Rigid relational systems: A general approach

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Introduction

An algebra or relational system usually is called *rigid* if its automorphism group or endomorphism monoid is trivial, i.e. consists of the indentity only. Sometimes constants are also considered as trivial what gives rise to the notion of a *C-rigid* system (cf. [Län-P 84]). In this paper, which is an extended version of a talk given in Budapest at the Conference on Ordered Structures and Universal Algebra in honour of L. Fuchs on the occasion of his 70^{th} birthday, we shall generalize the concept of rigidity simply by enlarging the set of functions to be considered as trivial. Thus, for a set H of functions, *H-rigid* means that the automorphisms, endomorphisms or in general the polymorphisms belong to H. Our aim is to give a criterion for H-rigidity. This will be done in terms of so-called H-minimal clones (see Theorem 2.1). Of course, for applications it remains the problem to determine these H-minimal clones. As an example, they are determined for some few special cases on a two- or three-element base set.

1 Basic notions and notations

1.1. Let A be a set and

$$O_A^{(n)} := \{f \mid f : A^n \longrightarrow A\}, \quad O_A := \bigcup_{n=1}^{\infty} O_A^{(n)}.$$

For $F \subseteq O_A$ we denote the set of its n-ary operations by $F^{(n)} := O_A^{(n)} \cap F$. Analogously, for m-ary relations $\varrho \subseteq A^m$ we use the notations

$$R_A^{(m)} := \{\varrho \mid \varrho \subseteq A^m\}, \quad R_A := \bigcup_{m=1}^{\infty} R_A^{(m)}.$$

The elements $r = (r(1), \ldots, r(m)) \in A^m$ $(m \in \{1, 2, \ldots\})$ sometimes will be considered as mappings $r: \underline{m} \to A: i \mapsto r(i)$ from $\underline{m} := \{1, \ldots, m\}$ into A. Let J_A denote the set of all projections, i.e. operations $e_i^n: A^n \to A: (a_1, \ldots, a_n) \mapsto a_i$ (for $i \in \underline{n},\ n \in \{1, 2, \ldots\}$).

1.2. An operation $f \in O_A^{(n)}$ *preserves* a relation $\varrho \in R_A^{(m)}$ (or ϱ is *invariant* for f) if $f[r_1, \ldots, r_n] \in \varrho$ for all $r_1, \ldots, r_n \in \varrho$ (where $f[r_1, \ldots, r_n]$ is defined by $f[r_1, \ldots, r_n](i) := f(r_1(i), \ldots, r_n(i)),\ i \in \underline{m}$).
Then, for $F \subseteq O_A$, $Q \subseteq R_A$,

$$Pol_A\, Q := \{f \in O_A \mid f \text{ preserves every } \varrho \in Q\}$$

is the set of so-called *polymorphisms* of Q and

$$Inv_A\, F := \{\varrho \in R_A \mid \varrho \text{ is invariant for every } f \in F\}$$

denotes the set of all invariant relations of F. Moreover, let

$$\begin{aligned} End_A\, Q &:= O_A^{(1)} \cap Pol_A\, Q, \\ wAut_A\, Q &:= S_A \cap End_A\, Q \\ Aut_A\, Q &:= \{f \in S_A \mid \{f, f^{-1}\} \subseteq End_A\, Q\} \end{aligned}$$

denote the *endomorphisms*, *weak automorphisms* and *automorphisms* of Q, respectively (S_A is the full symmetric group of all bijections $f: A \to A$). For finite A we have $wAut_A\, Q = Aut_A\, Q$.
It ist well-known that $Pol - Inv$ establishes a Galois connection between operations and relations and the Galois closed elements are exactly the (locally closed) clones of operations and relations, respectively. For more details we refer to e.g. [Pös-K 79], [Pös 79], [Pös 80]. Note that the following conditions are equivalent for $f \in O_A^{(n)}$, $\varrho \in R_A^{(m)}$:

- $f \in Pol\, \varrho$
- $\varrho \in Inv\, f$
- ϱ is a subalgebra of $\langle A; f\rangle^m$ (direct power of $\langle A; f\rangle$)
- $f: \langle A; \varrho\rangle^n \to \langle A; \varrho\rangle$ is a (relational) homomorphism.

