Clearly, every algebra from $\mathcal{P}$ hypersatisfies some Ψ_k, and hence, belongs to some $\mathcal{V}_k$. On the other hand, every finite algebra from each $\mathcal{V}_k$ belongs to $\mathcal{P}$. Thus $\mathcal{P}$ coincides with the set of all finite members of the union of all solid varieties $\mathcal{V}_k$. ∎

2 Pseudoidentities

2.1. We need the notion of a pseudoidentity. There are several approaches to this concept (see [23, 5, 18], [2, Section 3.5], [3, Section 1]), among which that based on the natural uniform structure of a free algebra appears to be most suitable for our purposes. We shall recall a few basic facts of the theory of uniform spaces, the details may be found, for example, in [6, 27].

2.2. Let Y be a set. A *covering* of Y is a family Γ of subsets of Y such that $Y = \bigcup_{U\in\Gamma} U$. Given two coverings Γ and Δ of Y, we write $\Gamma \leq \Delta$ if

$$\forall U \in \Gamma \ \exists V \in \Delta \ (U \subseteq V),$$

and we write $\Gamma \ll \Delta$ if

$$\forall U \in \Gamma \ \exists V \in \Delta \ \forall U' \in \Gamma \ (U' \cap U \neq \emptyset \Rightarrow U' \subseteq V).$$

Obviously, if Γ is a partition (the only case that is used in the sequel), then the conditions $\Gamma \leq \Delta$ and $\Gamma \ll \Delta$ are equivalent.

A *uniform structure* (*by coverings*) on a set Y is a family μ of coverings of Y (called *uniform*) such that

a) $\forall \Delta_1, \Delta_2 \in \mu \ \exists \Gamma \in \mu \ (\Gamma \ll \Delta_1$ and $\Gamma \ll \Delta_2)$;

b) $\forall \Gamma \in \mu \ \forall \Delta \ (\Gamma \leq \Delta \Rightarrow \Delta \in \mu)$.

A set Y endowed with a uniform structure is said to be a *uniform space*.

If ν is a family of coverings of Y satisfying condition (a) above, then

$$\mu = \{\Delta : \ \Gamma \leq \Delta \ \text{ for some } \ \Gamma \in \nu\}$$

obviously defines a uniform structure on Y, and we call ν a *basis* of the uniform structure.

If Y and Z are uniform spaces, then a function $\varphi : Y \longrightarrow Z$ is said to be *uniformly continuous* if the set $\{\varphi^{-1}V : \ V \in \Delta\}$ is a uniform covering of Y for any uniform covering Δ of Z. An isomorphism between two uniform spaces is a bijection φ between them such that φ and φ^{-1} are uniformly continuous functions.

If μ is a uniform structure on Y and $Z \subseteq Y$, then the *induced uniform structure* of Z is formed by coverings

$$\{U \cap Z : \ U \in \Gamma\}$$

where Γ runs over μ. (It is easy to check this really defines a uniform structure on Z.) If Y and Z are uniform spaces and $Z \subseteq Y$, then we say that Z is a *uniform subspace* of Y if the uniform structure of Z is that which is induced by that of Y.

Every uniform space Y has a natural topology associated with it: each point $x \in Y$ has as basis of neighborhoods the family of all subsets of Y of the form

$$\bigcap\{U: \ x \in U \ \text{ and } \ U \in \Gamma\}$$

where Γ is a uniform covering of Y. A uniform space is said to be *Hausdorff* if the topology associated with it is Hausdorff.

A *net* of a set Y is a function $\psi : I \longrightarrow Y$ where $\langle I; \leq\rangle$ is a directed set. A net is usually represented by $(x_i)_{i\in I}$ where $x_i = \psi(i)$. A net $(x_i)_{i\in I}$ of a uniform space Y with uniform structure μ is said to be a *Cauchy net* if

$$\forall\Gamma \in \mu \ \exists i_0 \in I \ \forall i, j \in I \ (i, j \geq i_0 \Rightarrow \exists U \in \Gamma : \ x_i, x_j \in U).$$

We say that $(x_i)_{i\in I}$ *converges* to $x \in Y$ if, for any neighborhood V of x,

$$\exists i_0 \in I \ \forall i \in I \ (i \geq i_0 \Rightarrow x_i \in V).$$

A uniform space is called *complete* if every Cauchy net of Y converges. We shall use the two following results concerning complete uniform spaces:

2.3. Proposition ([6, Section 3.6]) *Let T be a dense subspace of a uniform space Y and let Z be a complete Hausdorff uniform space. Then every uniformly continuous mapping $T \longrightarrow Z$ uniquely extends to a uniformly continuous mapping $Y \longrightarrow Z$.* ∎

2.4. Proposition ([6, Section 3.7]) *Given a Hausdorff uniform space Y, there exists (up to isomorphism) a unique complete Hausdorff uniform space $\overline{Y}$ (called the* completion *of Y) containing Y as a dense subspace.* ∎

2.5. We are ready to explain what we mean by a pseudoidentity. We fix a type τ. Let $X_n = \{x_1, \dots, x_n\}$ be a finite set and T_n the set of all terms of type τ over this set. The set T_n is the carrier set of the free n-generated algebra $\mathfrak{T}_n$ of type τ. The *natural uniform structure* on T_n is the uniform structure having as basis the set of all partitions into congruence classes of a congruence of a finite index on $\mathfrak{T}_n$. In other words, it is the smallest uniform structure on T_n such that all homomorphisms $\mathfrak{T}_n \longrightarrow \mathfrak{A}$ with $\mathfrak{A}$ being a finite (discrete) algebra of type τ are uniformly continuous. It is easy to see that every two terms in T_n are separated by a homomorphism onto a finite algebra; this means that the natural uniform structure on T_n is Hausdorff. Now consider the completion $\overline{T}_n$ of the uniform space T_n. The elements of $\overline{T}_n$ are called *n-ary pseudoterms* of type τ. A *pseudoidentity* is a pair of pseudoterms.

