Rigid relations

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Introduction

An algebra or relational system usually is called *rigid* if its automorphism group or endomorphism monoid is trivial, i.e. consists of the indentity only. Sometimes constants are also considered as trivial what gives rise to the notion of a *C-rigid* system (cf. [Län-P 84]). The next step of generalization of the concept of rigidity was done in [Gra-P 94] by enlarging the set of functions to be considered as trivial. Thus, for a set H of functions, *H-rigid* means that the automorphisms, endomorphisms or in general the polymorphisms belong to H. An H-rigidity criterion can be given in terms of so-called H-minimal clones. For special cases these H-minimal clones were determined in [Gra-P 94] and [Gra 95]. If we restrict our attention to a finite set $\mathcal{K}$ of relations with special properties there canonically arise the following problems:

(1) Does there exist a small set of relations (or even a single relation) which is H-rigid? What is the minimum number $t(\mathcal{K})$ of relations in $\mathcal{K}$ to be H-rigid?

(2) What is the maximum number $T(\mathcal{K})$ of relations in $\mathcal{K}$, which does not form an H-rigid relational system? (Then every system of $T(\mathcal{K}) + 1$ relations in $\mathcal{K}$ is H-rigid.)

These questions were asked in particular for the set $\mathcal{K} = Eq$ of all equivalence relations and the set $H = J_A$ of all projections in [Län-P 84], the answer $t(Eq) = 3$ was given by L. Zádory [Zad 84] and $T(Eq)$ was determined by I. Düntsch [Dün 94]. In the present paper we give an overview over basic results for H-rigid relational systems (section 1), present results for the numbers $t(Q)$ and $T(Q)$ for linear orders and tolerance relations and sketch some proofs (section 2). For more details we refer to [Gra-P 94] and

[Gra 95]. We restrict here to finite base sets A (however the definitions work also for arbitrary A).

1 H-rigid relational systems

1.1. Basic notions and notations. Let A be a finite set and

$$O_A^{(n)} := \{f \mid f : A^n \longrightarrow A\}, \quad O_A := \bigcup_{n=1}^{\infty} O_A^{(n)}.$$

For $F \subseteq O_A$ we denote the set of its n-ary operations by $F^{(n)} := O_A^{(n)} \cap F$. Analogously, for m-ary relations $\varrho \subseteq A^m$ we use the notations

$$R_A^{(m)} := \{\varrho \mid \varrho \subseteq A^m\}, \quad R_A := \bigcup_{m=1}^{\infty} R_A^{(m)}.$$

Let J_A denote the set of all projections, i.e. operations $e_i^n : A^n \to A : (a_1, \ldots, a_n) \mapsto a_i$ (for $i \in \underline{n} := \{1, \ldots, n\}$, $n \in \{1, 2, \ldots\}$).
An operation $f \in O_A^{(n)}$ *preserves* a relation $\varrho \in R_A^{(m)}$ (or ϱ is *invariant* for f) if $f[r_1, \ldots, r_n] \in \varrho$ for all $r_1, \ldots, r_n \in \varrho$ (where $f[r_1, \ldots, r_n]$ is defined by $f[r_1, \ldots, r_n](i) := f(r_1(i), \ldots, r_n(i)),\ i \in \underline{m}$).
Then, for $F \subseteq O_A$, $Q \subseteq R_A$,

$$Pol_A\, Q := \{f \in O_A \mid f \text{ preserves every } \varrho \in Q\}$$

is the set of so-called *polymorphisms* of Q and

$$Inv_A\, F := \{\varrho \in R_A \mid \varrho \text{ is invariant for every } f \in F\}$$

denotes the set of all invariant relations of Q. Moreover, let

$$\begin{aligned} End_A\, Q &:= O_A^{(1)} \cap Pol_A\, Q, \\ wAut_A\, Q &:= S_A \cap End_A\, Q \\ Aut_A\, Q &:= \{f \in S_A \mid \{f, f^{-1}\} \subseteq End_A\, Q\} \end{aligned}$$

denote the *endomorphisms*, *weak automorphisms* and *automorphisms* of Q, respectively (S_A is the full symmetric group of all bijections $f : A \to A$). For finite A we have $wAut_A\, Q = Aut_A\, Q$.

1.2. We recall the notion of a clone. $F \subseteq O_A$ is a (concrete) *clone* on A, notation $F \leq O_A$, if the following two conditions hold

(i) F contains all projections: $J_A \subseteq F$,

(ii) F is closed w.r.t composition, i.e. $f \in F^{(n)}$, $g_1, \ldots, g_n \in F^{(m)}$ $(n, m \in \{1, 2, \ldots\})$ implies $f[g_1, \ldots, g_n] \in F^{(m)}$ (the latter defined by $f[g_1, \ldots, g_n]\mathbf{a} := f(g_1\mathbf{a}, \ldots, g_n\mathbf{a})$, $\mathbf{a} = (a_1, \ldots, a_m) \in A^m$).

