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Reinhard Pöschel

A general Galois theory for operations and relations and concrete characterization of related algebraic structures

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ABSTRACT

The property of an operation to preserve a relation induces a Galois connection between sets of operations and relations, resp. This Galois connection (Pol - Inv) for operations and relations on an arbitrary set will be investigated in the present paper(part 1). The Galois closed sets can be characterized as local closures of clones of operations or relations, resp. These results are applied to concrete characterization problems (part 2). In particular, the concrete characterization of automorphism groups, endomorphism semigroups, subalgebra lattices and congruence lattices of universal (or relational) algebras is treated in detailed form.

ZUSAMMENFASSUNG

Die Eigenschaft einer Funktion, eine Relation zu bewahren induziert eine Galoisverbindung zwischen Funktionen- und Relationenmengen. In der vorliegenden Arbeit wird diese Galoisverbindung (Pol - Inv) für Operationen und Relationen auf einer beliebigen Menge untersucht (Teil 1). Die Galois-abgeschlossenen Mengen werden als lokale Abschließungen von Operationen- bzw. Relationenklons characterisiert. Diese Ergebnisse werden auf konkrete Charakterisierungsprobleme angewendet (Teil 2). Dabei wird besonders auf die konkrete Charakterisierung von Automorphismengruppe, Endomorphismenhalbgruppe, Unteralgebrenverband und Kongruenzrelationenverband universaler (z.T. auch relationaler) Algebren eingegangen.

РЕЗЮМЕ

Свойство функции сохранять отношение индуцирует соотношение Галуа между множествами функций и отношений. В данной работе исследуется это соотношение Галуа (Pol-Inv) для операций и отношений на произвольном множестве /часть 1/. Галуа-замкнутые множества характеризуются как локальные замыкания клонов операций или, соответственно, отношений. Эти результаты применяются для проблем конкретной характеризации /часть 2/. При этом особое внимание уделяется конкретной характеризации групп автоморфизмов, полугрупп эндоморфизмов, структур подалгебр и конгруенции универсальных /или релациональных/ алгебр.

CONTENTS

INTRODUCTION

Every universal algebra $\mathfrak{A} = \langle A;F \rangle$ is associated with so-called r e l a t e d s t r u c t u r e s , e.g., with the lattices Sub $\mathfrak{A}$ and Con $\mathfrak{A}$ of its subalgebras and congruences respectively or with its group Aut $\mathfrak{A}$ of automorphisms.

Such related structures have a common property: for instance, $B \in$ Sub $\mathfrak{A}$, $\Theta \in$ Con $\mathfrak{A}$ and $f \in$ Aut $\mathfrak{A}$ can be considered as relations of a special kind (here unary or equivalence relations and permutations $f: A \longrightarrow A$ considered as subsets of $A \times A$) which are preserved by all (fundamental) operations $g \in F$ of $\mathfrak{A} = \langle A;F \rangle$. Therefore, in general, we are interested in relations on A which are i n v a r i a n t for (i.e. preserved by) all $g \in F$.

The property "operation preserves relation" induces a Galois connection between sets of operations and relations, resp. For a set Q (F, resp.) of relations (operations), let Pol Q (Inv F) be the set of all operations (relations) which preserve (are invariant for) all relations in Q (operations in F, respectively). Then the operators Pol - Inv establish the mentioned Galois connection.

That what we call "General Galois theory for operations and relations" includes mainly the following topics:

a) Investigation of the Galois connection Pol - Inv (and several modifications and restrictions of it);
b) Characterization of the Galois closed sets;
c) Investigation of properties of operational systems by means of properties of related systems of invariant relations (and vice versa).

It turns out that there are close connections (in particular of b)) to so-called c o n c r e t e c h a r a c t e r i z a t i o n p r o b l e m s , that means, e.g., the characterization of the lattices Sub $\mathfrak{A}$, Con $\mathfrak{A}$ and of the group Aut $\mathfrak{A}$ for all algebras with underlying set A as sets of subsets of A , sets of partitions on A and sets of permutations on A , resp.

In fact, a set Q of relations is the set of all invariant relations of a universal algebra $\mathfrak{A} = \langle A;F \rangle$ iff Q = Inv Pol Q. The only problem now consists in finding suitable closure operations to define "clones of relations" [Q] in such a way that they coincide with the Galois closure Inv Pol Q.

Note that - by the above observations (for more details see §15) - every solution of a concrete characterization problem provides a characterization of a Galois closure and vice versa.

In the present paper we are mainly concerned with results on the Galois connection Pol - Inv (Part 1) and the application of these results to concrete characterization problems (Part 2). First of all we will list some references which also reflect the historical developement of our topic.

General Galois theory for operations and relations: +)

KRASNER(1938,...,1976)(mainly for permutation groups and transformation semigroups); KUZNECOV(1959); GEIGER(1968); BODNARČUK/KALUŽNIN/KOTOV/ROMOV(1969); PÖSCHEL/KALUŽNIN (1979)(for finite sets A); ROSENBERG(1972,...,1979); POIZAT(1971,...,1979), KRASNER/POIZAT(1976)(most general case); PÖSCHEL(1973)(for heterogeneous algebras); LECOMTE (1976/77); SAUER/STONE(1977/78); ROMOV(1977); PÖSCHEL(1979); FLEISCHER(1978).

Abstract characterization problems (i.e. characterization of related structures "up to isomorphism"):

BIRKHOFF(1946); BIRKHOFF/FRINK(1948); GRÄTZER/SCHMIDT (1963); E.T.SCHMIDT(1963/64); GRÄTZER/LAMPE(1967); JÓNSSON(1974); SCHEIN/TROHIMENKO(1979).

Concrete characterization problems

Automorphism group: KRASNER(1950); ARMBRUST/SCHMIDT(1970); JÓNSSON(1968); PŁONKA(1968); JÓNSSON(1972); GOULD(1972a) (for algebras of finite type); SZABÓ(1975), BREDIHIN(1976) (for local automorphisms).

+) cf. also §4, p.32.

Endomorphism semigroup (Problem 3 in [Gr]): LAMPE(1968); GRÄTZER/LAMPE(1968); JEŽEK(1972); STONE(1969/75); SZABÓ(1978); (cf.[Jón74]).

Subalgebra lattice: BIRKHOFF/FRINK(1948)(cf.[Jón72]); JOHNSON/SEIFERT(1967)(for unary algebras); GOULD(1968) (in particular for algebras of finite type).

Congruence lattice(Problem 2 in [Gr]): ARMBRUST(1970); S.BURRS/H.CRAPO/A.DAY/D.HIGGS/W.NICKOLS(cf.[Jón72(p. 174)]); QUACKENBUSH/WOLK(1971); JÓNSSON(1972(Thm. 4.4.1)); DRAŠKOVIČOVA(1974); WERNER(1974).

Automorphism group & subalgebra lattice: STONE(1972); GOULD(1972b)(for algebras of finite type).

Endomorphism semigroup & subalgebra lattice: SAUER/STONE(1977a); (cf.[Jón74]).

Automorphism group & congruence lattice: WERNER(1974)(Problem 4 and conjecture [We74(p. 452)]).

All structures together (automorphisms, endomorphisms, subalgebras, congruences): SZABÓ(1978); PÖSCHEL(1979).

General systems of relations (in particular subalgebra lattice of cartesian powers of algebras): BODNARČUK/KALUŽNIN/KOTOV/ROMOV(1969); DANTONI(1969); PÖSCHEL/KALUŽNIN(1979); SZABÓ(1978); PÖSCHEL(1979); ROSENBERG(1978).

Explicitely we want to mention the work of M. Krasner, D. Geiger and L. Szabó. M. Krasner was the first who systematically investigated Galois connections between (unary) operations and relations (therefore, in §16, we introduce the notion "Krasner-clone"). D. Geiger obtained the main results of [Bo/Kal], too, and outlined how to investigate the infinite case. L. Szabó characterized in [Sz78] clones of relations by a closure property with respect to so-called formula schemes and (independently) obtained essentially the same results concerning some of the concrete characterization problems.

We shortly outline now the content of the present paper.

In §1 we introduce or recall nearly all used notions and notations and give some preliminary results.

In §2 clones of operations are considered and some properties with respect to invariant relations are given.

In §3 clones of relations are introduced (and motivated). Some closure properties of these clones are given.

In §4 the Galois closed sets of operations and relations (with respect to Pol - Inv) are characterized as local closures of clones of operations and relations, respectively. Restrictions on the arity of operations and relations under consideration are investigated, too.

In §5 we consider the concrete characterization problem how to characterize those sets F of operations which are "related" to some relational system $\mathfrak{B} = \langle A; Q \rangle$ (related means e.g. $F = \mathrm{Aut}\,\mathfrak{B}$, $F = \mathrm{Hom}(\mathfrak{B}^m, \mathfrak{B})$, $F = \mathrm{Pol}\ Q$, etc.).

In §6 the following concrete characterization problem is considered: How to characterize those sets Q of relations which are "related" to some universal algebra $\mathfrak{A} = \langle A; F \rangle$ (related means e.g. $Q = \mathrm{Con}\,\mathfrak{A}$, $Q = \mathrm{Aut}\,\mathfrak{A}$, $Q = \mathrm{Sub}\,\mathfrak{A}^n$, $Q = \mathrm{Inv}\ F$). There are also some contributions to the case where $\mathfrak{A}$ is of bounded rank or has a finite system of (fundamental) operations.

The problems considered in §6 will be specialized in §7 where we answer the question for a simultaneous concrete characterization of automorphism group, endomorphism monoid, subalgebra and congruence lattices of universal algebras.

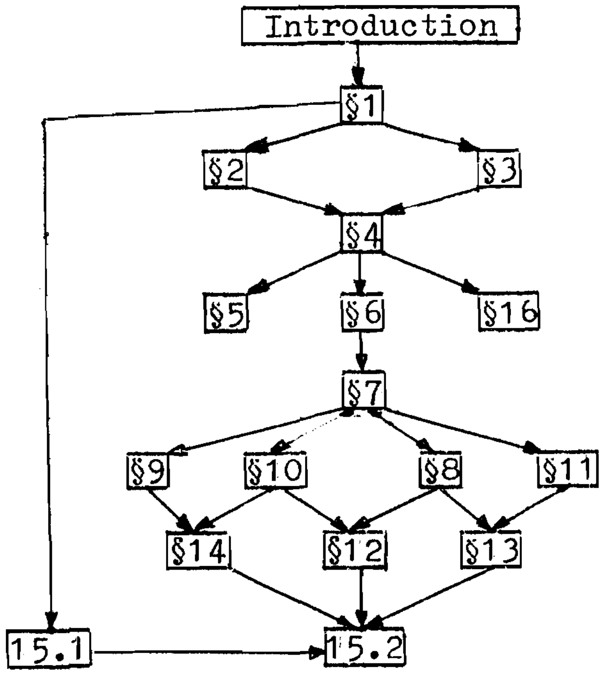
Of course, these special related structures are of great algebraic interest. Therefore the concrete characterization of (one or simultaneously two of) these structures will be treated in detailed form in §§8-14. A survey on many, partially well-known, results is given and the application of the General Galois theory is demonstrated. This yields sometimes new results, sometimes new proofs for known results. The results of §6 provide an answer to all concrete characterization problems of related structures of algebras in terms of clones of relations (or operations). Nevertheless we think these results not to be final ones because in special cases simpler characterizations might be obtained (and can be obtained as shown sometimes in §§8-14). Thus, more or less explicitely, a lot of problems for further investigations is contained in §§8-14.