Since every homomorphism $\varphi : \mathfrak{T}_n \longrightarrow \mathfrak{A}$ from the term algebra $\mathfrak{T}_n$ into a finite algebra $\mathfrak{A} = (A;\ f_i)$ is uniformly continuous, by Proposition 2.3 it has a unique extension to a uniformly continuous mapping $\bar{\varphi} : \overline{T}_n \longrightarrow A$. The algebra $\mathfrak{A}$ is said to *satisfy* a pseudoidentity $\pi \approx \rho$ if $\bar{\varphi}(\pi) = \bar{\varphi}(\rho)$ for all homomorphisms $\varphi : \mathfrak{T}_n \longrightarrow \mathfrak{A}$.

It is well known that pseudoidentities play the same role for characterizing pseudovarieties as usual identities do for varieties:

2.6. Proposition ([23, 5, 15, 16, 18]; [2, Section 3.5]). *For each set of pseudoidentities Ψ, the class of all finite algebras which satisfy Ψ is a pseudovariety. Conversely, for any pseudovariety $\mathcal{P}$, there exists a set of pseudoidentities Ψ such that $\mathcal{P}$ consists precisely of all finite algebras which satisfy Ψ.* ■

2.7. Now we want to extend the concept of hypersatisfaction to pseudoidentities. To do this in a natural way, we are to be able to apply hypersubstitutions as defined in Section 1 to pseudoterms. For a hypersubstitution σ and $n = 1, 2, \ldots$, denote by σ_n the restriction of the mapping $\hat{\sigma} : T(X) \longrightarrow T(X)$ to the set T_n of all n-ary terms. Since σ preserves arity, so does $\hat{\sigma}$, and therefore, σ_n maps T_n into T_n.

2.8. Lemma. *For any hypersubstitution σ and for any $n = 1, 2, \ldots$, the mapping σ_n is uniformly continuous with respect to the natural uniform structure of T_n.*

Proof. Let φ be a homomorphism of the term algebra $\mathfrak{T}_n$ onto a finite algebra $\mathfrak{A} = (A;\ f_i)$. It suffices to show that the set $U = \{\sigma_n^{-1}(\varphi^{-1}(a))\}_{a \in A}$ is a uniform covering of T_n. It is easy to check, however, that the composed mapping $\varphi \circ \sigma_n : T_n \longrightarrow A$ is a homomorphism of $\mathfrak{T}_n$ onto the derived algebra $\mathfrak{A}^\sigma = (A, t_i^\sigma)$. Thus, U is a partition of $\mathfrak{T}_n$ into congruence classes of a congruence of finite index, and therefore, it belongs to the basis of the natural uniform structure of T_n. ■

2.9. Since the uniform structure of T_n coincides with the restriction of that of $\overline{T}_n$ to T_n, the mapping σ_n remains uniformly continuous if considered as a mapping from T_n to $\overline{T}_n$. Therefore Proposition 2.3 applies yielding, for each $n = 1, 2, \ldots$, a unique uniformly continuous mapping $\bar{\sigma}_n : \overline{T}_n \longrightarrow \overline{T}_n$ extending σ_n. Clearly, the union $\bar{\sigma} = \bigcup_{n=1}^{\infty} \bar{\sigma}_n$ of all these mappings is a unique continuous transformation of the set of all pseudoterms $\overline{T}(X) = \bigcup_{n=1}^{\infty} \overline{T}_n$ (endowed with the inductive limit topology) extending the transformation $\sigma = \bigcup_{n=1}^{\infty} \sigma_n$ of the set of all terms $T(X) = \bigcup_{n=1}^{\infty} T_n$. We say that a finite algebra $\mathfrak{A}$ *hypersatisfies* a pseudoidentity $\pi \approx \rho$ if $\mathfrak{A}$ satisfies the pseudoidentity $\bar{\sigma}(\pi) \approx \bar{\sigma}(\rho)$ for every hypersubstitution σ.

To successfully apply this notion to characterizing solid pseudovarieties we need a "pseudo"-analogue of Lemma 1.6 which will be our closest aim. First we make a couple of technical observations. Let $\varphi : \mathfrak{T}_n \longrightarrow \mathfrak{A}$ be a homomorphism from the term algebra $\mathfrak{T}_n$ into a finite algebra $\mathfrak{A} = (A;\ f_i)$ and, as above, $\bar{\varphi}$ its unique extension to a uniformly continuous mapping from $\overline{T}_n$ into A. Recall that, for any hypersubstitution σ, the composed mapping $\varphi \circ \sigma_n : T_n \longrightarrow A$ is a homomorphism from $\mathfrak{T}_n$ into the derived algebra $\mathfrak{A}^\sigma = (A;\ t_i^\sigma)$. Thus the notation $\overline{\varphi \circ \sigma}_n$ makes sense. The uniformly continuous mappings $\overline{\varphi \circ \sigma}_n$ and $\bar{\varphi} \circ \bar{\sigma}_n$ are easily seen to coincide on the dense subspace T_n of the uniform space $\overline{T}_n$, and therefore, $\overline{\varphi \circ \sigma}_n = \bar{\varphi} \circ \bar{\sigma}_n$ on the whole $\overline{T}_n$. We are ready to prove