For finite A every clone F can be characterized as $F = Pol\,Q$ for a suitable set Q (e.g. $Q = Inv\,F$) of relations. For $F \subseteq O_A$ let $\langle F\rangle_{O_A}$ or $\langle F\rangle$ denote the *clone generated by* F. The clones on A form a lattice $\mathcal{L}_A$ with respect to inclusion.

1.3. Definition. Let H be a clone on A (or a submonoid of $O_A^{(1)}$ $(= A^A)$ or a subgroup of S_A, respectively). A relational system $Q \subseteq R_A$ is called *H-polyrigid*, *H-endorigid* or *H-autorigid*, resp., if $Pol\,Q \subseteq H$, $End\,Q \subseteq H$ or $Aut\,Q \subseteq H$, respectively.

1.4. Remarks. a) For $H = J_A$ (or $H = \{e\}$) definition 1.3 gives the usual notion of rigidity.

b) For $H = \mathcal{C} := \langle C\rangle$, where C is the set of all *constant operations* $c_a : A \to A : x \mapsto a$ $(a \in A)$, $\mathcal{C}$-rigid relational systems were investigated in [Län-P 84] and [Dün 94]. Note that $\mathcal{C}$ consists of all projections and n-ary constants $c_a^{(n)} : (x_1, \ldots, x_n) \mapsto a$, $n \in \mathbf{N}$.

c) From the point of view of a general Galois theory (induced by $Pol - Inv$ and outlined in e.g. [Pös-K 79]), H-rigid relational systems can be characterized equivalently by a kind of completeness property. Let $H = Pol\,Q_0$ for some set $Q_0 \subseteq R_A$. Then $Q \subseteq R_A$ is H-polyrigid iff the relational clone $[Q]_{R_A}$ generated by Q contains Q_0 (for $H = J_A$, $Q_0 = R_A$ this gives a completeness criterion for Q). For finite A, $[Q]_{R_A}$ can be described as the set of all relations definable by relations from Q by primitive positive formulas (cf. [Pös-K 79]).

1.5. Definition. Let $H \leq O_A$ be a clone. A clone $F \leq O_A$ is called *H-minimal* if F is not contained in H and does not contain properly another clone $F' \not\subseteq H$.

J_A-minimal clones are just the minimal clones, i.e. the atoms in the lattice $\mathcal{L}_A$. The following theorem characterizes H-polyrigid relational systems in terms of H-minimal clones. We shall restrict to clones H which can be characterized by a finite set $W \subseteq R_A$ of relations, i.e. $H = Pol\,T$.

1.6. Theorem. *Let $H = Pol\,W$ for some finite $W \subseteq R_A$.*

(1) *A relational system $Q \subseteq R_A$ is H-polyrigid iff for every H-minimal clone F and an (arbitrary chosen) $f \in F \setminus H$ there exist a relation $\varrho \in Q$ such that f does not preserve ϱ (i.e. $F \not\subseteq Pol\,\varrho$),* [Gra-P 94, 2.1].

(2) *The set of H minimal clones is finite,* [Gra-P 94, 2.4].

1.7. Remark. H-endorigid or H-autorigid systems can be treated in analogous manner.

1.8. Let F_0 be a set of functions containing some $f \in F \setminus H$ for every H-minimal clone F. Because of 1.6(2), F_0 can be chosen finite. Obviously, by 1.6(1), F_0 is a set of *test functions* for a rigidity check, i.e.

$$(*) \qquad Q \subseteq R_A \text{ is } H\text{-rigid} \iff \forall f \in F_0\ \exists \varrho \in Q:\ f \notin Pol\,\varrho.$$

Let β' be the number of H-minimal clones. Then any set F_0 of functions with property (*) has at least cardinality β', because F_0 must contain a function $f \in F \setminus H$ for any H-minimal clone F: In fact, assume there is an H-minimal clone F with $F_0 \cap (F \setminus H) = \emptyset$, hence $F_0 \cap F = \emptyset$ (F_0 has to be disjoint from H), and let $Q := Inv\,F$. Then Q is not H-rigid (because $Pol\,Q = Pol\,Inv\,F = F \not\subseteq H$ by the H-minimality of F) but for all $f \in F_0$ we have $f \notin F = Pol\,Q$ what contradicts property (*).

Let β be the least number with the following property: Every H-rigid system $Q \subseteq R_A$ contains an H-rigid subset $Q' \subseteq Q$ with $|Q'| \leq \beta$.