In §15 we will explaine once more explicitely the interdependences of concrete characterization problems and the characterization of Galois closed sets with respect to a canonically related Galois connection. From this point of view we summarize the results given in previous paragraphs(§§4-14) by listing which results characterize properly which Galois connection.

In §16 we investigate invariant relations of unary, in particular bijective operations. These relations (are characterized as Galois closed sets of the corresponding Galois connection Inv - End or Inv - Aut and) form so-called Krasner-clones (of 1st or 2nd kind). The inclusions Inv Aut Q $\supseteq$ $\supseteq$ Inv End Q $\supseteq$ Inv Pol Q suggest that Krasner-clones might be characterized by closure properties which are somewhat stronger than those for ordinary clones of relations. This will be clarified in §16.

In the paper, all definitions, propositions etc. are numbered consecutively; the first number which occurs on a page is marked also on the top of this page. The end of a proof (or of a statement with easy proof) is marked by ■. References are given in brackets, sometimes with some further informations in parenthesis, e.g. [Jón72(Thm. 3.6.4)].

The following picture shows the interdependence of the paragraphs:



The objective of the paper presented here is the introduction of a General Galois theory (for operations and relations) as a helpful background for concrete characterizations of related algebraic structures. However, there remains many things to do; e.g., the involved notion of a clone of relations needs much more detailed investigations in order to get conditions which are "easy" to check.

Almost all paragraphs of the present paper had been written up during 1977, but for nearly two years the author unfortunately could not find time enough to write up the final version. Some results were outlined in a lecture given at the conference on "Allgemeine Algebra" in Klagenfurt (Austria, May 1978); a short note (in which the present paper was refered to as a preprint) was published in the proceedings of this conference (cf. [Pö79]).

ACKNOWLEDGEMENT. I want to express my thanks to Professor L.A. Kalužnin who directed my attention to the investigation of a "General Galois theory" and showed a continuous interest on my work.

Part 1

CLONES AND THE GALOIS CONNECTION Pol-Inv

§1 Definitions and Preliminaries

1.1 Let A be an arbitrary set (with $|A| \geq 2$) and let $m,n \in \mathbb{N}$, where $\mathbb{N} = \{1,2,3,\ldots\}$ denotes the set of all natural numbers (without zero). Let $\underline{m} = \{0,1,\ldots,m-1\}$. With

$$O_A^{(n)} = \{f \mid f : A^n \longrightarrow A\} \text{ and}$$

$$R_A^{(m)} = \{\varrho \mid \varrho \subseteq A^m\} \text{ , resp.,}$$

we denote the sets of all n-ary operations and m-ary relations, resp., on the set A. Let

$$O_A = \bigcup_{n \in \mathbb{N}} O_A^{(n)} \quad \text{and} \quad R_A = \bigcup_{m \in \mathbb{N}} R_A^{(m)}.$$

Universal algebras $\langle A;(f_i)_{i\in I}\rangle$ or relational algebras (i.e. relational systems [Gr(p.8)]) $\langle A;(\varrho_i)_{i\in I}\rangle$ of some similiarity type are denoted shortly by $\langle A;F\rangle$ or $\langle A;Q\rangle$ where $F=\{f_i \mid i\in I\}$ and $Q=\{\varrho_i \mid i\in I\}$, respectively, because we are not interested in the type of these algebras.

All considerations will be restricted to finitary (except nullary) operations and relations. But it should be mentioned that most of the given results can be generalized to infinitary operations and relations (for this purpose one has to substitute $\mathbb{N}$ by a suitable chosen limit ordinal - or cardinal - number); we refer to [Kr/Poi],[Poi80] for this approach. To avoid some technical modifications, nullary operations (constants) are not considered in universal algebras.

For our purposes nullary operations can be replaced by unary constant operations.

The components of elements $x \in A^m$ of the cartesian power A^m of A are denoted by $x(i)$ ($i \in \underline{m}$), i.e.,

$$x = (x(i))_{i \in \underline{m}} \quad \text{or}$$

$$x = (x(0), \ldots, x(m-1)) \text{ (sometimes also } x = (x_0, \ldots, x_{m-1})\text{)}.$$

If $f \in O_A^{(n)}$, $x \in A^n$, we write fx for $f(x_0, \ldots, x_{n-1})$.
For $r_1, \ldots, r_n \in A^m$,

$$f(r_1, \ldots, r_n)$$

denotes the m-tupel $(f(r_1(i), \ldots, r_n(i)))_{i \in \underline{m}}$.

1.2 The most important notion for our investigations is the following one: A relation $\varrho \in R_A^{(m)}$ is said to be *invariant* for an operation $f \in O_A^{(n)}$ (or f *preserves* ϱ, f is a *polymorphism* of ϱ, f *admits* ϱ, ϱ is *stable* for f) if $f(r_1, \ldots, r_n)$ belongs to ϱ whenever $r_1, \ldots, r_n \in \varrho$. The empty relation $\emptyset$ is preserved by every operation $f \in O_A$.
Note that f preserves ϱ iff ϱ is (the base set of) a subalgebra of $\langle A; f\rangle^m$ or, equivalently, iff f is homomorphism of $\langle A; \varrho\rangle^n$ into $\langle A; \varrho\rangle$.

1.3 Sometimes, for $f \in O_A^{(n)}$, we consider the relation

$$f^{\bullet} = \{(x_0, \ldots, x_{n-1}, x_n) \in A^{n+1} \mid f(x_0, \ldots, x_{n-1}) = x_n\}$$

instead of the operation f. Thus, for $F \subseteq O_A$, the "dotted"

$$F^{\bullet} = \{f^{\bullet} \mid f \in F\}$$

will remind us that we have to handle the operation like relations. It is well known that $g \in O_A$ preserves $f^{\bullet} \in O_A^{\bullet}$ iff g and f commute.

<u>1.4</u> For $F \subseteq O_A$ and $Q \subseteq R_A$ we use the following notations:

$F^{(n)} := F \cap O_A^{(n)}$, $Q^{(m)} := Q \cap R_A^{(m)}$ $(n,m \in \mathbb{N})$,

$\mathrm{Pol}\ Q = \mathrm{Pol}_A Q := \{f \in O_A \mid f \text{ preserves all } \varrho \in Q\}$
(*polymorphisms* of Q),

$\mathrm{Pol}_A F := \mathrm{Pol}_A F^{\bullet}$,

$\mathrm{Inv}\ F = \mathrm{Inv}_A F := \{\varrho \in R_A \mid \varrho \text{ is invariant for all } f \in F\}$
(*invariants* of F),

$\mathrm{Pol}^{(n)} Q := (\mathrm{Pol}\ Q)^{(n)}$, $\mathrm{Inv}^{(m)} F := (\mathrm{Inv}\ F)^{(m)}$,

$\mathrm{End}_A Q := \mathrm{Pol}_A^{(1)} Q$ (*endomorphisms* of $\langle A;Q\rangle$),

$\mathrm{Pol}_A^{(1-1)} Q := \{f \in \mathrm{End}\ Q \mid f \text{ is injective}\}$,

$\text{w-Aut}_A Q := S_A \cap \mathrm{Pol}_A Q$ (*weak automorphisms**)),

$\mathrm{Aut}_A Q := \{f \in S_A \mid \{f, f^{-1}\} \subseteq \mathrm{Pol}_A Q\}$ (*automorphisms*),

where S_A denotes the full *symmetric group* (of all permutations) on A. We write shortly $\mathrm{Pol}\ \varrho$, $\mathrm{Inv}\ f$,... for $\mathrm{Pol}\{\varrho\}$, $\mathrm{Inv}\{f\}$,... $(\varrho \in R_A,\ f \in O_A)$. Clearly we have:

$$\mathrm{Pol}\ Q \supseteq \dots \supseteq \mathrm{Pol}^{(n+1)} Q \supseteq \mathrm{Pol}^{(n)} Q \supseteq \dots \supseteq \mathrm{End}\ Q \supseteq \mathrm{Pol}^{(1-1)} Q \supseteq$$
$$\supseteq \text{w-Aut}\ Q \supseteq \mathrm{Aut}\ Q\ ,$$
$$\mathrm{Inv}\ F \supseteq \dots \supseteq \mathrm{Inv}^{(n+1)} F \supseteq \mathrm{Inv}^{(n)} F \supseteq \dots \supseteq \mathrm{Inv}^{(1)} F\ .$$

<u>1.5</u> <u>Remarks.</u> (i) $\mathrm{Pol}^{(n)} Q$ is the set of all homomorphisms of the relational algebra $\langle A;Q\rangle^n$ into $\langle A;Q\rangle$.

(ii) $\mathrm{Inv}^{(m)} F$ is the set of all (base sets of) subalgebras of $\langle A;F\rangle^m$; in particular, $\mathrm{Inv}^{(1)} F$ is the set $\mathrm{Sub}\,\mathfrak{A}$ of all subalgebras of $\mathfrak{A} = \langle A;F\rangle$.

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*) In [Pö79] they are called automorphisms.

(iii) $(\mathrm{Pol}\ F)^{\bullet} = O_A^{\bullet} \cap \mathrm{Inv}\ F$.

(iv) $(\mathrm{Pol}^{(n)}F)^{\bullet} \subseteq \mathrm{Inv}^{(n+1)}F$, $n \in \mathbb{N}$.

1.6 We define, $f \in S_A$ strongly preserves $\varrho \in R_A^{(m)}$ if $\{f(r) \mid r \in \varrho\} = \varrho$, i.e., if f is an isomorphism(cf.[Gr]) of the relational system $\langle A;\varrho\rangle$ onto $\langle A;\varrho\rangle$. Then we have:

a) $\mathrm{Aut}_A Q$ is the set of all $f \in S_A$ which strongly preserve each $\varrho \in Q \subseteq R_A$.

(Proof : $\{f(r) \mid r \in \varrho\} = \varrho \implies f^{-1}(r) \in \varrho$ for all $r \in \varrho$, i.e. f^{-1} also preserves $\varrho \in Q$. ■). One easily sees that $\mathrm{Aut}_A Q$ is a subgroup of S_A.

b) For finite A we have w-Aut Q = Aut Q (since $f \in$ w-Aut Q implies $f^n = f \circ \ldots \circ f \in$ w-Aut Q and $[\exists n \in \mathbb{N}: f^{-1} = f^n$ for all $f \in S_A)]$, but

c) in general, w-Aut Q properly contains Aut Q.