2.10. Lemma. *Let $\mathfrak{A}$ be an algebra and σ a hypersubstitution. The algebra $\mathfrak{A}^\sigma$ satisfies a pseudoidentity $\pi \approx \rho$ if and only if the algebra $\mathfrak{A}$ satisfies the pseudoidentity $\bar{\sigma}(\pi) \approx \bar{\sigma}(\rho)$.*

Proof. Let the pseudoterms π and ρ be, say, n-ary. For any homomorphism $\varphi : \mathfrak{T}_n \longrightarrow \mathfrak{A}$ there is a unique homomorphism $\vartheta : \mathfrak{T}_n \longrightarrow \mathfrak{A}^\sigma$ such that $\varphi(x_j) = \vartheta(x_j)$ for all $j = 1, \ldots, n$, and vice versa. Since $\hat{\sigma}(x_j) = x_j$, we have $\varphi(\sigma_n(x_j)) = \vartheta(x_j)$, and therefore, ϑ actually equals the composition $\varphi \circ \sigma_n$. The fact that the algebra $\mathfrak{A}$ satisfies the pseudoidentity $\bar{\sigma}(\pi) \approx \bar{\sigma}(\rho)$ means by definition that, for any φ,

$$\bar{\varphi}(\bar{\sigma}(\pi)) = \bar{\varphi}(\bar{\sigma}(\rho)). \tag{1}$$

Since π and ρ are n-ary pseudoterms, $\bar{\sigma}(\pi) = \bar{\sigma}_n(\pi)$ and $\bar{\sigma}(\rho) = \bar{\sigma}_n(\rho)$. As mentioned before the formulation of the lemma, $\bar{\varphi} \circ \bar{\sigma}_n = \overline{\varphi \circ \sigma}_n = \bar{\vartheta}$. We see that (1) holds for all φ if and only if $\bar{\vartheta}(\pi) = \bar{\vartheta}(\rho)$ for all ϑ, that is, if and only if the algebra $\mathfrak{A}^\sigma$ satisfies the pseudoidentity $\pi \approx \rho$. ∎

Now we can give a description of solid pseudovarieties in the language of pseudoidentities.

2.11. Theorem. *For any class $\mathcal{P}$ of finite algebras, the following conditions are equivalent:*

(i) *$\mathcal{P}$ is a solid pseudovariety;*

(ii) *$\mathcal{P}$ is a pseudovariety which hypersatisfies every pseudoidentity it satisfies;*

(iii) *there exists a set of pseudoidentities Ψ such that $\mathcal{P}$ consists precisely of all finite algebras which hypersatisfy Ψ.*

Proof. (i) $\Rightarrow$ (ii). Let $\pi \approx \rho$ be any pseudoidentity holding in $\mathcal{P}$. Take any algebra $\mathfrak{A} \in \mathcal{P}$ and any hypersubstitution σ. Since $\mathcal{P}$ is solid, the derived algebra $\mathfrak{A}^\sigma$ belongs to $\mathcal{P}$; this implies that $\mathfrak{A}^\sigma$ satisfies $\pi \approx \rho$. Then by Lemma 2.10 the algebra $\mathfrak{A}$ satisfies $\bar{\sigma}(\pi) \approx \bar{\sigma}(\rho)$. Thus, $\mathfrak{A}$ hypersatisfies the pseudoidentity $\pi \approx \rho$.

(ii) $\Rightarrow$ (iii). By Proposition 2.6 there is a set of pseudoidentities Ψ such that $\mathcal{P}$ consists precisely of all finite algebras which satisfy Ψ. In view of our condition (ii), every algebra of $\mathcal{P}$ in fact hypersatisfies Ψ. Thus, $\mathcal{P}$ consists precisely of all finite algebras which hypersatisfy Ψ.

(iii) $\Rightarrow$ (i). Let Ψ be a set of pseudoidentities and $\mathcal{P}$ the class of all finite algebras which hypersatisfy Ψ. Clearly, $\mathcal{P}$ is a pseudovariety. Let us check that it is closed under the formation of derived algebras. Let $\mathfrak{A} = (A;\ f_i)$ be an arbitrary algebra of $\mathcal{P}$ and $\mathfrak{A}'$ any of its derived algebras. Then $\mathfrak{A}'$ has the form $\mathfrak{A}^\sigma$ for a suitable hypersubstitution σ. Now take any pseudoidentity $(\pi \approx \rho) \in \Psi$ and an arbitrary hypersubstitution ξ. Consider the mapping $\overline{\sigma \circ \xi}$ where the hypersubstitution $\sigma \circ \xi$ is defined, as in the proof of Theorem 1.8, by letting $(\sigma \circ \xi)(\mathbf{f}_i) = \hat{\sigma}(\xi(\mathbf{f}_i)$ for each fundamental operation symbol $\mathbf{f}_i$. Since the algebra $\mathfrak{A}$ hypersatisfies the pseudoidentity $\pi \approx \rho$, it should also satisfy the pseudoidentity $(\overline{\sigma \circ \xi})(\pi) \approx (\overline{\sigma \circ \xi})(\rho)$. It is easy to see that $\overline{\sigma \circ \xi} = \bar{\sigma} \circ \bar{\xi}$ (where the product on the right is understood as the usual composition of transformations) as two continuous mappings coinciding on a dense subspace of their common domain. Therefore $\mathfrak{A}$ satisfies the pseudoidentity $\bar{\sigma}(\bar{\xi}(\pi)) \approx \bar{\sigma}(\bar{\xi}(\rho))$. By Lemma 1.6 this means that $\mathfrak{A}^\sigma$ satisfies the pseudoidentity $\bar{\xi}(\pi) \approx \bar{\xi}(\rho)$. Thus, the algebra $\mathfrak{A}^\sigma$ hypersatisfies $\pi \approx \rho$, and therefore, belongs to the pseudovariety $\mathcal{P}$. ∎