Problem: Determine β and β' for concrete H. Obviously

$$\beta \leq \beta'.$$

Under which condition equality holds? (cf. [Län-P 84, Problem 2])

1.9. Remarks. For the case $A = \{0,1\}$ we have 7 minimal clones [Pos 21] and 5 $\mathcal{C}$-minimal clones [Gra 95]; for a three-element base set $A = \{0,1,2\}$ there are 84 minimal clones [Csá 83] and 96 $\mathcal{C}$-minimal clones. The minimal clones for $|A| > 3$ are unknown, but for $|A| = 4$ there are 224 $\mathcal{C}$-minimal but not minimal clones (for details see [Gra 95]).

2 Rigid systems of chains and tolerance relations

2.1. Definition. Let χ-*rigid* denote one of the rigidity notions (e.g. $\chi \in \{poly, endo, auto, \mathcal{C}-endo, \ldots\}$) and let $\mathcal{K} \subseteq R_A$ be a special fixed set of relations. Then we define

$$\begin{aligned} t_\chi(\mathcal{K}) &:= \min\{|Q| \mid Q \subseteq \mathcal{K} \text{ is } \chi\text{-rigid}\} \\ T_\chi(\mathcal{K}) &:= \max\{|Q| \mid Q \subseteq \mathcal{K} \text{ is not } \chi\text{-rigid}\} \end{aligned}$$

If the minimum or maximum does not exist or is not finite we put the corresponding values $t_\chi(\mathcal{K})$ or $T_\chi(\mathcal{K})$ equal ∞. If every subset of $\mathcal{K}$ is χ-rigid we have $T_\chi(\mathcal{K}) = 0$.

2.2. Let Cha and Tol be the set of all chains (linear order relations) and tolerance (i.e. reflexive and symmetric) relations on A, respectively. Thus, both, Cha and Tol are contained in $R_A^{(2)}$. Because of the reflexivity all relations from Cha and Tol are invariant for any constant function.
Moreover, for $|A| \geq 3$ and a set Q of reflexive relations holds: Q is $\mathcal{C}$-polyrigid iff Q is $\mathcal{C}$-endorigid ([Län-P 84]).
Hence, the cases $\chi \in \{poly, endo\}$ are always trivial and for $|A| \geq 3$ the values for $\chi = \mathcal{C}$-*poly* and $\chi = \mathcal{C}$-*endo* coincide. Let $n := |A|$, then we have (note $|Cha| = n!$ and $|Tol| = 2^{\binom{n}{2}}$):

$\chi=$	$t_\chi(Cha)$	$T_\chi(Cha)$	$t_\chi(Tol)$	$T_\chi(Tol)$
poly	∞	$n!$	∞	$2^{\binom{n}{2}}$
endo	∞	$n!$	∞	$2^{\binom{n}{2}}$
auto	1	0	∞ $(n=2)$ 2 $(3 \leq n \leq 5)$ 1 $(n \geq 6)$	2 $(n=2)$ $2^{\binom{n-1}{2}+1}$ $(n \geq 3)$
$\mathcal{C}$-poly				
$n=2$:	∞	2	∞	2
$n \geq 3$:	see case $\mathcal{C}$-endo			
$\mathcal{C}$-endo				
$n=2$:	1	0	∞	2
$n \geq 3$:	3 $(n=3)$ 2 $(n \geq 4)$	$2(n-1)!$	3 $(n=3,4)$ 2 $(n \geq 5)$	$2^{\binom{n}{2}-1} + 2^{\binom{n-1}{2}}$

For the Proof we refer to [Gra 95] and give here only some hints how to find the values above in the non-trivial cases.

The H-minimal monoids and H-minimal groups and hence the corresponding sets F_0, F_1 of test functions (cf. 1.8), are well-known for the two notions

of rigidity needed in the proof:
For autorigidity we can use (cf. e.g. [Pös-K 79])

$$F_0 = \{f \in S_A \setminus \{e_1^1\} \mid f^q = e_1^1,\ q \text{ prime}\}$$

(this are all non-trivial permutations which have only cycles of lenght q and 1 in their cycle representation).
For $\mathcal{C}$-endorigidity we can use ([Län-P 84])

$$F_1 = \{f \in O_A^{(1)} \setminus \mathcal{C} \mid f^2 = f \text{ or } f^2 \in \mathcal{C} \text{ or } f^q = e_1^1 \ (q \text{ prime})\}.$$

2.3. Chains.
Case $n = 2$:

(1) $t_{\mathcal{C}\text{-}endo}(Cha) = t_{auto}(Cha)$ and $T_{\mathcal{C}\text{-}endo}(Cha) = T_{auto}(Cha)$, since the unary functions on a two-element set are the permutations and constants only.

(2) no set of chains is $\mathcal{C}$-polyrigid, since each chain is preserved e.g. by the corresponding binary operations min and max.