1.3. We recall the notion of a clone. $F \subseteq O_A$ is a (concrete) *clone* on A, notation $F \leq O_A$, if the following two conditions hold

(i) F contains all projections: $J_A \subseteq F$,

(ii) F is closed w.r.t composition, i.e. $f \in F^{(n)}, g_1, \ldots, g_n \in F^{(m)}$ $(n, m \in \{1, 2, \ldots\})$ implies $f[g_1, \ldots, g_n] \in F^{(m)}$ (the latter defined by $f[g_1, \ldots, g_n]\mathbf{a} := f(g_1\mathbf{a}, \ldots, g_n\mathbf{a})$, $\mathbf{a} = (a_1, \ldots, a_m) \in A^m$).

As mentioned above (in 1.2), for finite A every clone F can be characterized as $F = Pol\, Q$ for a suitable set Q (e.g. $Q = Inv\, F$) of relations. For $F \subseteq O_A$ let $\langle F \rangle_{O_A}$ or $\langle F \rangle$ denote the clone generated by F.
Clones on A form a lattice $\mathcal{L}_A$ with respect to inclusion, where $F_1 \wedge F_2 = F_1 \cap F_2$ and $F_1 \vee F_2 = \langle F_1 \cup F_2 \rangle$.
We do not give here an internal definition of a relational clone but use the following characterizations as definitions (this works for finite A only). Let A be finite and $Q \subseteq R_A$ then

$$[Q]_{R_A} := Inv\, Pol\, Q, \quad [Q]_{R'_A} := Inv\, End\, Q, \quad [Q]_{R''_A} := Inv\, Aut\, Q$$

is called the *relational clone*, the *weak Krasner clone* and the *Krasner clone* generated by Q (what coincides with the function algebra, Krasner algebra of first kind and Krasner algebra of second kind, resp., in [Pös-K 79]).

1.4. We recall the following facts concerning the Galois connection $Pol - Inv$ (for details see [Pös-K 79]).

a) For finite A, $F \subseteq O_A$, we have $\langle F \rangle_{O_A} = Pol\, Inv\, F$.

b) For $\varrho \in R_A$, $|\varrho| \leq m$ and a clone $F \leq O_A$ we have $F \subseteq Pol\, \varrho \iff F^{(m)} \subseteq Pol\, \varrho$.

Now we come to the crucial definition of this paper.

1.5. Definition. Let H be a clone on A (or a submonoid of $O_A^{(1)}$ $(= A^A)$ or a subgroup of S_A, respectively). A relational system $Q \subseteq R_A$ is called *H-polyrigid*, *H-endorigid* or *H-autorigid*, resp., if $Pol\, Q \subseteq H$, $End\, Q \subseteq H$ or $Aut\, Q \subseteq H$, respectively.

1.6. Remarks. a) For $H = J_A$ $(H = \{e\})$ definition 1.5 gives the usual notion of rigidity.
b) For $H = \mathcal{C} := \langle C \rangle$, where C is the set of all constant operations $c_a : A \to A : x \mapsto a$ $(a \in A)$, $\mathcal{C}$-rigid relational systems were investigated in [Län-P 84] and [Dün 94]. Note that $\mathcal{C}$ consists of all projections and n-ary constants

$c_a^{(n)} : (x_1, \ldots, x_n) \mapsto a$, $n \in \mathbf{N}$.

c) Special cases of H-rigid relational systems are treated (as semirigid systems) e.g. in [Dem-M-R-S-S 90], [Las-M-N-P-R 93], [Miy-N-P-R 92] and [Pog-N-M-R 92]. There are also several other papers dealing with the existence of concrete H-rigid systems (e.g. [Vop-P-H 65], [Zád 84], for further references see also [Miy-N-P-R 92]).

d) From the point of view of a general Galois theory (induced by $Pol - Inv$ and outlined in e.g. [Pös-K 79]), H-rigid relational systems can be characterized equivalently by a kind of completeness property. Let $H = Pol\, Q_0$ for some set $Q_0 \subseteq R_A$. Then $Q \subseteq R_A$ is H-polyrigid iff the relational clone $[Q]_{R_A}$ generated by Q contains Q_0 (for $H = J_A$, $Q_0 = R_A$ this gives a completeness criterion for Q). In fact, $Q_0 \subseteq [Q]_{R_A} = Inv\, Pol\, Q$ is equivalent to $H = Pol\, Q_0 \supseteq Pol\, Inv\, Pol\, Q = Pol\, Q$. Analogously, H-endorigid and H-autorigid relational systems Q are characterized by completeness criteria for the weak Krasner clone and the Krasner clone generated by Q.