2.12. Concluding this section, we briefly discuss another possible approach to solid pseudovarieties. Namely, we have embedded the set of all terms $T(X)$ in the bigger set of all pseudoterms $\overline{T}(X)$. Why should a hypersubstitution take values in $T(X)$ and not in $\overline{T}(X)$? Indeed, we can define a *pseudohypersubstitution* as a mapping which assigns a *pseudoterm* of the appropriate arity to each fundamental operation. Then it can be shown that every pseudohypersubstitution σ can be extended in a natural way to a continuous transformation $\bar{\sigma}$ of the whole set $\overline{T}(X)$, and we can define the corresponding version of a hypersatisfaction: a finite algebra $\mathfrak{A}$ *pseudohypersatisfies* a pseudoidentity $\pi \approx \rho$ if $\mathfrak{A}$ satisfies the pseudoidentity $\bar{\sigma}(\pi) \approx \bar{\sigma}(\rho)$ for every *pseudohypersubstitution* σ. This generalization, however, gives no real advantage in our situation for we are able to prove that a finite algebra pseudohypersatisfies and hypersatisfies exactly the same pseudoidentities.

3 Lattices of solid pseudovarieties

3.1. For any variety or pseudovariety, the collection of its subvarieties (respectively, subpseudovarieties) forms a complete lattice under inclusion. The meet $\mathcal{X} \cap \mathcal{Y}$ of two (pseudo)varieties $\mathcal{X}$ and $\mathcal{Y}$ in this lattice is merely their intersection as classes while the join $\mathcal{X} \vee \mathcal{Y}$ is the least (pseudo)variety containing both $\mathcal{X}$ and $\mathcal{Y}$. If we denote by $\mathbb{H}$, $\mathbb{S}$, $\mathbb{P}$ the operators of taking respectively homomorphic images, subalgebras, direct products, then it easily follows Birkhoff's HSP-theorem that the join of two varieties $\mathcal{X}$ and $\mathcal{Y}$ is equal to the class $\mathbb{H}\mathbb{S}\,\mathbb{P}\,(\mathcal{X}\cup\mathcal{Y})$. Analogously, if we additionally denote by $\mathbb{P}_{fin}$ the operator of forming finitary direct products, then we can express the join of two pseudovarieties $\mathcal{X}$ and $\mathcal{Y}$ as $\mathbb{H}\mathbb{S}\,\mathbb{P}_{fin}\,(\mathcal{X}\cup\mathcal{Y})$.

Let $\mathbb{D}$ denote the operator of taking derived algebras. It is observed in the proof of [24, Theorem 1.4.3] (see also [14, Theorem 4.3]) that, for any class $\mathcal{C}$ of algebras, the following inclusions hold:

$$\mathbb{D}\mathbb{H}\,(\mathcal{C})\subseteq\mathbb{H}\mathbb{D}\,(\mathcal{C});\quad \mathbb{D}\mathbb{S}\,(\mathcal{C})\subseteq\mathbb{S}\,\mathbb{D}(\mathcal{C});\quad \mathbb{D}\mathbb{P}\,(\mathcal{C})\subseteq\mathbb{P}\mathbb{D}\,(\mathcal{C}). \tag{2}$$

This obviously implies that the join of a family of solid varieties is again a solid variety and hence, for any solid variety $\mathcal{V}$, the collection $S_v(\mathcal{V})$ of all its solid subvarieties forms a complete sublattice of the subvariety lattice of $\mathcal{V}$ [24, Corollary 4.1.7]. Lattices of solid varieties have been investigated in several papers, see, for example [13, 14, 25]. In the same manner, one can easily check that, for any class $\mathcal{C}$ of finite algebras, $\mathbb{D}\mathbb{P}_{fin}\,(\mathcal{C})\subseteq\mathbb{P}_{fin}\,\mathbb{D}\,(\mathcal{C})$ and deduce from this that also the family $S_p(\mathcal{P})$ of all solid subpseudovarieties of a solid pseudovariety $\mathcal{P}$ is a complete sublattice of the subpseudovariety lattice of $\mathcal{P}$. The aim of this section is to show that the relation between the lattices of solid varieties and solid pseudovarieties of finite type is similar to that found in [1] and [19] for the lattices of arbitrary varieties and pseudovarieties.