(Example : $A = Z = \{\ldots,-1,0,1,2,\ldots\}$, $f: x \mapsto x+1$, $\varrho = \mathbb{N} \in R_A^{(1)}$, $f \in$ w-Autϱ but $f^{-1} \notin$ w-Autϱ , i.e. w-Aut$\varrho \supsetneq$ Autϱ .)

d) Nevertheless, we have

$$\text{w-Aut}_A Q = \mathrm{Aut}_A Q \quad \text{also for infinite } A$$

if for all non-trivial $\varrho \in Q$ holds $|\varrho| < \aleph_0$ or $\varrho = g^{\bullet}$ for some $g \in O_A$. or $\neg\varrho \in Q$

(The proof is left to the reader as an easy exercise.) ■

1.7 Proposition. The operators Pol and Inv define a Galois connection between the subsets of R_A and the subsets of O_A. In particular we have

$$F \subseteq F' \subseteq O_A \implies \mathrm{Inv}\ F \supseteq \mathrm{Inv}\ F'$$

$$Q \subseteq Q' \subseteq R_A \implies \mathrm{Pol}\ Q \supseteq \mathrm{Pol}\ Q',$$

$$F \subseteq \mathrm{Pol}\ \mathrm{Inv}\ F, \qquad Q \subseteq \mathrm{Inv}\ \mathrm{Pol}\ Q,$$

$$\mathrm{Pol}\ Q = \mathrm{Pol}\ \mathrm{Inv}\ \mathrm{Pol}\ Q, \qquad \mathrm{Inv}\ F = \mathrm{Inv}\ \mathrm{Pol}\ \mathrm{Inv}\ F,$$

$$\mathrm{Pol} \bigcup_{i\in I} Q_i = \bigcap_{i\in I} \mathrm{Pol}\ Q_i, \qquad \mathrm{Inv} \bigcup_{i\in I} F_i = \bigcap_{i\in I} \mathrm{Inv}\ F_i. \ \blacksquare$$

1.8 For $F \subseteq O_A$, $Q \subseteq R_A$ and $\sigma \subseteq A^m (m \in \mathbb{N})$ we define

$$\Gamma^Q(\sigma) := \bigcap \{\varrho \in R_A^{(m)} \mid \sigma \subseteq \varrho \in Q\},$$

$$\Gamma_F(\sigma) := \bigcap \{\varrho \in R_A^{(m)} \mid \sigma \subseteq \varrho \in \mathrm{Inv}_A F\}.$$

$\Gamma_F(\sigma)$ is (the base set of) the subalgebra of $\langle A;F\rangle^m$ generated by σ and belongs to Inv F (since Inv F is closed under arbitrary intersections), cf. 1.5(ii). Thus

$$\Gamma_F(\varrho) = \varrho = \bigcup_{\substack{\sigma\subseteq\varrho \\ |\sigma|<\aleph_0}} \Gamma_F(\sigma) \quad \text{for all} \quad \varrho \in \mathrm{Inv}\ F.$$

Moreover, $\Gamma_F(\varrho) = \varrho \iff \varrho \in \mathrm{Inv}\ F$. ■

1.9 For the investigation of "local properties" of operations and relations we define the following *local closure operators* ($s \in \mathbb{N}$, $F \subseteq O_A$, $Q \subseteq R_A$):

$$s\text{-Loc } F := \{f \in O_A^{(n)} \mid \forall B \subseteq A^n, |B| \leq s, \exists g \in F: f|B = g|B; \ n \in \mathbb{N}\}.$$

This is the set of all n-ary operations ($n \in \mathbb{N}$) with the property that for every subset B of A^n with at most s elements there exists a member of F that agrees with f on B.

$$\mathrm{Loc}\ F := \bigcap_{s\in\mathbb{N}} s\text{-Loc } F,$$

i.e., f belongs to Loc F iff f agrees with some $g \in F$ on every finite subset.

$$s\text{-LOC } Q := \{\varrho \in R_A \mid \forall B \subseteq \varrho, |B| \leq s \ \exists \sigma \in Q: B \subseteq \sigma \subseteq \varrho\}$$

This is the set of all m-ary relations ϱ ($m \in \mathbb{N}$) such that for every subset B of ϱ with at most s elements there exists a member of Q that agrees with ϱ on B and is contained in ϱ.

$$\text{LOC } Q := \bigcap_{s \in \mathbb{N}} s\text{-LOC } Q.$$

Remarks. (i) $(\text{Loc } F)^{(n)} = \text{Loc } F^{(n)}$, $(\text{LOC } Q)^{(n)} = \text{LOC } Q^{(n)}$ ($n \in \mathbb{N}$).
(ii) A relation ϱ belongs to s-LOC Q (if and) only if for all $B \subseteq A^m$ with $|B| \leq s$ there exists a $\sigma \in Q$ with $\varrho \cap B \subseteq \sigma \subseteq \varrho$. We have $\emptyset \in s\text{-LOC } Q \iff \emptyset \in Q$.

In the next propositions we collect some properties of these local operators.

1.10 Proposition. Let $F \subseteq F' \subseteq O_A$, $Q \subseteq Q' \subseteq R_A$, $s, s' \in \mathbb{N}$. Then:
(i) $1\text{-Loc } F \supseteq \ldots \supseteq s\text{-Loc } F \supseteq (s+1)\text{-Loc } F \supseteq \ldots \supseteq \text{Loc } F \supseteq F$.
(ii) $1\text{-LOC } Q \supseteq \ldots \supseteq s\text{-LOC } Q \supseteq (s+1)\text{-LOC } Q \supseteq \ldots \supseteq \text{LOC } Q \supseteq Q$.
(iii) All operators defined in 1.9 are closure operators; in particular we have:

$s\text{-Loc } F \subseteq s\text{-Loc } F'$, $\quad s\text{-LOC } Q \subseteq s\text{-LOC } Q'$,

$\text{Loc Loc } F = \text{Loc } F$, $\quad \text{LOC LOC } Q = \text{LOC } Q$,

$s\text{-Loc } s'\text{-Loc } F = s''\text{-Loc } F$, $s\text{-LOC } s'\text{-LOC } Q = s''\text{-LOC } Q$, where $s'' = \min\{s, s'\}$,

$\text{Loc } s\text{-Loc } F = s\text{-Loc Loc } F = s\text{-Loc } F$,

$\text{LOC } s\text{-LOC } Q = s\text{-LOC LOC } Q = s\text{-LOC } Q$.

(iv) For finite A we have

$\text{Loc } F = F$ and $\text{LOC } Q = Q$ for all $F \subseteq O_A$, $Q \subseteq R_A$.

(v) $s\text{-LOC } F^\bullet = F^\bullet$ *for* $s \geqq 2$.

The **proofs** easily follow from the definitions. ■

1.11 Proposition.

(a) $\mathrm{Loc}(\mathrm{Pol}\ Q) = \mathrm{Pol}\ Q$ *for* $Q \subseteq R_A$.

(b) $s\text{-Loc}(\mathrm{Pol}\ Q) = \mathrm{Pol}\ Q$ *for* $Q \subseteq R_A^{(1)} \cup \dots \cup R_A^{(s)}$, $s \in \mathbb{N}$.

(a') $\mathrm{LOC}(\mathrm{Inv}\ F) = \mathrm{Inv}\ F$ *for* $F \subseteq O_A$.

(b') $s\text{-LOC}(\mathrm{Inv}\ F) = \mathrm{Inv}\ F$ *for* $F \subseteq O_A^{(1)} \cup \dots \cup O_A^{(s)}$, $s \in \mathbb{N}$.

(c) *We mention here that, in general,* Loc G *is not contained in* S_A *if* $G \subseteq S_A$. *But all* $f \in \mathrm{Loc}\ S_A$ *are injective.*

Proof. For (a)-(b') we have to show that the left side is contained in the right one.

(b): Let $f \in (s\text{-Loc Pol } Q)^{(n)}$ and $\varrho \in Q^{(s')}$, $s' \leqq s$. For $r_1, \dots, r_n \in \varrho$ and $B = \{(r_1(i), \dots, r_n(i)) \mid i \in \underline{s'}\} \subseteq A^n$ there exists a $g \in \mathrm{Pol}\ Q$ such that $f|B = g|B$, consequently $f(r_1, \dots, r_n) = g(r_1, \dots, r_n) \in \varrho$. Thus $f \in \mathrm{Pol}\ Q$. (The proof of (a) runs with the same argument).

(b'): Let $\varrho \in (s\text{-LOC Inv } F)^{(m)}$ and $f \in F^{(s')}$, $s' \leqq s$. For $r_1, \dots, r_{s'} \in \varrho$ and $B = \{r_1, \dots, r_{s'}\} \subseteq \varrho$ there exists a $\sigma \in \mathrm{Inv}\ F$ such that $\{r_1, \dots, r_{s'}\} \subseteq \sigma \subseteq \varrho$, consequently $f(r_1, \dots, r_{s'}) \in \sigma \subseteq \varrho$. Thus $\varrho \in \mathrm{Inv}\ F$. ((a') can be proved in the same manner).

(c): Example: $f : \mathbb{N} \longrightarrow \mathbb{N} : x \longmapsto x+1 \in (\mathrm{Loc}\ S_{\mathbb{N}}) \setminus S_{\mathbb{N}}$. ■

Another characterization of the operator LOC is given in the next propositions. First of all we need the following

1.12 Definition. Recall, a set $\mathcal{T}$ of sets is called **directed** (upwards) if for all $X, Y \in \mathcal{T}$ there exists a $Z \in \mathcal{T}$

such that $X \cup Y \subseteq Z$. Now, let us call a set $\mathcal{T}$ of sets to be s-directed ($s \in \mathbb{N}$) if for all $X_1, \dots, X_s \in \mathcal{T}$ and $r_1 \in X_1, \dots, r_s \in X_s$ there exists a $Z \in \mathcal{T}$ such that $\{r_1, \dots, r_s\} \subseteq Z$. (In the terminology of M.Gould[Go68(p.472)], $\mathcal{T}$ has the s-ary containment property.)

1.13 Proposition(cf.[Go68(Th.1.1)]). Let $Q \subseteq R_A$ be closed under arbitrary intersections. Then the following holds:

(i) $\mathrm{LOC}\ Q = \{\bigcup \mathcal{T} \mid \emptyset \neq \mathcal{T} \subseteq Q$ and $\mathcal{T}$ is directed$\}$.

(ii) $s\text{-}\mathrm{LOC}\ Q = \{\bigcup \mathcal{T} \mid \emptyset \neq \mathcal{T} \subseteq Q$ and $\mathcal{T}$ is s-directed$\}$, $s \in \mathbb{N}$.

Thus Q is closed under (s-)LOC iff Q is closed under taking arbitrary unions of (s-)directed systems of relations of Q. In particular, 1-LOC Q is closed under arbitrary unions.

Proof. (i):"$\Rightarrow$": Let $\varrho \in \mathrm{LOC}\ Q$. Then for finite $B \subseteq \varrho$ there exists at least one $\sigma \in Q$ with $B \subseteq \sigma \subseteq \varrho$, and we can define

$$\sigma_B := \bigcap \{\sigma \in Q \mid B \subseteq \sigma \subseteq \varrho\}.$$

$\mathcal{T} = \{\sigma_B \mid B \subseteq \varrho$, B finite$\}$ is a directed system of elements of Q (since $\sigma_B \cup \sigma_{B'} \subseteq \sigma_{B \cup B'}$). Now, since $\varrho = \bigcup\{B \mid B \subseteq \varrho$, B finite$\} \subseteq \bigcup \mathcal{T} \subseteq \varrho$, ϱ is contained in the right side of (i).