The following notions are needed for the characterization theorem 2.1.

1.7. Definition. Let $H \leq O_A$ be a clone. A clone $F \leq O_A$ is called *H-minimal* if F is not contained in H and does not contain properly another clone $F' \nsubseteq H$.

1.8. Lemma. *Let $H \leq O_A$ be a clone. A clone $F \leq O_A$ is H-minimal iff*

(1) $F \setminus H \neq \emptyset$ *and*

(2) $\forall f \in F \setminus H : \langle f \rangle = F$.

Proof. "$\Rightarrow$": The first property is clear by definition. Let $f \in F \setminus H$. Then $\langle f \rangle \leq F$ and $\langle f \rangle \nsubseteq H$. By minimality of F we have $\langle f \rangle = F$.
"$\Leftarrow$": Let $F \nsubseteq H$ be not H-minimal, i.e. there is a clone F' with $F' < F$ and $F' \nsubseteq H$. In particular, there is an $f \in F' \subseteq F$, $f \notin H$ with $\langle f \rangle \leq F' < F$ what contradicts (2). $\square$

1.9. Remark. J_A-minimal clones are just the minimal clones, i.e. the atoms in the lattice $\mathcal{L}_A$.

2 Characterization of H-polyrigid relational systems

The following theorem characterizes H-polyrigid relational systems in terms of H-minimal clones. We shall restrict to clones H which can be characterized by a finite set $T \subseteq R_A$ of relations, i.e. $H = Pol\,T$. In that case the filter in $\mathcal{L}_A$ generated by H is atomic (cf. e.g. [Pös-K 79]). Moreover we shall see that there are only finitely many H-minimal clones. This makes the following theorem applicable provided that one can fix a function $f \in F \setminus H$ for every H-minimal clone F. These picked out functions constitute a test set for rigidity:

2.1. Theorem. *Let $H = Pol\,T$ for some finite $T \subseteq R_A$. Then a relational system $Q \subseteq R_A$ is H-polyrigid iff for every H-minimal clone F and an (arbitrary chosen) $f \in F \setminus H$ there exist a relation $\varrho \in Q$ such that f does not preserve ϱ (i.e. $F \not\subseteq Pol\,\varrho$).*

For the proof of 2.1 we need the following lemma:

2.2. Lemma. *Let $H = Pol\,T$ for finite $T \subseteq R_A$ and let $F_0 \geq F_1 \geq F_2 \geq \dots$ be a descending chain of clones with $\bigcap_{i=0}^{\infty} F_i \leq H$. Then there is some $n_0 \in \mathbf{N}$ with $F_{n_0} \leq H$.*

Proof. Via the Galois connection $Pol - Inv$ we get

$$H = Pol\,T \supseteq \bigcap_{i=0}^{\infty} F_i = \bigcap_{i=0}^{\infty} Pol\,Inv\,F_i = Pol\,(\bigcup_{i=0}^{\infty} Inv\,F_i)$$

and therefore

$$T \subseteq [T]_{R_A} = Inv\,Pol\,T \subseteq [\bigcup_{i=0}^{\infty} Inv\,F_i\,]_{R_A} = \bigcup_{i=0}^{\infty} Inv\,F_i$$

(the latter is because $Inv\,F_i$ form an ascending chain). Since T is finite, there is some $n_0 \in \mathbf{N}$ with $T \subseteq Inv\,F_{n_0}$, consequently $H = Pol\,T \supseteq Pol\,Inv\,F_{n_0} = F_{n_0}$. $\square$

2.3. Lemma. *Every clone $F \not\subseteq H$ contains an H-minimal clone.*

Proof. Let $\mathcal{M} := \{F' \mid F' \leq F \text{ is a clone, } F' \not\subseteq H\}$. Because $F \in \mathcal{M}$ we have $\mathcal{M} \neq \emptyset$.
Claim: *Every descending chain in $\mathcal{M}$ has a lower bound in M.*
In fact, let $F_1 \geq F_2 \geq \dots$ be a chain of clones $F_i \in \mathcal{M}$ and $\bigcap_{i=1}^{\infty} F_i =: F^*$. Then $F^* \not\subseteq H$. In fact, if $F^* \subseteq H$ then $F_{n_0} \subseteq H$ for some n_0 (by lemma 2.2) in contradiction to $F_{n_0} \in \mathcal{M}$. Thus F^* is a lower bound in M and the

claim is proved.
We apply ZORN's Lemma to $\langle \mathcal{M}, \subseteq \rangle$: There is a minimal element in $\mathcal{M}$ which is contained in F and therefore H-minimal. $\square$