3.2. Theorem. *Let $\mathcal{V}$ be a solid variety of finite type and $\mathcal{V}_{fin}$ the pseudovariety of all finite algebras of $\mathcal{V}$. Then the lattice $S_p(\mathcal{V}_{fin})$ is isomorphic to a homomorphic image of a sublattice of the lattice of all ideals of the lattice $S_v(\mathcal{V})$.*

Proof. Recall that a variety is said to be *locally finite* if all its finitely generated algebras are finite. Birkhoff's HSP-theorem easily implies that the lattice join of a finite number of locally finite varieties is again locally finite whence the collection of all locally finite solid subvarieties of the variety $\mathcal{V}$ forms an ideal L of the lattice $S_v(\mathcal{V})$. For any ideal I of $S_v(\mathcal{V})$ contained in L, we define $\mathcal{P}_I$ as the set of all finite algebras of the (class-theoretical) union $\bigcup I$ of all varieties from I. Since every lattice ideal is by definition closed under finitary lattice joins, the union $\bigcup I$ is directed so it is a solid generalized variety and the set $\mathcal{P}_I$ of its finite members is a solid pseudovariety. Thus we have defined a mapping $\vartheta : I\mapsto\mathcal{P}_I$ from a sublattice of the lattice of all ideals of the lattice $S_v(\mathcal{V})$ into the lattice $S_p(\mathcal{V}_{fin})$.

Let I and J be two ideals of $S_v(\mathcal{V})$ contained in the ideal L. Then it is easy to see that $\vartheta(I\cap J)=\mathcal{P}_I\cap\mathcal{P}_J$. To verify that $\vartheta(I\vee J)=\mathcal{P}_I\vee\mathcal{P}_J$, let $\mathfrak{A}$ be a finite algebra in $\bigcup(I\vee J)$. Recall that the ideal $I\vee J$ consists of all varietal joins of the form $\mathcal{X}\vee\mathcal{Y}$ with $\mathcal{X}\in I$, $\mathcal{Y}\in J$. Thus the algebra $\mathfrak{A}$ belongs to some join of the form $\mathcal{X}\vee\mathcal{Y}$ and is therefore a quotient of the free finitely generated algebra $\mathfrak{F}$ of the latter variety. From the proof of HSP-theorem it follows that $\mathfrak{F}$ is a subdirect product of an algebra $\mathfrak{X} \in \mathcal{X}$ with an algebra $\mathfrak{Y} \in \mathcal{Y}$. Since the variety $\mathcal{X} \vee \mathcal{Y}$ is contained in L, it is locally finite whence the finitely generated algebras $\mathfrak{X}$ and $\mathfrak{Y}$ are finite and belong to the pseudovarieties $\mathcal{P}_I$ and $\mathcal{P}_J$ respectively. Being a quotient of a subdirect product of $\mathfrak{X}$ and $\mathfrak{Y}$, the algebra $\mathfrak{A}$ belongs to the pseudovariety join $\mathcal{P}_I \vee \mathcal{P}_J$. This proves that $\vartheta(I \vee J) \subseteq \mathcal{P}_I \vee \mathcal{P}_J$, and the inverse inclusion is obvious. We have shown that the mapping ϑ is a lattice homomorphism.

It remains to check that ϑ maps the lattice of all ideals of $S_v(\mathcal{V})$ contained in L onto the lattice $S_p(\mathcal{V}_{fin})$. Take any solid subpseudovariety $\mathcal{P}$ of the pseudovariety $\mathcal{V}_{fin}$ and, for each algebra $\mathfrak{A} \in \mathcal{P}$ consider the least solid variety $\mathcal{V}_{\mathfrak{A}}$ containing $\mathfrak{A}$. From the inclusions (2) and the HSP-theorem it follows that this variety is in fact the variety generated by the collection of all derived algebras of the algebra $\mathfrak{A}$. Since the type is finite, there exist only finitely many non-isomorphic derived algebras of $\mathfrak{A}$, and the variety $\mathcal{V}_{\mathfrak{A}}$ is locally finite as every variety generated by a finite set of finite algebras. Let now I be the collection of all solid subvarieties of the variety $\mathcal{V}$ contained in a variety of the form $\mathcal{V}_{\mathfrak{A}}$ for some algebra $\mathfrak{A} \in \mathcal{P}$. Since the join $\mathcal{V}_{\mathfrak{A}} \vee \mathcal{V}_{\mathfrak{B}}$ is in view of (2) contained in (in fact, even equal to) the variety $\mathcal{V}_{\mathfrak{A} \times \mathfrak{B}}$, the set I is easily seen to be an ideal of the lattice $S_v(\mathcal{V})$, and moreover, since I consists of locally finite varieties, it is contained in the ideal L. Now let us calculate $\vartheta(I)$. On one hand, every algebra $\mathfrak{A} \in \mathcal{P}$ lies in "its" variety $\mathcal{V}_{\mathfrak{A}}$ so it belongs to $\bigcup I$ whence $\vartheta(I) \supseteq \mathcal{P}$. On the other hand, every finite algebra $\mathfrak{B}$ from $\bigcup I$ belongs to a variety of the form $\mathcal{V}_{\mathfrak{A}}$ for some algebra $\mathfrak{A} \in \mathcal{P}$ whence it lies in the class $\mathbb{H}\,\mathbb{S}\,\mathbb{P}_{fin}\,\mathbb{D}(\mathfrak{A}) \subseteq \mathcal{P}$ since the pseudovariety $\mathcal{P}$ is solid. Thus we have $\vartheta(I) = \mathcal{P}$, and the homomorphism ϑ is indeed surjective. ∎