"$\Leftarrow$": Let $\varrho = \bigcup \mathcal{T} \in R_A^{(m)}$ for a directed system $\mathcal{T}$ of relations of Q and let $B = \{r_1, \dots, r_t\} \subseteq \varrho$ ($t \in \mathbb{N}$). Then there exist $\varrho_1, \dots, \varrho_t \in \mathcal{T}$ such that $r_i \in \varrho_i$ ($1 \leq i \leq t$). Because $\mathcal{T}$ is directed there exists a $\sigma \in \mathcal{T} \subseteq Q$ such that $B \subseteq \varrho_1 \cup \dots \cup \varrho_t \subseteq \sigma \subseteq \varrho$. Thus, by definition, $\varrho \in \mathrm{LOC}\ Q$.

(ii):"$\Rightarrow$": Let $\varrho \in s\text{-}\mathrm{LOC}\ Q$, $B \subseteq \varrho$, $|B| \leq s$, σ_B as above. Then $\mathcal{T} = \{\sigma_B \mid B \subseteq \varrho, |B| \leq s\}$ is an s-directed system: in fact, for $r_1 \in \sigma_{B_1}, \dots, r_s \in \sigma_{B_s}$ and $B = \{r_1, \dots, r_s\} \subseteq \varrho$ we have $\{r_1, \dots, r_s\} \subseteq \sigma_B \in \mathcal{T}$.

Thus $\varrho = \bigcup\mathcal{T}$ belongs to the right side of (ii). The opposite direction can be proved analogously to that of (i). ■

1.14 Proposition. Let $Q \subseteq R_A$ be closed under arbitrary intersections. Then the following conditions are equivalent ($s \in \mathbb{N}$ fixed):

(a) $Q = s\text{-LOC}\, Q$.

(b) $B \in Q$ if (and -clearly- only if) $\Gamma^Q(X) \subseteq B$ for all $X \subseteq B$ with $|X| \leq s$. (Notation cf. 1.8)

Proof. (a)⇒(b) follows from $B = \bigcup\{\Gamma^Q(X) \mid X \subseteq B, |X| \leq s\}$ by 1.13(ii) and the fact that all the $\Gamma^Q(X)$ form an s-directed system. (b)⇒(a): If $\mathcal{T} \subseteq Q$ is s-directed and $X \subseteq B := \bigcup\mathcal{T}$ with $|X| \leq s$ then, by definition, $X \subseteq Z$ for some $Z \in \mathcal{T}$, consequently, $\Gamma^Q(X) \subseteq \Gamma^Q(Z) = Z \subseteq \bigcup\mathcal{T} = B$. By (b) we get $\bigcup\mathcal{T} \in Q$, thus $s\text{-LOC}\, Q \subseteq Q$ by 1.13. ■

1.15 Remarks. The local operators introduced in 1.9 will reflect those properties (of universal or relational algebras) which are caused by the finitarity of the operations and relations under consideration. If we consider also infinitary operations and relations most of all further results (§§4-6) remain valid by deleting these local operators (cf. also [Kr/Poi], [Poi80]).

The investigation of algebraic structures by means of "local" properties (together with closure properties w.r.t. composition) is very obvious and often used in the literature (cf. for instance such notions as locally primal, locally affine complete(cf.[Pö/Kal(§5.5)]), interpolation properties for algebras(cf. [Pix], [Ist/K/Pix],[Hu/Nö])).

§2 Clones of operations

2.1 Let us recall the well-known notion of a clone: A set $F \subseteq O_A$ is called a *clone of operations* on A,

notation: $F \leq O_A$,

if

(i) F contains all *projections* e_i^n ($i \in \underline{n}$, $n \in \mathbb{N}$) defined by $e_i^n(x_0, \dots, x_{n-1}) = x_i$; and

(ii) For $g \in F^{(n)}$, $f_1, \dots, f_n \in F^{(m)}$ ($n, m \in \mathbb{N}$) the operation $g(f_1, \dots, f_n)$ (*composition*) defined by $g(f_1, \dots, f_n)x := g(f_1 x, \dots, f_n x)$, $x \in A^m$, also belongs to F .

For $F \subseteq O_A$, the clone generated by F is denoted by

$\langle F \rangle_{O_A}$, shortly $\langle F \rangle$.

This is the least clone of operations containing F. The set of all projections is a clone contained in every clone. The $f \in \langle F \rangle$ are also called *superpositions* of F .

With $F \leq S_A$ (or $F \leq O_A^{(1)}$)

we denote that F is a subgroup (subsemigroup with unit) of S_A (or $O_A^{(1)}$).

2.2 Remark. There are some other equivalent definitions of a clone (cf. e.g.[Schm.J.]); e.g., one can consider the *full function algebra* $\underline{O}_A = \langle O_A; e_0^2, \zeta, \tau, \Delta, \circ \rangle$ (cf. [Ma66,76],[Pö/Kal], where ζ, τ or Δ can produce any permutation or identification of variables of each $f \in O_A$ and $\circ$ is a special composition of two functions) and the clones are exactly the subalgebras of $\underline{O}_A$. By this way one can avoid the infinitely many superpostion operations in 2.1(ii).

<u>2.3</u> S_A and $O_A^{(1)}$ form a group and semigroup, resp., with respect to composition. For $F \subseteq S_A$ or $F \subseteq O_A^{(1)}$ the subgroup or subsemigroup (with unit e_0^1) generated by F will be denoted by $\langle F \rangle_{S_A}$ or $\langle F \rangle_{O_A^{(1)}}$, respectively.

For finite A (but not in general) we have $\langle F \rangle_{S_A} = \langle F \rangle_{O_A^{(1)}}$ for $F \subseteq S_A$.

Now we are able to give an "inner" characterization of $\Gamma_F(\sigma)$ (cf. 1.8) which does not use the invariants of F:

<u>2.4</u> Proposition([Pö/Kal(1.1.19)],[Gr(§9,Lemma 3)]). <u>Let</u> $F \subseteq O_A$ <u>and</u> $\sigma \in R_A$. <u>Then</u> (cf. 1.8)

$$\Gamma_F(\sigma) = \{g(r_1, \dots, r_n) \mid g \in \langle F \rangle^{(n)}, \{r_1, \dots, r_n\} \subseteq \sigma, n \in \mathbb{N}\}.$$

For the sake of completeness we sketch the well-known <u>proof</u>: Denote the right side with γ. One easily checks that γ is invariant for all $f \in F$, and because $\langle F \rangle$ contains all projections we have $\sigma \subseteq \gamma$, thus $\Gamma_F(\sigma) \subseteq \gamma$. At the other hand, $\Gamma_F(\sigma) \in$ Inv F implies $g(r_1, \dots, r_n) \in \Gamma_F(\sigma)$, i.e. $\gamma \subseteq \Gamma_F(\sigma)$. ∎

The following two lemmata show that the clone generating process and the local closure do not change invariant relations and that the local closure preserves the property of being a clone.

2.5 Lemma. *Let* $Q \subseteq R_A$ *and* $F \subseteq O_A$. *Then*

(i) $Pol_A Q$ *is a clone of operations, i.e.* $\langle Pol_A Q \rangle_{O_A} = Pol_A Q$;

(ii) $s\text{-}Loc\langle F \rangle$ *is a clone of operations* ($s \in \mathbb{N}$);

(iii) $Loc\langle F \rangle$ *is a clone of operations*.

Proof. (i): The composition of functions preserving $\varrho \in Q$ also preserves ϱ. (ii): Let $g \in s\text{-}Loc\, F^{(n)}$, $f_1, \dots, f_n \in s\text{-}Loc\, F^{(m)}$, $B = \{b_1, \dots, b_t\} \subseteq A^m$, $t \leq s$, and $B' := \{(f_1 b_i, \dots, f_n b_i) \mid 1 \leq i \leq t\}$. Then there exist $g' \in F^{(m)}$, $f_1', \dots, f_n' \in F^{(n)}$ such that $g'|B' = g|B'$, $f_i'|B = f_i|B$ $(1 \leq i \leq t)$. Therefore $g(f_1, \dots, f_n)$ coincides with $g'(f_1', \dots, f_n')$ on B, consequently $g(f_1, \dots, f_n) \in s\text{-}Loc\langle F \rangle$. (iii): Clearly, with $s\text{-}Loc\langle F \rangle$ also the intersection $Loc\langle F \rangle$ of all these is a clone. ∎

2.6 Lemma. *Let* F O_A. *Then we have:*

(i) $Inv_A^{(m)} F = Inv_A^{(m)} \langle F \rangle = Inv_A^{(m)} Loc\langle F \rangle = Inv_A^{(m)} s\text{-}Loc\langle F \rangle$ *for* $1 \leq m \leq s \in \mathbb{N}$;

(ii) $Inv_A F = Inv_A \langle F \rangle = Inv_A Loc\langle F \rangle$.

Proof. (i): Let $\varrho \in Inv_A^{(m)} F$. Because the sets in (i) must form (from left to right) a decreasing chain (cf. 1.7) it suffices to show $\varrho \in Inv^{(m)} s\text{-}Loc\ F$. Clearly $\varrho \in Inv^{(m)} \langle F \rangle$ (since superpositions of F also preserve ϱ (2.5(i))). Now, let $f \in s\text{-}Loc\langle F \rangle^{(n)}$ and $r_1, \dots, r_n \in \varrho$. By definition of the s-local closure (1.9, note $m \leq s$), there exists a $g \in \langle F \rangle$ such that $f(r_1, \dots, r_n) = g(r_1, \dots, r_n)$, consequently $f(r_1, \dots, r_n) \in \varrho$, i.e., f preserves ϱ. Thus $\varrho \in Inv\ s\text{-}Loc\langle F \rangle$.

(ii) is a direct consequence of (i). ∎

§3 Clones of relations

3.1 Motivations. There does not exist a fixed notion of a "clone of relations". Something like a "clone" is given with Suzlin's theory of projective sets and, moreover, there are close connections to the theory of cylindric algebras (Tarski), polyadic algebras (Halmos) and the axiomatization of enlargements (Robinson)(I wish to thank Prof. J.Schmidt for this hint).
The definition we are going to provide now shall serve as an analogon to clones of operations with respect to the Galois connection Pol-Inv. There are two approaches to get canonically the notion of a clone of relations:

a) The clone of operations generated by $F \subseteq O_A$ consists of all term functions (i.e. superpositions of F) of the algebra $\langle A;F\rangle$(cf.[Gr], 2.1). Therefore, for a relational algebra $\langle A;Q\rangle$($Q \subseteq R_A$) one can try to consider "term relations" which are built up from the elements of Q by using the (set theoretic) composition of relations (including such operations like permutations, identifications or delating of variables). If we consider relations as predicates (over A) then this means that firstly we have to take a set Φ of formulas (with some predicate symbols) and secondly we must define: The clone generated by Q is the set of all predicates which are derivable from elements of Q by means of formulas in Φ ("internal definition" of a clone). E.g., the composition of relations leads to the set Φ of all existential formulas without negation or disjunction. Clearly, the choice of Φ should depend on our algebraic purpose, i.e. on the question: what we want to do with clones of relations. Roughly speaking, the above mentioned formulas are suitable for describing general systems of operations. In special cases however, namely for unary (or bijective) operations the set Φ must be extended by allowing also disjunction (or disjunctions and negations) of formulas.