Now we are ready to prove Theorem 2.1:

PROOF of 2.1. Let Q be H-polyrigid, i.e. $Pol\, Q \subseteq H$. Obviously (by definition 1.5), $Pol\, Q$ does not contain an H-minimal clone F, i.e. $\exists f \in F$: f does not preserve Q. Conversely, if $F \nsubseteq Pol\, Q$ for every H-minimal clone F, then $Pol\, Q \subseteq H$ (since otherwise $Pol\, Q$ would contain an H-minimal clone by 2.3). $\square$

2.4. Theorem. *Let $H = Pol\, T$ for finite $T \subseteq R_A$. Then the set of H-minimal clones is finite.*

PROOF. Let F be H-minimal, $m_0 = max\{|\varrho| \mid \varrho \in T\}$. Then $F \nsubseteq H$ and $F^{(m_0)} \nsubseteq H$ by 1.4b. Then, by 2.3, $\langle F^{(m_0)} \rangle$ contains an H-minimal clone, thus by minimality of F we have $F = \langle F^{(m_0)} \rangle$. Consequently every H-minimal clone is generated by a set of m_0-ary operations, and for finite A, there are only finitely many such sets $F^{(m_0)}$. $\square$

We conclude with some remarks on H-endorigid and H-autorigid systems.

2.5. The characterization of H-autorigid and H-endorigid relational systems can be done in full analogy to H-polyrigid systems. Moreover, we have to deal only with the finitely many unary operations.
Let H be a submonoid of $O_A^{(1)}$. A submonoid $M \leq O_A^{(1)}$ is called *H-minimal* if $M \nsubseteq H$ and M does not contain properly another monoid $M' \nsubseteq H$.
Note that M is H-minimal iff $\langle M \rangle_{O_A}$ is $\langle H \rangle_{O_A}$-minimal in the sense of 1.5 (however, not every $\langle H \rangle_{O_A}$-minimal clone is of the form $\langle M \rangle_{O_A}$, i.e. generated by unary operations). Since $O_A^{(1)}$ is finite for finite A, all above results, in particular Lemma 2.3 and Theorem 2.1 remain valid (without any assumption on H) if "polyrigid" is substituted by "endorigid" and "H-minimal clone" is substituted by "H-minimal submonoid".

2.6. Now, by 2.5, it is obvious how to treat H-autorigid relational systems: Substitute "$O_A^{(1)}$" and "submonoid" by "S_A" and "subgroup", respectively, and use the canonically given notion of an H-minimal subgroup for $H \leq S_A$.

In order to apply Theorem 2.1 it remains the following

2.7. Main problem. For given $H \leq O_A$ ($O_A^{(1)}$, S_A, resp.) find all H-minimal clones (submonoids, subgroup, resp.).

The next section gives some few results concerning this problem.

2.8. Remark. A clone F is H-minimal iff $F \wedge H$ is maximal in F and F is $\vee$-irreducible in the lattice $\mathcal{L}_A$.

3 $\mathcal{C}$-minimal clones

Let $\mathcal{C} = \langle C \rangle$, where C is the set of all constant functions (cf. 1.6b). In this section we determine all $\mathcal{C}$-minimal clones on the base sets $A = \{0,1\}$ and $A = \{0,1,2\}$. Some facts are given for arbitrary finite A.

3.1. In case $A = \{0,1\}$ the lattice $\mathcal{L}_A$ is completely described by L.A. Post (cf. e.g. [Jab-G-K 70]). This description immediatly gives the H-minimal clones for an arbitrary clone H. E.g., for $H = \mathcal{C}$ we have the following $\mathcal{C}$-minimal clones:

$$\langle \vee \rangle, \langle \wedge \rangle, \langle \neg \rangle, \langle h \rangle, \langle g \rangle$$

where $\vee, \wedge, \neg$ denote logical OR, AND and NOT, respectively, and h, g are ternary boolean functions defined by $h(x,y,z) := (x \wedge y) \vee (y \wedge z) \vee (x \wedge z)$ and $g(x,y,z) := x \oplus y \oplus z$ ($\oplus$ denotes the sum mod 2).
In case $|A| > 2$ there are uncountably many clones and one cannot expect a full description of all H-minimal clones for arbitrary H. We give here only partial results.