4 Examples

4.1. We start by considering solid pseudovarieties of semigroups. It is worth mentioning that solid *varieties* of semigroups have been intensively studied by several authors (see, for example, [7, 10, 21, 22]). Every solid variety of semigroups should, in particular, hypersatisfy the associativity law $(xy)z \approx x(yz)$. Substituting here the term operation xyx for the multiplication, we get the identity $xyxzxyx \approx xyzyx$. The semigroup variety given by this identity is locally finite, as easily follows from a general result by Sapir (see, for example, [17, Theorem 3.8]). Thus every solid semigroup variety is locally finite. This observation immediately leads to the following result:

4.2. Proposition. *Every solid semigroup pseudovariety $\mathcal{P}$ coincides with the class of all finite members of a solid semigroup variety.*

Proof. The semigroup variety $\mathcal{V}$ generated by the pseudovariety $\mathcal{P}$ is solid (as any variety generated by a class closed under forming derived algebras, see [24, Theorem 1.4.3]). Take any finite semigroup $\mathfrak{S} \in \mathcal{V}$. It is a homomorphic image of a finitely generated free semigroup $\mathfrak{F}$ of $\mathcal{V}$ being, in its turn, a subdirect product of some semigroups from $\mathcal{P}$. As mentioned, $\mathcal{V}$ is locally finite; in particular, the semigroup $\mathfrak{F}$ is finite. However finite subdirect products of semigroups from $\mathcal{P}$ are easily seen to belong to $\mathcal{P}$. Thus $\mathfrak{F}$ lies in $\mathcal{P}$, and so does $\mathfrak{S}$. We have proved that the class of all finite semigroups from $\mathcal{V}$ is contained in $\mathcal{P}$, and the opposite inclusion is obvious. ∎

4.3. In contrast, let us exhibit a transparent example of a *proper solid pseudovariety*, that is, a solid pseudovariety which is not a finite part of any variety. In fact, we shall construct a family of proper solid pseudovarieties that has the cardinality of the continuum.

Consider *unars*, that is, algebras with one unary operation. Let P be a non-empty set of prime numbers and $\langle P \rangle$ the submonoid of the multiplicative monoid of positive integers generated by P. A unar $\mathfrak{U} = (U;\ f)$ is said to be *P-cyclic* if, for any $a \in U$, there exists a number $k > 1$ in $\langle P \rangle$ such that $f^k(a) = a$. In the standard representation of a unar as a graph with the vertex set U and edges of the form $(a, f(a))$, $a \in U$, being P-cyclic means that the graph is a disjoint union of cycles whose lengths belong to $\langle P \rangle$, see Figure 1.

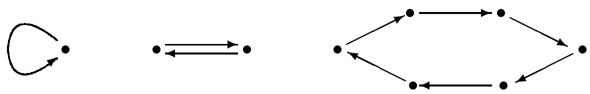

Figure 1: A $\{2, 3\}$-cyclic unar

Let $\mathcal{C}_P$ denote the class of all finite P-cyclic unars. It is easy to see that, for P and Q being different sets of primes, $\mathcal{C}_P \neq \mathcal{C}_Q$. (Indeed, if there is a prime number $p \in P \setminus Q$ then the cycle of length p lies in $\mathcal{C}_P \setminus \mathcal{C}_Q$.) Therefore the family $\{\mathcal{C}_P\}$ where P runs over all non-empty sets of primes has the cardinality of the continuum.

4.4. Example. *For any non-empty set of primes P, the class $\mathcal{C}_P$ of all finite P-cyclic unars is a proper solid pseudovariety.*

PROOF. The class $\mathcal{C}_P$ is obviously closed under taking subunars and homomorphic images. Let $\mathfrak{U}_j = (U_j;\ f)$, $j = 1, \ldots, n$ be arbitrary unars from $\mathcal{C}_P$ and $\mathfrak{U} = \mathfrak{U}_1 \times \cdots \times \mathfrak{U}_n$. Take any element a in the unar $\mathfrak{U}$ and write it as $a = (a_1, \ldots, a_n)$ where $a_j \in U_j$, $j = 1, \ldots, n$. There exists a number $k_j > 1$ in $\langle P \rangle$ such that $f^{k_j}(a_j) = a_j$. Let $k = k_1 \cdots k_n$; then $k > 1$, $k \in \langle P \rangle$, and $f^k(a) = (f^k(a_1), \ldots, f^k(a_n)) = (a_1, \ldots, a_n) = a$. Therefore the unar $\mathfrak{U}$

belongs to the class $\mathcal{C}_P$. We proved that $\mathcal{C}_P$ is a pseudovariety. To check that this pseudovariety is solid, take into account that any term operation in unars is the m^{th} superposition f^m of the fundamental operation f for some non-negative integer m. Hence, for any unar $\mathfrak{U} = (U;\ f) \in \mathcal{C}_P$, all its derived unars are of the form $\mathfrak{U}^{(m)} = (U;\ f^m),\ m = 0, 1, \ldots$. For any $a \in U$, there exists a number $k > 1$ in $\langle P \rangle$ such that $f^k(a) = a$. Clearly, we then have $(f^m)^k(a) = a$ for any $m = 0, 1, \ldots$ as well, and this means that the derived unar $\mathfrak{U}^{(m)}$ also belongs to the pseudovariety $\mathcal{C}_P$. Thus, $\mathcal{C}_P$ is solid, and it remains to verify that $\mathcal{C}_P$ is a proper pseudovariety. Suppose that $\mathcal{C}_P$ coincides with the set of all finite unars of a variety $\mathcal{V}$. Take a prime number $p \in P$ and, for any $n = 1, 2, \ldots$, consider the unar being the cycle of length p^n. These unars belong to $\mathcal{C}_P$ and so to $\mathcal{V}$, and therefore, their direct product lies in the variety $\mathcal{V}$. It is easy to see, however, that every element in this direct product generates a subunar isomorphic to the unar of positive integers with the successor function as the unary operation. Among finite homomorphic images of the latter subunar, there are unars which are not in $\mathcal{C}_P$, for example, the two-element chain $(\{a, b\};\ f)$ with $f(a) = f(b) = b$. Thus, we have found a finite unar in $\mathcal{V} \setminus \mathcal{C}_P$, a contradiction. ∎

It is interesting to compare Example 4.4 with the fact (proved in [9, Proposition 5.1]) that there are only countably many solid varieties of unars.