We will not develop a formula calculus here because – equivalently (cf.[Pö/Kal(§2.1)]) – we use some operations on R_A instead of formulas.

b) For a moment let us forget the finitarity of the relations under consideration. Then one can prove (cf.[Ros72,75], [Kr/Poi]) that the clones of operations $\langle F\rangle$ are exactly the Galois closed sets Pol Inv F ($F \subseteq O_A$). In other words we have the external definition: $\langle F\rangle$ is the greatest set F' of operations for which $\langle A;F\rangle^n$ and $\langle A;F'\rangle^n$ have the same subalgebras for all n (n can be infinite, too), cf. 1.5(ii).

The counterpart (i.e. the dual with respect to Pol-Inv, cf. 1.2) of the notion "subalgebra" is the notion "polymorphism". Therefore the polymorphisms should play the central role for relational algebras. This motivates the following "external definition" (in full analogy to clones of operations):

The clone of relations [Q] generated by $Q \subseteq R_A$ is the greatest set Q' of relations for which $\underline{A} = \langle A;Q\rangle$ and $\underline{A'} = \langle A;Q'\rangle$ have the same n-ary polymorphisms (i.e. $\mathrm{Hom}(\underline{A}^n,\underline{A}) = \mathrm{Hom}(\underline{A'}^n,\underline{A'})$) for all n. In other words, [Q] should be equal to the Galois closed set Inv Pol Q.

Clearly, the notion of a clone depends on the range of n, i.e. whether we allow to consider infinitary operations (or relations) or not. Because we want to deal with finitary relations and operations only, we shall seperate those properties of the Galois closed sets of relations which are caused by the finitarity of the operations. For this reason we introduced the local operator LOC (analogeously Loc for operations) which describes the influence of finitarity while the notion of the clone to be defined is the essential part of the Galois closure (and works also in case of infinitary relations).

Thus we try to avoid any infinitarity in the internal definition of clones of relations and we shall define the clones by suitable chosen set theoretic operations on *finitary* relations. (However, the attentive reader will find that, in fact, we cannot escape from a clandestine use of infinitary relations – because we need infinite many $\exists$-quantifiers – and probably one can prove that this has to be.)

Now we are going to define (set theoretic) operations on R_A some of which will be used to define the closure properties of clones.

3.2 Definitions.

(R0) Diagonal relations: The relations $\delta_m^\tau \in R_A^{(m)}$, where $m \in \mathbb{N}$ and τ is an equivalence relation on $\underline{m}$, defined by

$$\delta_m^\tau = \{(x_0, \dots, x_{m-1}) \in A^m \mid (i,j) \in \tau \implies x_i = x_j\}$$

are called trivial or diagonal relations. Let D_A be the set of all diagonal relations together with the empty relation $\emptyset$. The elements of $R_A \setminus D_A$ are called to be nontrivial.

(R1) Substitutions: For a mapping $\pi : \underline{n} \longrightarrow \underline{m}$, $\varrho \in R_A^{(m)}$ and $\sigma \in R_A^{(n)}$ we define $\pi(\varrho) \in R_A^{(n)}$ and $\pi^{-1}(\sigma) \in R_A^{(m)}$ as follows $(n, m \in \mathbb{N})$:

contravariant substitution functor:

$$\pi(\varrho) := \{(a_{\pi(0)}, \dots, a_{\pi(n-1)}) \in A^n \mid (a_0, \dots, a_{m-1}) \in \varrho\},$$

covariant substitution functor:

$$\pi^{-1}(\sigma) := \{(a_0, \dots, a_{m-1}) \in A^m \mid (a_{\pi(0)}, \dots, a_{\pi(n-1)}) \in \sigma\}.$$

By a special choice of π we obtain many well-known operations on R_A, e.g.:

(R1a) Permutation of coordinates:

Take $\pi(\varrho)$ for $\pi : \underline{n} \longrightarrow \underline{n} \in S_n$;

(R1b) Delating of coordinates :

For $\pi : \underline{(m-1)} \longrightarrow \underline{m} : i \longmapsto i$ we get

$$\pi(\varrho) = \{(a_0,\dots,a_{m-2}) \mid \exists a : (a_0,\dots,a_{m-2},a)\in\varrho\};$$

or more general

(R1c) Projections onto coordinates :

For injective $\pi : \underline{n} \longrightarrow \underline{m} : i \longmapsto j_i$ ($i\in\underline{n}$) we get

$$\pi(\varrho) = pr_{j_0,\dots,j_{n-1}}(\varrho)$$
$$:= \{(a_{j_0},\dots,a_{j_{n-1}}) \mid \exists a_i (i\in\underline{m}\setminus\{j_0,\dots,j_{n-1}\}): (a_0,\dots,a_{m-1})\in\varrho\};$$

(R1d) Doubling of coordinates :

For $\pi : \underline{(m+1)} \longrightarrow \underline{m} : i \longmapsto i$ ($i\in\underline{m}$), $m \longmapsto m-1$, we get

$$\pi(\varrho) = \{(a_0,\dots,a_{m-1},a_{m-1}) \mid (a_0,\dots,a_{m-1})\in\varrho\};$$

(R1e) Identification of coordinates :

For $\pi : \underline{n} \longrightarrow \underline{(n-1)} : i \longmapsto i (i\in\underline{n-1}), n-1 \longmapsto n-2$, we get

$$\pi^{-1}(\sigma) = \{(a_0,\dots,a_{n-2}) \mid (a_0,\dots,a_{n-2},a_{n-2})\in\sigma\};$$

(R1f) Adjoining fictive coordinates :

For $\pi : \underline{n} \longrightarrow \underline{(n+1)} : i \longmapsto i$, we get

$$\pi^{-1}(\sigma) = \{(a_0,\dots,a_{n-1},a) \mid (a_0,\dots,a_{n-1})\in\sigma,\ a\in A\}.$$

Note that the definition of substitutions keeps its sense also for infinite (ordinal numbers) n and m .

(R2) Intersection: For $\varrho_i\in R_A^{(m)}$, $i\in I$,

$$\bigcap\{\varrho_i \mid i\in I\} := \{(a_j)_{j\in\underline{m}} \mid \forall i\in I : (a_j)_{j\in\underline{m}}\in\varrho_i\}$$

is the intersection of all ϱ_i, $i\in I$.

(R3) Composition : For $\varrho\in R_A^{(m)}$, $\sigma\in R_A^{(n)}$, the composition $\varrho\circ\sigma$ is the following (m+n-2)-ary relation:

$$\varrho \circ \sigma = \{(a_0, \dots, a_{m+n-3}) \mid \exists a \in A : (a_0, \dots, a_{m-2}, a) \in \varrho \text{ and } (a, a_{m-1}, \dots, a_{m+n-3}) \in \sigma\},$$

(For m=n=1 we define $\varrho \circ \sigma = \emptyset$).

(R4) General superposition : For $\varrho_i \in R_A^{(m_i)}$, $\pi_i : \underline{m_i} \longrightarrow \underline{\alpha}$ (α arbitrary ordinal number, $\underline{\alpha} = \{\beta \mid \beta < \alpha\}$), $i \in I$ (index set), and $\pi : \underline{m} \longrightarrow \underline{\alpha}$ we define

$$\bigwedge\nolimits^{\pi}_{(\pi_i)_{i\in I}} (\varrho_i)_{i \in I}$$

to be the relation

$$\{(a_{\pi(0)}, \dots, a_{\pi(m-1)}) \in A^m \mid \exists (a_i)_{i \in \underline{\alpha} \setminus \pi \underline{m}} : \text{For all } i \in I \ (a_{\pi_i(0)}, \dots, a_{\pi_i(m_i-1)}) \in \varrho_i\}.$$

For $\varrho_i \in Q$, the relations $\bigwedge\nolimits^{\pi}_{(\pi_i)}(\varrho_i)$ are called (general) superpositions of (elements of) Q.

Remarks: We can take the identity ($i \mapsto i$) for π if we allow substitutions of coordinates.

With the notations given in (R1) and (R2) we have

$$\bigwedge\nolimits^{\pi}_{(\pi_i)}(\varrho_i) = \pi\Big(\bigcap_{i \in I} \pi_i^{-1}(\varrho_i)\Big).$$

Note that the ϱ_i need not be different.

(R5) Special superposition: For infinite A, $\alpha := |A|$, let I_n be the set of all monotone injective mappings $\pi : \underline{n} \longrightarrow \underline{\alpha}$, $n \in \mathbb{N}$. For a family $Q = (\varrho_\pi)_{\pi \in I_n}$ of relations (with $\varrho_\pi \in R_A^{(n)}$ for $\pi \in I_n$) we define the m-special superposition of Q as follows:

$$\bigcap\nolimits^m Q := \{(a_0, \dots, a_{m-1}) \mid \exists (a_i)_{i \in \underline{\alpha} \setminus \underline{m}} \ \forall \varrho_\pi \in Q : (a_{\pi(0)}, \dots, a_{\pi(n-1)}) \in \varrho_\pi\}.$$

Wrong:
In some cases
$\alpha = 2^{|A|}$ is needed

3.3 Remarks. It is easy to see that the operations (R1) - (R3), (R5) are special cases of (R4) - one has to specialize α, π, π_i, I. The general superposition was defined and studied also in [Sz78] by means of so-called formula schemes which in effect are nothing else than the tripel $((\varrho_i, \pi_i)_{i\in I}, \alpha, \pi)$.

3.4 Notations. The elements $a=(a(i))_{i\in \underline{n}} \in A^n$ can be considered as functions $a : \underline{n} \longrightarrow A : i \longmapsto a(i)$. For $\pi : \underline{m} \longrightarrow \underline{n}$ the composition of π and a – notation πa (first π then a) – is an element of A^m. Thus the notions in 3.2 can be defined shortly as follows:

(R1) $\pi(\varrho) = \{\pi a \mid a \in \varrho\}$, $\pi^{-1}(\sigma) = \{a \in A^m \mid \pi a \in \sigma\}$;

(R4) $\bigwedge^{\pi}_{(\pi_i)_{i\in I}} (\varrho_i)_{i\in I} = \{\pi a \mid a \in A^{\alpha} \text{ with } \pi_i a \in \varrho_i \text{ for all } i \in I\}$.

3.5 Definition. A set $Q \subseteq R_A$ is called a *clone of relations* on A

– notation: $Q \leq R_A$ –

if Q contains the trivial relations $\emptyset$ and $A \in R_A^{(1)}$ and is closed with respect to general superposition (R4).

For arbitrary $Q \subseteq R_A$, the clone (of relations) generated by Q (i.e. the least clone containing Q) will be denoted by $[Q]_{R_A}$, shortly $[Q]$.

There are some equivalent definitions.

3.6 Proposition. Let $Q \subseteq R_A$. The following conditions are equivalent (cf. 3.2, 3.3):

(i) Q is a clone of relations: $Q \leq R_A$.

(ii) Q is closed with respect to (R0) (i.e. Q contains all trivial relations), (R1), (R2), (R3), (R4) and (R5).

(iii) Q is closed with respect to (R0), (R1), (R2), (R5).* (* for $\alpha = 2^{|A|}$)

(iv) Q is closed with respect to (R0), (R1a), (R1e), (R2), (R5).

For finite A, (i) is also equivalent to each of the following conditions:

(v) Q is closed with respect to (R1), (R3) and contains A.