3.2. Definition. A function $f \in O_A^{(k)}$ is called *H-sharp* if $f \notin H$ and $\forall l < k:\ \langle f \rangle^{(l)} \subseteq H$.

3.3. Lemma. *A clone $F \leq O_A$ is H-minimal iff there are $k \in \mathbf{N}$ and $f \in O_A^{(k)} \setminus H$ with $F = \langle f \rangle$ such that*

(1) *f is H-sharp (i.e. $\forall l < k : \langle f \rangle^{(l)} \subseteq H$),*

(2) $\forall l \geq k, f' \in \langle f \rangle^{(l)} \setminus H : f \in \langle f' \rangle$.

Remark: (2) *can be splitted and is equivalent to*

(2a) $\forall f' \in \langle f \rangle^{(k)} \setminus H : f \in \langle f' \rangle$,

(2b) $\forall l > k,\ f' \in \langle f \rangle^{(l)} : f'$ *is not H-sharp.*

Proof. "$\Rightarrow$": Let F be H-minimal. Choose $k := min\{n \in \mathbf{N} \mid F^{(n)} \setminus H \neq \emptyset\}$ and $f \in F^{(k)} \setminus H$. Then (1) is satisfied and (2) follows from 1.8.
"$\Leftarrow$": The conditions (1),(2) imply both conditions in 1.8. $\square$

3.4. How to determine H-minimal clones $F = \langle f \rangle$? Obviously, every minimal clone F' with $F' \not\subseteq H$ is H-minimal, too. However, all minimal clones in O_A are known only for $|A| \in \{2, 3\}$ ([Csá 83]). The cases $|A| > 3$ are a hard open problem. Therefore, "modulo minimal clones", we shall restrict our considerations to clones $F = \langle f \rangle$ which do not contain a minimal clone F' with $F' \not\subseteq H$.
Further, by 3.3, the H-minimal clones are among those generated by H-sharp functions. In order to determine H-sharp functions $f \in O_A$ one has to solve the following problems (for $l = 1, 2, 3, \ldots$):

(A) Determine $\langle f \rangle^{(l)}$;

(B) Check $\langle f \rangle^{(l)} \subseteq H$.

The set $\langle f \rangle^{(l)}$ is a Menger algebra. For k-ary f, the Menger algebra $\langle f \rangle^{(k)}$ can be constructed by the following straigthforward algorithm: Starting with $M_0 := J_A^{(k)} \cup \{f\}$ we put $M_{i+1} := M_i \cup \{f[g_1, \ldots, g_k] \mid g_1, \ldots, g_k \in M_i\}$. Then $M_0 \subseteq M_1 \subseteq M_2 \subseteq \ldots$ and $\langle f \rangle^{(k)} = \bigcup\limits_{i=0}^{\infty} M_i$ (cf. e.g. [Pös-K 79; 4.2.2, 4.2.3]).
The following two lemmata deal with problem (B) in the special case $H = \mathcal{C}$.

3.5. Lemma. *Let $F = \langle f \rangle$, $f \in O_A^{(k)} \setminus \mathcal{C}$, $k \geq 2$ and let F does not contain a minimal clone $F' \not\subseteq \mathcal{C}$. Then the following are equivalent:*

(1) *f is $\mathcal{C}$-sharp,*

(2) *For all $f' \in F^{(k)} \setminus J_A$ there is a $c_{f'} \in A$ such that*

$$\forall 1 \leq i < j \leq k : f'(x_1, \ldots, x_{j-1}, x_i, x_{j+1}, \ldots, x_k) = c_{f'},$$

Proof. (2) $\Rightarrow$ (1): Let $g \in F^{(l)} \setminus J_A$, $l < k$. Then $g \in \langle f \rangle$ can be obtained from the formally k-ary operation $f'(x_1, \ldots, x_k) := g(x_1, \ldots, x_l)$ by identification of the variables $x_l = \ldots = x_k$. Because of (2) we have $g = c_{f'} \in \mathcal{C}$.
(1) $\Rightarrow$ (2): Let $f' \in F^{(k)}$. Identification of two variables (say x_i, x_j) must yield a function f'_{ij} which is a constant or projection (since $f'_{ij} \in F^{(k-1)} \subseteq \mathcal{C}$), i.e. we have

(*) $\quad f'(x_1, \ldots, x_{j-1}, x_i, x_{j+1}, \ldots, x_k) = c_{ij}$, $(c_{ij} \in A)$ or

(**) $f'(x_1, \ldots, x_{j-1}, x_i, x_{j+1}, \ldots, x_k) = x_{ij}$, $(x_{ij} \in \{x_1, \ldots, x_k\})$.