4.5. Let us continue with a bit more involved example which was promised in Section 1. This is an example of a solid generalized variety which is not equal to any directed union of solid varieties. We again consider unary algebras, this time with countably many unary operations. For brevity, we call such algebras *multiunars*. A multiunar $\mathfrak{M} = (M;\ f_1, f_2, \ldots, f_i, \ldots)$ is said to be *degenerate* if all its operations work as the identical function on M, in other words, if $\mathfrak{M}$ satisfies the identities $f_i(x) \approx x$ for all $i = 1, 2, \ldots$. We say that a multiunar $\mathfrak{M} = (M;\ f_1, f_2, \ldots, f_i, \ldots)$ has *finite character* if its operations induce only finitely many different functions on M. This condition can be expressed in the language of identities as follows: there exists a finite partition α of the set $\mathbb{N}$ of all positive integers such that, for any i, j from the same α-class, the multiunar $\mathfrak{M}$ satisfies the identity $f_i(x) \approx f_j(x)$. We need also two technical restrictions, namely, we want our multiunars to be *commutative* and 2-*terminating*, that is, to satisfy, for all $i, j = 1, 2, \ldots$, the identities

$$f_i f_j(x) \approx f_j f_i(x) \tag{3}$$

and

$$f_i^2(x) \approx f_i^3(x) \tag{4}$$

respectively. We denote by $\mathcal{M}$ the class of all commutative 2-terminating multiunars of finite character.

4.6. Example. *The class $\mathcal{M}$ is a solid generalized variety such that all its solid subvarieties consist of degenerate multiunars only. In particular, the generalized variety $\mathcal{M}$ is not a union of any directed set of solid varieties.*

PROOF. The class $\mathcal{M}$ is obviously closed under the formation of subalgebras, homomorphic images, and direct powers. Let $\mathfrak{M}_k = (M_k;\ f_1, f_2, \ldots, f_i, \ldots)$, $k = 1, \ldots, n$, be arbitrary multiunars from $\mathcal{M}$ and $\mathfrak{M} = \mathfrak{M}_1 \times \cdots \times \mathfrak{M}_n$. For each $k = 1, \ldots, n$, there exists a finite partition α_k of $\mathbb{N}$ such that, for any i, j from the same α_k-class, the multiunar $\mathfrak{M}_k$ satisfies the identity $f_i(x) \approx f_j(x)$. Let $\alpha = \bigcap\limits_{k=1}^{n} \alpha_k$. Then α is again a finite partition of $\mathbb{N}$ and, for every $k = 1, \ldots, n$, the multiunar $\mathfrak{M}_k$ satisfies the identity $f_i(x) \approx f_j(x)$ whenever i, j are in the same α-class, and so does the unar $\mathfrak{M}$. Thus the class $\mathcal{M}$ is a generalized variety.

Now let $\mathfrak{M} = (M;\ f_1, f_2, \ldots, f_i, \ldots)$ be a multiunar from $\mathcal{M}$ and $\mathfrak{M}' = (M;\ g_1, g_2, \ldots, g_i, \ldots)$ any of its derived multiunars, where g_i are some term operations. Since every two fundamental operations f_i and f_j induce commuting functions on M, so do every two term operations g_i and g_j, and this means that $\mathfrak{M}'$ is a commutative multiunar. The identities (3) also guarantee that, for any term operation g, there is some positive integer n and some non-negative integers r_j, $j = 1, 2, \ldots, n$ such that the multiunar $\mathfrak{M}$ satisfies the identity

$$g(x) \approx f_1^{r_1} f_2^{r_2} \cdots f_n^{r_n}(x). \tag{5}$$

In view of the identities (4) we may additionally assume that $r_j \in \{0, 1, 2\}$ for all $j = 1, 2, \ldots, n$.

Using (3), we can deduce from (5) that

$$g^2(x) \approx f_1^{2r_1} f_2^{2r_2} \cdots f_n^{2r_n}(x)$$

and

$$g^3(x) \approx f_1^{3r_1} f_2^{3r_2} \cdots f_n^{3r_n}(x)$$

whence $g^2(x) \approx g^3(x)$ in view of the identities (4). In particular, the multiunar $\mathfrak{M}'$ satisfies the identities $g_i^2(x) \approx g_i^3(x)$ for all $i = 1, 2, \ldots$, in other words, it is 2-terminating. Finally, there exists a partition α of $\mathbb{N}$ into finitely many (say, ℓ) classes such that the multiunar $\mathfrak{M}$ satisfies the identity $f_i(x) \approx f_j(x)$ whenever i and j belong to the same α-class. Let A be any cross-section of α, that is, an ℓ-element set containing exactly one integer from each α-class. Using (3) and (4), we may then choose for any term operation g of $\mathfrak{M}$ an identity of the kind (5) with the additional property that $r_j = 0$ for all $j \notin A$; in other words, any $g(x)$ defines the same function on M as an operation of the form $f_i^{r_i} \cdots f_j^{r_j} \cdots f_k^{r_k}(x)$, where $i < \ldots < j < \ldots < k$, $\{i, \ldots, j, \ldots, k\} = A$, and $r_i, \ldots, r_j, \cdots, r_k \in \{0, 1, 2\}$. Therefore there are no more than 3^ℓ functionally different term operations on M. In particular, the derived multiunar $\mathfrak{M}'$ has finite character and belongs to the generalized variety $\mathcal{M}$. We have proved that $\mathcal{M}$ is solid.