Case $n > 2$:

(1) $t_{\mathcal{C}\text{-}endo}(Cha)$: For $n = 3$ two chains cannot form a $\mathcal{C}$-endorigid system, but the three chains $a_1 \to a_2 \to a_3$, $a_2 \to a_1 \to a_3$ and $a_1 \to a_3 \to a_2$ do so.
For $n > 3$ the set of the following two chains is $\mathcal{C}$-endorigid:
For n even: $a_1 \to a_2 \to \ldots \to a_n$ and
$a_2 \to a_4 \to \ldots \to a_n \to a_1 \to a_3 \to \ldots \to a_{n-1}$.
For n odd: $a_1 \to a_2 \to \ldots \to a_n$ and
$a_{n-1} \to a_2 \to a_4 \to \ldots \to a_{n-3} \to a_n \to a_1 \to a_3 \to \ldots \to a_{n-2}$.

(2) $T_{\mathcal{C}\text{-}endo}(Cha)$: We have to find an $f \in O_A^{(1)} \setminus \mathcal{C}$, such that the number of chains preserved by f is maximal. It is sufficient to consider functions f from the set F_1 of test functions (see above):
Let $m := |Im(f)|$, $Im(f) = \{x_1, \ldots, x_m\}$ and $m_i := |f^{-1}(x_i)|$.

   (a) For $f^2 = f$ there are $m! \prod_{i=1}^{m} m_i!$ chains preserved by f.

   (b) For $f^2 \in \mathcal{C} \setminus J_A$ there are $\prod_{i=1}^{m} m_i!$ chains preserved by f.

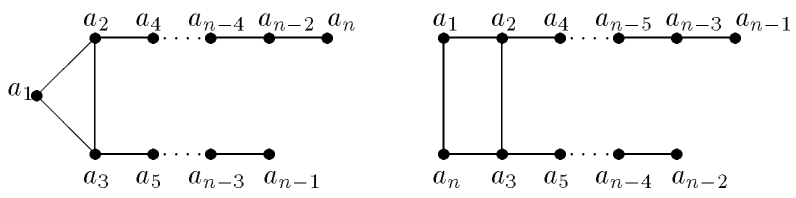
(c) The maximum of this values among all $f \in F_0$ is $2(n-1)!$ for an f with $f^2 = 2$ and $m = 2$, $m_1 = n-1$ ($\Longrightarrow m_2 = 1$).

2.4. Tolerances. $n = 2$: each tolerance relation is invariant for any function $f \in O_A$. Thus let $n \geq 3$:

(a) $t_{auto}(Tol)$: For $n \leq 5$ one can easily test that all tolerances are preserved by an $f \in S_A \setminus \{e_1^1\}$. But the two tolerances

are autorigid.
For $n \geq 6$ a single tolerance relation is autorigid, namely:

for n even for n odd



(b) $T_{auto}(Tol)$: We have to find an $f \in F_0$ (see above), such that the number of tolerances preserved by f is maximal:
Let $f \in S_A \setminus J_A$, $f^q = e_1^1$ (q prime) and k the number of non-trivial cycles in the cycle representation of f.

- For $q \geq 2$ there are 2^w with $w = \frac{1}{q}(n-kq)kq + \binom{n-kq}{2} + \frac{1}{q}k\binom{q}{2} + q\binom{k}{2}$ tolerance relations preserved by F, and for $q = 2$ there are 2^w with $w = \frac{1}{2}(n-2k)2k + \binom{n-2k}{2} + k + 2\binom{k}{2}$ tolerance relations preserved by f.
- The maximum of this values among all $f \in F_0$ is $2^{\binom{n-1}{2}+1}$ for $f \in S_A$ with $f^q = e_1^1$ (q prime) and $k = 1$, $q = 2$.

(c) $t_{\mathcal{C}\text{-}endo}(Tol)$: For $|A| \geq 5$ the following two tolerances form a $\mathcal{C}$-endorigid system (we represent an element (a, b) by $a - b$):
$a_1 - a_2 - \ldots - a_{n-1} - a_n$ and
$a_{n-1} - a_2 - a_4 - \ldots - a_{n-3} - a_n - a_1 - a_3 - \ldots - a_{n-4} - a_{n-2}$ (n odd)
or
$a_2 - a_4 - a_6 - \ldots - a_n - a_1 - a_{n-1} - a_{n-3} - \ldots - a_5 - a_3$ (n even).

(d) $T_{C-endo}(Tol)$: Again we have to find an $f \in F_1$ such that the number of tolerances preserved by f is maximal:
The maximum is $2^{\binom{n}{2}-1} + 2^{\binom{n-1}{2}}$, e.g. for the function f^* with $f^*(a_1) = a_1,\ f^*(a_i) = a_2,\ i = 2, \ldots, n$.

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