Because $c_{ij} = f'(x, x, \ldots, x) = c_{lm}$ for $i, j, l, m \in \underline{k}$ ($i \neq j$, $l \neq m$), all the c_{ij} are equal.
If case (*) occur for at least one pair i, j then $f'(c', c', \ldots, c') = c$ for any $c' \neq c$, thus case (**) cannot occur, hence (2) holds.
Otherwise, if (**) holds for all pairs $i \neq j$ then $\langle f' \rangle$ (and hence F) contains a minimal clone $F' \not\subseteq \mathcal{C}$ (in contradiction to the assumption on F) or f' is a projection [Csá 83]. $\square$

3.6. Lemma. *Let $F = \langle f \rangle$, $f \in O_A^{(k)} \setminus \mathcal{C}$, $k \geq 2$ and let F does not contain a minimal clone $F' \not\subseteq \mathcal{C}$. Then the following are equivalent:*

(1) *f is $\mathcal{C}$-sharp,*

(2) *There is some $c \in A$ such that*

$$\begin{aligned} \forall 1 \leq i < j \leq k \quad &: \quad f(x_1, \ldots, x_{j-1}, x_i, x_{j+1}, \ldots, x_k) = c, \quad \textit{and} \\ \forall 1 \leq j \leq k \quad &: \quad f(x_1, \ldots, x_{j-1}, c, x_{j+1}, \ldots, x_k) = c. \end{aligned}$$

Proof. (1) $\Rightarrow$ (2): Because of 3.5 it remains to prove that (1) implies the second part of (2). In fact, let $f''(x_1, \ldots, x_k) := f(f(x_1, \ldots, x_k), x_2, \ldots, x_k)$, then $f'' \in \langle f \rangle$ and by lemma 3.5 we obtain $f'' \in J_A$ or

$$c_{f''} = f''(x_2, x_2, \ldots, x_k) = f(f(x_2, x_2, \ldots, x_k), x_2, \ldots, x_k) = f(c, x_2, \ldots, x_k).$$

In the latter case we have $c_{f''} = f(c, c, \ldots, c) = c$ and therefore (2) holds (here for $j = 1$, but $j > 1$ can be done analogously).
The case $f'' \in J_A$ cannot occur, otherwise, for $k \geq 3$, we have

$$c' = f''(c', c', \ldots, c') = f(f(c', c', \ldots, c'), c', \ldots, c') = f(c, c', \ldots, c') = c$$

for all $c' \in A$ in contradiction to $|A| \geq 2$. The case $k = 2$ needs an additional argument. One can show that f^* defined by $f^*(x_1, x_2, x_3) := f(f(x_1, x_2), x_3)$ is J_A-sharp (in case $f'' \in J_A$). Thus $\langle f^* \rangle$ does not contain a constant and therefore must contain a minimal clone $F' \not\subseteq \mathcal{C}$, in contradiction to the assumption.
(2) $\Rightarrow$ (1): Let $g \in F^{(l)} \setminus J_A$, $l < k$. Then $g \in \langle f \rangle$ can be represented by a term built up from f using composition. For any subterm of the form $f(y_1, \ldots, y_k)$ with $\{y_1, \ldots, y_k\} \subseteq \{x_1, \ldots, x_l\}$, at least two variables coincide, i.e. the subterm becomes the constant c by (2), consequently g is also constant by the second part of (2). $\square$

3.7. It remains to consider the case $k = 1$, i.e. unary $\mathcal{C}$-sharp operations f. Here one can explicitely describe all $\mathcal{C}$-minimal clones $\langle f\rangle$ (see [Län-P 84]): *A function* $f \in O_A^{(1)} \setminus C$ *generates a* $\mathcal{C}$*-minimal clone iff* $f^2 \in C$ *or* $f^2 = f$ *or there is a prime* p *with* $f^p = e_1^1$.