(vi) Q is closed with respect to (R0), (R1a), (R1e), (R3).

(vii) Q is closed with respect to (R1a), (R1e), (R3) and contains δ_3^{ε} (where $\varepsilon = \{(0,0),(0,1),(1,0),(1,1),(2,2)\}$).

(viii) Q is closed with respect to (R1a), (R1e), (R1f), (R3) and contains $A \in R_A^{(1)}$.

P r o o f. We give some hints only and will not go into technical details. The construction (R4) can be transformed into the form (R5) using (R0), (R1) and (R2) - and vice versa. Every diagonal can be obtained from $A^m \in D_A^{(m)}$ using (R1), while $A \in R_A^{(1)}$ generates A^m via (R1f). All substitutions (R1) can be generated with (R1a) and (R1e) using (R0) and (R3) (the latter is derivable from (R5)). For finite A, the operations (R4) can be reduced to finite I and α (see [Sz78 (Lemma 2)]) and can be expressed by (R0), (R1) and (R3) (see [Pö/Kal (1.1.9)]). (■)

3.7 Remarks. (i) The set D_A of all diagonals is a clone of relations contained in every clone.

(ii) Clones of relations can be considered as the subalge- of the algebra $\langle R_A; \delta, \zeta, \tau, \Delta, \nabla, \cap, (\cap^m)_{m \in \mathbb{N}} \rangle$ of type

$\langle 0,1,1,1,1,2^{|A|},(|A|)_{m\in\mathbb{N}}\rangle$, where $\delta=\{(a,a)\mid a\in A\}\in D_A^{(2)}$, $\zeta(\varrho)=\pi_1(\varrho)$, $\tau(\varrho)=\pi_2(\varrho)$ for $\varrho\in R_A^{(n)}$ and the permutations $\pi_1=(0\ 1\ \dots\ n-1)$, $\pi_2=(0\ 1)$ of S_A, $\Delta=$(R1e), $\nabla=$(R1f), $\bigcap$ denotes the intersection of a family (of relations) of cardinality $2^{|A|}$ and $\bigcap^m$ is defined as in 3.2(R5). The proof follows from 3.6(iv) and the fact that $\delta,\zeta,\tau,\Delta,\nabla$ generate all (R0) and (R1). For finite A, the above algebra can be chosen much simpler: The clones are exactly the subalgebras of the algebra $\langle R_A;\ A,\zeta,\tau,\Delta,\nabla,\circ\rangle$ of type $\langle 0,1,1,1,1,2\rangle$ (cf, [Pö/Kal(p.43)]).

(iii) From the point of view of logic, clones of relations are those sets of predicates (on A) which are closed under first order formulas containing $\exists,\wedge,=$ but not $\forall,\neg,\vee$, and under infinite intersections and $\exists$-quantifications (i.e. we can use positive first order formulas with infinite many existential quantifiers and conjunctions), cf. [Kr/Poi].

In analogy to 2.5 and 2.6 we have the following properties.

<u>3.8</u> Proposition. <u>Let</u> $Q\subseteq R_A$ <u>and</u> $F\subseteq O_A$. <u>Then</u>

(i) $\mathrm{Inv}_A F$ <u>is a clone of relations</u>, <u>i.e.</u> $[\mathrm{Inv}_A F]=\mathrm{Inv}_A F$.

(ii) s-LOC[Q] <u>is a clone of relations</u> ($s\in\mathbb{N}$).

(iii) LOC[Q] <u>is a clone of relations</u>. (cf. 1.9)

Proof. (i): $\bigwedge^{\pi}_{(\pi_i)}(\varrho_i)_{i\in I}$ (cf. 3.2(R4)) preserves an operation $f\in O_A$ whenever each ϱ_i preserves f .

(ii) and (iii) will follow e.g. from theorem 4.2 but the proof can be done easily also by checking the definitions. ∎

<u>3.9</u> Proposition. <u>Let</u> $Q\subseteq R_A$. <u>Then we have</u>:

(i) $\mathrm{Pol}_A^{(n)}Q=\mathrm{Pol}_A^{(n)}[Q]=\mathrm{Pol}_A^{(n)}\mathrm{LOC}[Q]=\mathrm{Pol}^{(n)}\text{s-LOC}[Q]$ <u>for</u> $1\leq n\leq s\in\mathbb{N}$.

(ii) $Pol_A Q = Pol_A[Q] = Pol_A LOC[Q]$.

P r o o f. (i): Let $f \in Pol^{(n)} Q$. Because the sets in (i) must form a decreasing (from left to right) chain it suffices to show that $f \in Pol^{(n)}$ s-LOC$[Q]$. Clearly $f \in Pol^{(n)}[Q]$ because superpositions of Q are also invariant for f (easy proof or use 3.8(i)). Let $\varrho \in$ s-LOC$[Q]^{(m)}$ and $r_1, \dots, r_n \in \varrho$. By 1.9 (note $n \leq s$), there exists a $\sigma \in [Q]$ such that $\{r_1, \dots, r_n\} \subseteq \sigma \subseteq \varrho$. Consequently $f(r_1, \dots, r_n) \in \sigma \subseteq \varrho$, i.e. f preserves ϱ.
(ii) immediately follows from (i). ■

§4 The Galois connection Pol - Inv

The following two theorems are basic for a "General Galois Theory of operations and relations"(cf. introduction) because they characterize the Galois closed sets of operations and relations, resp. The propositions 1.11, 2.5 and 3.8 suggest that these Galois closed sets might be exactly the local closed clones, i.e., the "external" and "internal" definitions of clones might coincide (cf. 3.1b).

4.1 Theorem(cf.[Gei],[Ba/Pix(Lemma 3.1)],[Rom77b]).
Let $F \subseteq O_A$. Then we have:
(a) $Loc\langle F\rangle = Pol_A Inv_A F$.
(b) s-$Loc\langle F\rangle = Pol_A Inv_A^{(s)} F$ for $s \in \mathbb{N}$ (for s=1 cf.[Schm J. (Thm.1.6)]).

4.2 Theorem. Let $Q \subseteq R_A$. Then we have:

(a) $LOC[Q] = Inv_A Pol_A Q$ (cf. [Sz78(Lemma 4)],[Gei(p.99)]).

(b) $s\text{-}LOC[Q] = Inv_A Pol_A^{(s)} Q$ for $s \in \mathbb{N}$ (for unary invariants cf.[Go68]).

Remarks. Results concerning the characterization of Galois closed sets (w.r.t. Pol-Inv) of relations can be found - with more or less modifications and restrictions - also in:

[Ros75],[Sz78] (Inv Pol Q for finitary relations and operations);

[Kr/Poi] (Inv Pol Q (coalgebrès de Post) for infinitary relations and operations);

[Bo/Kal],[Pö/Kal] ($Inv_A Pol_A Q$ (Post coalgebras, Relationenalgebren), $Inv_A Pol_A^{(1)} Q$, $Inv_A Aut_A Q$ (Krasner algebras) for finite A);

[Kr68],[Kr76a] (Inv Aut Q, Inv $Pol^{(1)}Q$);

[Pö73] (Inv Pol Q for operations and relations defined on a family of finite sets);

[Ros78] ($Inv^{(n)}Pol$ Q, cf. 10.5);

[Sa/St77b] (End Pol f, $f \in O_A$, note $(\text{End Pol})^{\bullet} = (O_A^{(1)})^{\bullet} \cap$ Inv Pol);

[Sa/St77c] (End Pol S, $S \subseteq O_A^{(1)}$); (cf. also [Je])

[Sa/St78] (Pol End F, $F \subseteq O_A$).

Proof of 4.1: (b): By 1.7 and 2.6 we have $s\text{-}Loc\langle F\rangle \subseteq \text{Pol Inv } s\text{-}Loc\langle F\rangle \subseteq \text{Pol Inv}^{(s)} s\text{-}Loc\langle F\rangle = \text{Pol Inv}^{(s)}F$. To show the opposite inclusion let $f \in Pol^{(n)} Inv^{(s)} F$. We prove $f \in s\text{-}Loc\langle F\rangle$. Let $B = \{b_0, \dots, b_{t-1}\} \subseteq A^n$, $t \leq s$, $r_i := (b_0(i), \dots, b_{t-1}(i))$, $i \in \underline{n}$, and $\sigma = \{r_i \mid i \in \underline{n}\}$. Since $\Gamma_F(\sigma) \in Inv^{(t)}F$ (and $f \in \text{Pol Inv}^{(s)}F \subseteq \text{Pol Inv}^{(t)}F$) there exists (cf. 2.4) an $g \in \langle F\rangle$ such that $f(r_0, \dots, r_{n-1}) = g(r_0, \dots, r_{n-1})$, i.e. $f|B = g|B$, hence $f \in s\text{-}Loc\langle F\rangle$. (a) follows froms (b) since $Loc\langle F\rangle = \bigcap_{s \in \mathbb{N}} s\text{-}Loc\langle F\rangle = \bigcap_{s \in \mathbb{N}} \text{Pol Inv}^{(s)}F = \text{Pol} \bigcup_{s \in \mathbb{N}} Inv^{(s)}F = \text{Pol Inv } F$. ∎

P r o o f   of 4.2: (b): By 1.7 and 3.9 we have $s\text{-LOC}[Q] \subseteq \text{Inv Pol } s\text{-LOC}[Q] \subseteq \text{Inv Pol}^{(s)} s\text{-LOC}[Q] = \text{Inv Pol}^{(s)} Q$. Now, for $\varrho \in \text{Inv}^{(m)}\text{Pol}^{(s)}Q$ ($m \in \mathbb{N}$), we are going to prove $\varrho \in s\text{-LOC}[Q]$. Note that $\varrho = \bigcup \mathcal{T}$ for the s-directed (1.12) system $\mathcal{T} = \{\Gamma_F(B) \mid B \subseteq \varrho, |B| \leq s\}$ where $F = \text{Pol}^{(s)}Q$. By the next lemma (4.3b) we have $\Gamma_F(B) \in [Q]$, thus $\varrho = \bigcup \mathcal{T} \in s\text{-LOC}[Q]$ (cf. 1.13) and we are done.

(a) follows from (b) since $\text{LOC}[Q] = \bigcap_{s \in \mathbb{N}} s\text{-LOC}[Q] = \bigcap_{s \in \mathbb{N}} \text{Inv Pol}^{(s)}Q = \text{Inv} \bigcup_{s \in \mathbb{N}} \text{Pol}^{(s)}Q = \text{Inv Pol } Q$. ∎

The hard core of the proof of 4.2 is the following lemma.

4.3 Lemma (cf. [Sz78(Lemma 2)]). For $Q \subseteq R_A$ and $F = \text{Pol}_A Q$ we have:

a) $\Gamma_F(B) \in [Q]$ for all finite $B \subseteq A^n$ ($n \in \mathbb{N}$).

b) $\Gamma_{F^{(s)}}(B) \in [Q]$ for all $B \subseteq A^n$ with $|B| \leq s$ ($s, n \in \mathbb{N}$).