Now suppose that $\mathcal{V}$ is a solid variety contained in the generalized variety $\mathcal{M}$. Let $\mathfrak{F}$ be the $\mathcal{V}$-free multiunar over a non-empty set. Since $\mathfrak{F}$ belongs to $\mathcal{M}$, it is a multiunar of finite character, and therefore, it satisfies an identity of the kind $f_i(x) \approx f_j(x)$ for some $i \neq j$. This identity then holds in the whole variety $\mathcal{V}$ and, since $\mathcal{V}$ is solid, it is a hyperidentity. Take now any $k = 1, 2, \ldots$ and consider a hypersubstitution σ such that $\sigma(\mathbf{f}_i) = f_k$ and $\sigma(\mathbf{f}_j)$ is the identity operation. Applying σ to the identity $f_i(x) \approx f_j(x)$, we get that $\mathcal{V}$ satisfies the identity $f_k(x) \approx x$ for every k. Thus $\mathcal{V}$ consists of degenerate multiunars only. Clearly, there are non-degenerate multiunars in $\mathcal{M}$, and therefore, the generalized variety $\mathcal{M}$ is not a union of any directed set of solid varieties. ■

4.7. Our last example serves to show that the weak and the strong versions of the notion of ultimate hypersatisfaction introduced in 1.9 are indeed different. This example again deals with multiunars. For every $n = 1, 2, \ldots$, consider the multiunar $\mathfrak{C}_n$ with the carrier set $\{a_n^0, a_n^1, a_n^2, \ldots, a_n^{n-1}\}$ and the unary operations $f_1, f_2, \ldots, f_i, \ldots$ defined as follows:

$$f_i(a_n^j) = \begin{cases} a_n^j \ \text{if} \ \ i \neq n, \\ a_n^{(j+1) \bmod n} \ \ \text{if} \ \ i = n. \end{cases}$$

We denote by $\mathfrak{C}$ the disjoint union of all multiunars $\mathfrak{C}_n$, $n = 1, 2, \ldots$. It is clear that the multiunar $\mathfrak{C}$ satisfies all identities (3) and all the identities

$$f_n^n(x) \approx x. \tag{6}$$

4.8. Example. *The multiunar $\mathfrak{C}$ ultimately hypersatisfies the sequence of identities*

$$f_1(x) \approx x, \ f_1^2(x) \approx x, \ f_1^6(x) \approx x, \ldots, f_1^{m!}(x) \approx x, \ldots \tag{7}$$

but it does not ultimately strongly hypersatisfy this sequence.

Proof. As in Example 4.6, we can use the identities (3) to show that, for any term operation g, the multiunar $\mathfrak{C}$ satisfies the identity (5) for some positive integer n and non-negative integers $r_1, r_2, \ldots, r_n$. From (3) and (6) one can easily deduce that $g^{n!}(x) \approx x$ (and therefore, $g^{m!}(x) \approx x$ for all $m \geq n$) in $\mathfrak{C}$ whenever (5) holds. Thus, for any hypersubstitution σ, we can apply this argument to the term operation $\sigma(\mathbf{f}_1)$ to show that the multiunar $\mathfrak{C}$ satisfies all but finitely many identities

$$\sigma(\mathbf{f}_1)(x) \approx x, \ (\sigma(\mathbf{f}_1))^2(x) \approx x, \ (\sigma(\mathbf{f}_1))^6(x) \approx x, \ldots, (\sigma(\mathbf{f}_1))^{m!}(x) \approx x, \ldots \ .$$

Now suppose that our multiunar $\mathfrak{C}$ ultimately strongly hypersatisfies the sequence (7), that is, there exists a positive integer n such that $(\sigma(\mathbf{f}_1))^{m!}(x) \approx x$

for all $m \geq n$ and for every hypersubstitution σ. Take a prime number $p > n$ and consider a hypersubstitution σ such that $\sigma(\mathbf{f}_1) = f_p$. By definition $f_p^{n!}(a_p^0) = a_p^{n! \bmod p} \neq a_p^0$ for $n! \not\equiv 0 \pmod p$. Therefore the identity $(\sigma(\mathbf{f}_1))^{n!}(x) \approx x$ fails in $\mathfrak{C}$, a contradiction. ■

4.9. We remark that the phenomena occurring in Examples 4.4, 4.6 and 4.8 could be better understood from the point of view of deep interrelations existing between solid (pseudo)varieties of unary algebras on the one side and (pseudo)varieties of monoids on the other side. These interrelations in fact expand to similar links between solid (pseudo)varieties of algebras of arbitrary type and (pseudo)varieties of *abstract clones*, see [8, 9] (where the variety case was studied). We are not going to discuss this direction here for we plan to devote a separate paper to it.

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