3.8. The above results suffice in order to dertermine all $\mathcal{C}$-minimal clones on $A = \{0, 1, 2\}$. The minimal clones $F' \not\subseteq \mathcal{C}$ are taken from [Csá 83], the unary functions generating a $\mathcal{C}$-minimal clone can easily be found by 3.7.
It remains to consider binary operations (since the conditions in 3.6(2) can be satisfied only for $\mathcal{C}$-sharp functions of arity less than three).
Thus it suffices to check $\mathcal{C}$-sharp functions $f \in O_A^{(2)}$ whether they generate a $\mathcal{C}$-minimal clone. Assume that $\langle f\rangle$ does not contain a minimal clone $F' \not\subseteq \mathcal{C}$. By Lemma 3.5(2), for C-sharp f we have $f(x, x) = c$ for some $c \in A$. Moreover, by 3.6, it remains to check only those functions with $f(x, c) = f(c, x) = c$ which fulfil condition 3.3(2a) (since 3.3(2b) is satisfied trivially). By 3.3, these functions generate the $\mathcal{C}$-sharp-minimal clones.
All neccessary checks can be done by a computer (for condition 3.3(2a) using the algorithm mentioned in 3.4).

3.9. The $\mathcal{C}$-minimal clones on $A = \{0, 1, 2\}$.
For a permutation $\sigma : A \to A : a \mapsto a^\sigma$ the mapping

$$(\ )^\sigma : O_A \to O_A : f \mapsto f^\sigma,$$

where f^σ is defined by

$$f^\sigma(x_1, \ldots, x_n) := (f(x_1^{\sigma^{-1}}, \ldots, x_n^{\sigma^{-1}}))^\sigma,$$

is an inner automorphism of the clone O_A. Two clones $F, F' \le O_A$ are called *dual* if $F' = F^\sigma := \{f^\sigma \mid f \in F\}$ for some permutation $\sigma \in S_A$, i.e. if they are isomorphic under some inner automorphism of O_A (then the algebras with clones F, F' are weakly isomorphic). Thus all clones dual to F have the same structure as F and are completely determined by F.
Therefore, for every class of dual clones we shall mention only one representative. There are 26 $\mathcal{C}$-minimal clones up to duality. They can be generated by the following functions (most of them are minimal clones and taken from [Csá 83]), in brackets we add the number of dual clones:

(A) *Clones generated by unary operations:*

$$u_1(\#3),\ u_2(\#6),\ u_7(\#3),\ u_{15}(\#1),$$

where u_i is defined by $i = 9u_i(0) + 3u_i(1) + u_i(2)$.

(B) *Clones generated by binary operations:*

$$b_0(\#3),\ b_8(\#6),\ b_{10}(\#6),\ b_{11}(\#3),\ b_{16}(\#6),\ b_{17}(\#6),\ b_{26}(\#3),$$
$$b_{33}(\#3),\ b_{35}(\#6),\ b_{68}(\#3),\ b_{178}(\#2),\ b_{624}(\#1),\ b'_1(\#6),\ b'_5(\#3),$$

where b_i denotes the idempotent binary operation with $i = 3^5 b_i(0,1) + 3^4 b_i(0,2) + 3^3 b_i(1,0) + 3^2 b_i(1,2) + 3b_i(2,0) + b_i(2,1)$ and b'_i is given by $i = 3b'_i(1,2) + b'_i(2,1)$ and $b'_i(x,y) = 0$ for all other arguments.

(C) *Clones generated by ternary operations:*

$$m_0(\#3),\ m_{109}(\#3),\ m_{624}(\#1),$$
$$s_0(\#3),\ s_8(\#6),\ s_{26}(\#3),\ s_{76}(\#3),\ s_{424}(\#1),$$

where m_i is defined by

$$i = 3^5 m_i(0,1,2) + 3^4 m_i(0,2,1) + 3^3 m_i(1,0,2) + 3^2 m_i(1,2,0) + 3m_i(2,0,1) + m_i(2,1,0)$$

and $m_i(x,x,y) = m_i(x,y,x) = m_i(y,x,x) = x$,
while s_i is defined by

$$i = 3^5 s_i(0,1,2) + 3^4 s_i(0,2,1) + 3^3 s_i(1,0,2) + 3^2 s_i(1,2,0) + 3s_i(2,0,1) + s_i(2,1,0)$$

and $s_i(x,y,z) = x$ for all other arguments.

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