P r o o f. b) follows from a) since $\Gamma_F(B) = \Gamma_{F^{(s)}}(B)$ for $|B| \leq s$ (cf. 2.4).

a): Extending the proof given in [Bo/Kal] for finite A to infinite A – what was already done independently by L.Szabó [Sz78] – and following [Sz78(proof of lemma 2)] we construct a $\sigma_B \in [Q]$ and show $\sigma_B = \Gamma_F(B)$ (for finite $B \subseteq A^n$).

Let $B = \{b_0, \dots, b_{s-1}\} \subseteq A^n$, $z_i = z_i(B) := (b_0(i), \dots, b_{s-1}(i)) \in A^s$ ($i \in \underline{n}$). For $\varrho \in Q^{(m)}$ ($m \in \mathbb{N}$) let I_ϱ be the set of all matrices $M = (r_j(i))_{(i,j) \in \underline{m} \times \underline{s}}$, the rows of which we denote by $z_i(M) = (r_0(i), \dots, r_{s-1}(i))$ ($i \in \underline{m}$), such that $\{r_0, \dots, r_{s-1}\} \subseteq \varrho$ (i.e. the colums belong to ϱ). For all $\varrho \in Q^{(m)}$ and $M \in I_\varrho$ we define the mapping

$$\pi_\varrho^M : \underline{m} \longrightarrow A^s : i \mapsto z_i(M) \quad (i\in\underline{m}), \text{ and}$$

let $\quad \pi : \underline{n} \longrightarrow A^s : i \mapsto z_i(B) \quad (i\in\underline{n})$.

Then, by 3.5,

$$\sigma_B = \{(a_{z_0},\dots,a_{z_{n-1}})\in A^n \mid \exists (a_z)_{z\in A^s\setminus\{z_i \mid i\in\underline{n}\}} : (a_{z_0(M)},\dots,a_{z_{m-1}(M)})\in\varrho \text{ for all } M\in I_\varrho, \varrho\in Q^{(m)}, m\in\mathbb{N}\},$$

i.e. (3.4) $\sigma_B = \{\pi a \mid a\in A^{A^s} \text{ with } \pi_\varrho^M a\in\varrho \text{ for all } M\in I_\varrho,\ \varrho\in Q\}$,

belongs to $[Q]$. We are going to show $\sigma_B = \Gamma_F(B)$.

We observe: If $a=(a_z)_{z\in A^s}$ fulfilles the condition

(*) $\qquad \pi_\varrho^M a\in\varrho$ for all $M\in I_\varrho$ and $\varrho\in Q$,

then $f: A^s \longrightarrow A : z\mapsto a_z$ preserves all $\varrho\in Q$. In fact, $M=\{r_0,\dots,r_{s-1}\}\subseteq\varrho$ implies $f(r_0,\dots,r_{s-1}) = \pi_\varrho^M a\in\varrho$. Conversely, if $f: A^s\longrightarrow A$ preserves all $\varrho\in Q$, then $a=(f(z))_{z\in A^s}$ fulfilles (*). Therefore

$\sigma_B = \{(a_{z_0},\dots,a_{z_{n-1}}) \mid (a_z)_{z\in A^s} \text{ fulfilles } (*)\} =$

$\quad = \{(f(z_0),\dots,f(z_{n-1})) \mid f\in F = \mathrm{Pol}\ Q\}$. Thus, by definition of the z_i and 2.4, we get $\sigma_B = \{f(b_0,\dots,b_{s-1}) \mid f\in F\} = \Gamma_F(B)$. ∎

The following propositions which are corollaries to 4.1 and 4.2 give the characterization of clones of operations and relations, resp., via the Galois connection Pol-Inv.

<u>4.4</u> Proposition. <u>For</u> $F\subseteq O_A$, <u>the following conditions are equivalent</u>:

(i) $\quad F\leq O_A$ (<u>i.e.</u> $F=\langle F\rangle_{O_A}$) <u>and</u> $\mathrm{Loc}\ F = F$,

(ii) $\quad F = \mathrm{Pol}_A \mathrm{Inv}_A F$,

(iii) $\exists Q \subseteq R_A$: $F = \mathrm{Pol}_A Q$.

<u>Proof</u>. (i)$\Rightarrow$(ii) by 4.1a, (ii)$\Rightarrow$(iii) obvious, (iii)$\Rightarrow$(i) by 2.5(i) and 1.11(a). ∎

<u>4.5</u> Proposition. <u>For</u> $Q \subseteq R_A$, <u>the following conditions are equivalent</u>:

(i) $Q \leq R_A$ (<u>i.e.</u> $Q = [Q]_{R_A}$) <u>and</u> LOC $Q = Q$,

(ii) $Q = \mathrm{Inv}_A \mathrm{Pol}_A Q$,

(iii) $\exists F \subseteq O_A$: $Q = \mathrm{Inv}_A F$.

<u>Proof</u>. (i)$\Rightarrow$(ii) by 4.2(a), (ii)$\Rightarrow$(iii) obvious, (iii)$\Rightarrow$(i) by 1.11(a') and 3.8(i). ∎

Moreover, 4.4 and 4.5 answer the question under which conditions a set F or Q is representable as Pol Q' or Inv F' respectively. Such concrete characterization problems will be treated in the next paragraphs. In particular we have for the group case:

<u>4.6</u> Proposition. <u>For</u> $G \subseteq S_A$, <u>the following conditions are equivalent</u>:

(i) $G \leq S_A$ (<u>i.e.</u> $G = \langle G \rangle_{S_A}$) <u>and</u> $G = S_A \cap \mathrm{Loc}\ G$,

(ii) $G = \mathrm{Aut}_A \mathrm{Inv}_A G$,

(iii) $\exists Q \subseteq R_A$: $G = \mathrm{Aut}_A Q$.

<u>Proof</u>. (i)$\Rightarrow$(ii): $G \subseteq \mathrm{Aut\ Inv}\ G \subseteq S_A \cap \mathrm{Pol\ Inv}\ G = S_A \cap \mathrm{Loc}\langle G \rangle_{O_A} = S_A \cap \mathrm{Loc}\ G = G$. (ii)$\Rightarrow$(iii) obvious.

(iii)$\Rightarrow$(i): $G=\mathrm{Aut}\ Q$ is a subgroup of S_A (cf. 1.6a). Moreover, for $f\in S_A\cap \mathrm{Loc}\ G$ we have $f^{-1}\in S_A\cap\mathrm{Loc}\ G$ (this easily follows from the definitions) and by 1.11a every $f\in\mathrm{Loc}\ G$ preserves all $\varrho\in Q$. Thus $S_A\cap\mathrm{Loc}\ G = \mathrm{Aut}\ Q$. ■

Remark: For $G\leqq S_A$ we have $S_A\cap\mathrm{Loc}\ G = \mathrm{Loc}^{(1/-1)}G := \{f\in S_A \mid \{f,f^{-1}\}\subseteq \mathrm{Loc}\ G\}$.

In 4.6, (i) could be replaced by

$$\text{(i)'}\quad G=\langle G\rangle_{O_A^{(1)}} \quad\text{and}\quad G=\mathrm{Loc}^{(1/-1)}G\ . \ ■$$

4.7 Lemma. For $F\subseteq O_A$ and $G\subseteq S_A$ we have:

$$(\mathrm{Loc}\langle F\rangle_{O_A})^{\bullet}\subseteqq \mathrm{LOC}[F^{\bullet}] \quad\text{(notation cf. 1.3)},$$

$$(\mathrm{Loc}\langle G\rangle_{S_A})^{\bullet}\subseteqq \mathrm{LOC}[G^{\bullet}].$$

Proof. $(\mathrm{Loc}\langle F\rangle)^{\bullet}=(\mathrm{Pol\ Inv}\ F)^{\bullet}\subseteqq(\mathrm{Pol\ Pol}\ F^{\bullet})^{\bullet}\subseteq\mathrm{Inv\ Pol}\ F^{\bullet} = \mathrm{LOC}[F^{\bullet}]$ (cf. 4.1, 1.5(iii), 4.2). Since $(f^{-1})^{\bullet}\in[f^{\bullet}]$ for $f\in S_A$ we have $\langle G\rangle_{S_A}^{\bullet}\subseteqq[G^{\bullet}]$ what implies the second inclusion of the lemma. ■

Part 2

CONCRETE CHARACTERIZATIONS OF RELATED ALGEBRAIC STRUCTURES

§5 Concrete characterizations I. (Characterization of operational systems via relational ones)

In this paragraph we investigate the problem whether for a given permutation group G, a transformation semigroup H or similiar "operational structures" does exist a relational algebra $\mathcal{L} = \langle A; Q\rangle$ such that $G = (w-)\mathrm{Aut}\,\mathcal{L}$, $H = \mathrm{End}\,\mathcal{L}$ or "something like this" respectively. All these problems will be covered by the following general problem:

5.1 Concrete characterization problem:

Given a set A and $F_i \subseteq E_i \subseteq O_A$ $(i \in I)$, does there exist a relational algebra $\mathcal{L} = \langle A; Q\rangle (Q \subseteq R_A)$ such that

$$F_i = E_i \cap \mathrm{Pol}_A Q \;?$$

Under which conditions one can choose the relations of Q to be of bounded rank (i.e. of bounded arity) ?

Specializing E_i we get e.g. the following characterization problems:

E_i =	yields the characterization of the
S_A	weak automorphisms
$O_A^{(1)}$	endomorphisms
$O_A^{(n)}$	n-ary polymorphisms
O_A	polymorphisms
	of relational algebras $\mathcal{L}=\langle A;Q\rangle$.

Moreover, we can get a simultaneous characterization of these structures. For automorphism groups we need little modifications.

The answer to 5.1 is given in the following theorem:

<u>5.2</u> Theorem. <u>Let</u> $F_i \subseteq E_i \subseteq O_A$ ($i\in I$ <u>index set</u>) <u>and</u> $F = \bigcup_{i\in I} F_i$. <u>There exists a relational algebra</u> $\langle A;Q\rangle$ <u>with</u>

α) $Q \subseteq R_A$ <u>or</u>

β) $Q \subseteq R_A^{(1)} \cup \ldots \cup R_A^{(s)}$ ($s\in\mathbb{N}$), <u>resp.</u>,

<u>such that</u>

$$F_i = E_i \cap \mathrm{Pol}_A Q \quad (i\in I)$$

<u>if and only if</u>

α) $F_i = E_i \cap \mathrm{Loc}\langle F\rangle$ ($i\in I$) <u>or</u>

β) $F_i = E_i \cap s\text{-}\mathrm{Loc}\langle F\rangle$ ($i\in I$), <u>resp.</u>

<u>Remark</u>. If we want to consider automorphisms $F_i = E_i \cap \mathrm{Aut}_A Q$ then $\mathrm{Loc}\langle F\rangle$ above must be replaced by $\mathrm{Loc}^{(1/-1)}\langle F\rangle = \{f\in S_A \mid \{f,f^{-1}\} \subseteq \mathrm{Loc}\langle F\rangle\}$ (analogeously for $s\text{-}\mathrm{Loc}\langle F\rangle$).

<u>Proof</u>. α)"⟹": Since $F_i \subseteq \mathrm{Pol}\ Q$ we have $F \subseteq \mathrm{Pol}\ Q$ and

(by 2.5(i),1.11a) $\mathrm{Loc}\langle F\rangle \subseteq \mathrm{Loc}\langle \mathrm{Pol}\ Q\rangle = \mathrm{Pol}\ Q$. Thus $F_i \subseteq E_i \cap \mathrm{Loc}\langle F\rangle \subseteq E_i \cap \mathrm{Pol}\ Q = F_i$ (note $F_i \subseteq \mathrm{Loc}\langle F\rangle$ since $F_i \subseteq F$).

"$\Leftarrow$": Take $Q := \mathrm{Inv}\ F$. Then $F_i = E_i \cap \mathrm{Loc}\langle F\rangle = E_i \cap \mathrm{Pol}\ Q$ by 4.1a.

β) can be proved analogeously (using 4.1b).

The remark follows from the observation (cf. 4.6(remark)) that $\mathrm{Loc}^{(1/-1)}\langle F\rangle = \mathrm{Aut}\ \mathrm{Inv}\ F$. ■

5.3 Remark. Note that for 5.2 case β) we could take Q to be a set of s-ary relations only because every relation $\varrho \in R_A^{(i)}$ $(i \leq s)$ can be extended to a $\bar{\varrho} \in R_A^{(s)}$ (by adjoining fictive coordinates, cf. 3.2(R1f)) such that $\mathrm{Pol}\,\varrho = \mathrm{Pol}\,\bar{\varrho}$.

5.4 Definition and notations. The algebra $\langle O_A^{(n)}; (e_i^n)_{i\in \underline{n}}, \circledast\rangle$ of type $\langle (0)_{i\in\underline{n}}, n+1\rangle$ where $\circledast(g,f_1,\dots,f_n) := g(f_1,\dots,f_n)$ (cf. 2.1(ii)) is called the full Menger algebra of n-ary operations. Thus $F \subseteq O_A^{(n)}$ is a Menger subalgebra of $O_A^{(n)}$ iff F contains all projections and is closed with respect to $\circledast$. This can be proven to be equivalent to $F = \langle F\rangle_{O_A^{(n)}}$. Therefore, for $F \subseteq O_A$, $F^{(n)}$ is a subalgebra of the Menger algebra $O_A^{(n)}$; in particular, $F^{(1)}$ is a subsemigroup of $\langle O_A^{(1)}; e, \cdot\rangle$ (e identity, $\cdot$ composition).

For $f \in O_A^{(1)}$, let $f^{\nabla n}$ be the n-ary operation which is equal to f up to fictive variables and which is defined as follows:

$$f^{\nabla n}(x_0,\dots,x_{n-1}) = f(x_0).$$

For $H \subseteq O_A^{(1)}$ and $F \subseteq O_A^{(n)}$ we put

$$H^{\nabla n} = \{f^{\nabla n} \mid f \in H\},$$
$$F^{[1/-1]} = \{f \in S_A \mid \{f^{\nabla n}, (f^{-1})^{\nabla n}\} \subseteq F\}.$$

As a generalization of results in §4 and as a specialization of 5.2 we get:

5.5 Corollary. Let $G \subseteq G' \subseteq S_A$, $H \subseteq O_A^{(1)}$ and $F \subseteq O_A^{(n)}$ $(n \in \mathbb{N})$. Then there exists a relational algebra $\mathcal{L} = \langle A; Q \rangle$ with

α) $Q \subseteq R_A$ or

β) $Q \subseteq R_A^{(s)}$, resp.,

such that

$G = \mathrm{Aut}\,\mathcal{L}$ (=$\mathrm{Aut}_A Q$ automorphisms)

$G' = \mathrm{w\text{-}Aut}\,\mathcal{L}$ (=$\mathrm{w\text{-}Aut}_A Q$ weak automorphisms)

$H = \mathrm{End}\,\mathcal{L}$ (=$\mathrm{End}_A Q$ endomorphisms)

$F = \mathrm{Hom}(\mathcal{L}^n, \mathcal{L})$ (=$\mathrm{Pol}_A^{(n)} Q$ n-ary polymorphisms)

if and only if

(i) $G = F^{[1/-1]}$ (and G is a subgroup of S_A),

(ii) $G'^{\nabla n} = F \cap S_A^{\nabla n}$

(iii) $H^{\nabla n} = F \cap O_A^{(1)\nabla n}$ (and H is a subsemigroup of $O_A^{(1)}$).

(iv) F is a Menger subalgebra of $O_A^{(n)}$ (cf. 5.4)

(v) α) $F = \mathrm{Loc}\, F$ or

β) $F = s\text{-}\mathrm{Loc}\, F$, resp., $s \in \mathbb{N}$.

Proof by 5.2 (note $\mathrm{Aut}\, Q = (\mathrm{Pol}^{(n)} Q)^{[1/-1]}$, $(\mathrm{w\text{-}Aut}\, Q)^{\nabla n} = \mathrm{Pol}\, Q \cap S_A^{\nabla n}$, $(\mathrm{End}\, Q)^{\nabla n} = \mathrm{Pol}\, Q \cap O_A^{(1)\nabla n}$). ∎

Remark. Clearly, because of (iv), the parts of (i) and (iii) included in parentheses are superfluous.

At the end of this paragraph we ask whether the wanted relational algebra $\mathcal{L} = \langle A; Q \rangle$ may be chosen to be of finite similiarity type, i.e. whether Q may be finite.

For finite A the answer is nearly trivial; we have:

for finite A,

5.6 Proposition. A clone $F \leq O_A$ is the set of all polymorphisms of a finite relational algebra $\langle A; Q \rangle$ of finite type iff there exists an $s \in \mathbb{N}$ such that s-Loc F = F.

Proof by 5.2, note that $R_A^{(s)}$ is finite (since A finite). ∎

Besides this internal characterization (a second one can be found in [Pö/Kal(Satz 4.1.9)]) there exists also an external characterization for polymorphisms of finite relational algebras of finite type by means of a chain condition in the lattice of all clones of operations (cf. [Pö/Kal(4.1.3)]). In case of infinite A we can prove only the following weaker proposition:

5.7 Proposition. Let $F \leq O_A$. For the conditions

(i) $\mathrm{Loc}\langle F \rangle = F$ and, for every down-directed system $\{F_i \mid i \in I\}$ of local closed clones of operations (i.e., $F_i = \mathrm{Loc}\langle F_i \rangle$, $\forall i,j \in I\ \exists k \in I: F_i \cap F_j \geq F_k$) $\bigcap_{i \in I} F_i = F$ implies the existence of a finite subset $I' \subseteq I$ such that $\bigcap_{i \in I'} F_i = F$.

(ii) There exists a finite set Q of finitely generated $\varrho \in R_A$ (i.e. $\forall \varrho \in Q\ \exists$ B finite : $\varrho = \Gamma_F(B)$) such that F = Pol Q.

(ii)' There exists a finite set $Q \subseteq R_A$ such that F = Pol Q.

(iii) $\exists\, s \in \mathbb{N} : \; F = s\text{-Loc}\langle F\rangle$.

<u>the following implications hold</u>:

(i) $\Rightarrow$ (ii) $\Rightarrow$ (ii)' $\Rightarrow$ (iii) .

<u>Proof</u>. (ii)' $\Rightarrow$ (iii) follows from 5.2 since $\exists s \in \mathbb{N} : Q \subseteq \bigcup_{i \leq s} R_A^{(i)}$.
(ii) $\Rightarrow$ (ii)' is trivial.
(i) $\Rightarrow$ (ii): Let $I = \{Q \mid Q \subseteq \text{Inv}_A F,\ Q \text{ finite},\ \forall \varrho \in Q\ \exists\, B \text{ finite}: \varrho = \Gamma_F(B)\}$
and let $F_Q = \text{Pol } Q$ for $Q \in I$. Then $\{F_Q \mid Q \in I\}$ is a down-directed set of local closed (1.11a) clones (2.5(i))(since $F_Q \cap F_{Q'} = F_{Q \cup Q'}$) and
$$\bigcap_{Q \in I} F_Q = \bigcap_{Q \in I} \text{Pol } Q = \text{Pol} \bigcup_{Q \in I} Q \overset{(+)}{=} \text{Pol Inv } F \overset{(4.1)}{=} \text{Loc}\langle F\rangle = F.$$
(The equation (+) follows from $\text{Pol}\{\varrho\} = \text{Pol}\,\{\Gamma_F(B) \mid B \text{ finite},\ B \subseteq \varrho\}$ (cf. 1.8)). Thus, by (i), there exists a finite $I' \subseteq I$ such that $\bigcap_{Q \in I'} F_Q = F$, i.e., $F = \text{Pol} \bigcup_{Q \in I'} Q$ where $\left|\bigcup_{Q \in I'} Q\right| \leq \sum_{Q \in I'} |Q| < \aleph_0$. ∎

<u>5.8 Remark</u>. The sufficient condition 5.7(i) becomes also necessary for (ii) if one takes into consideration infinitary operations. This can be done analogeously to the result 6.7 shown in the next paragraph. Therefore we will not go into further details here.
For finite A, all conditions in 5.7 are equivalent.

§6 Concrete characterizations II. (Characterization of relational systems via universal algebras)

In §5 we asked for the characterization of related (operational) structures (like Aut $\mathcal{L}$, Pol $\mathcal{L}$) of relational algebras $\mathcal{L}$. Now, we are interestested in the dual question (which had been much more investigated in the literature): How to characterize related relational structures (like Con $\mathfrak{A}$, Inv $\mathfrak{A}$) of universal algebras $\mathfrak{A}$?
This question includes the characterization of related operational systems (like Aut $\mathfrak{A}$, End $\mathfrak{A}$) because operations can be considered as relations (cf. 1.3).

These problems will be covered (cf. §§7-14) by the following characterization problem:

6.1 Concrete characterization problem:
Given a set A and $Q_i \subseteq E_i \subseteq R_A$ (i I), does there exist a universal algebra $\mathfrak{A} = \langle A;F\rangle$ ($F \subseteq O_A$) such that

$$Q_i = E_i \cap \mathrm{Inv}_A F \ ?$$

Under which conditions one can choose the operations of F to be of bounded rank (i.e. of bounded arity) ?

The solution looks like follows:

6.2 Theorem (cf.[Sz78(Thm.6)]). Let $Q_i \subseteq E_i \subseteq R_A$ $(i \in I)$ and $Q = \bigcup_{i \in I} Q_i$. There exists a universal algebra $\mathfrak{A} = \langle A;F \rangle$ with

(a) $F \subseteq O_A$ or

(b) $F \subseteq O_A^{(1)} \cup \dots \cup O_A^{(s)}$, resp., $(s \in \mathbb{N})$

such that

$$Q_i = E_i \cap \mathrm{Inv}_A F \quad (i \in I)$$

if and only if

(a) $Q_i = E_i \cap \mathrm{LOC}[Q]$ $(i \in I)$ or

(b) $Q_i = E_i \cap s\text{-LOC}[Q]$ $(i \in I)$, resp.

P r o o f. (a)"$\Rightarrow$": Since $Q_i \subseteq \mathrm{Inv}\ F$ we have $Q \subseteq \mathrm{Inv}\ F$ and, by 3.8(i),1.11a', $\mathrm{LOC}[Q] \subseteq \mathrm{LOC}[\mathrm{Inv}\ F] = \mathrm{Inv}\ F$. Thus $E_i \cap \mathrm{LOC}[Q]$ $\subseteq E_i \cap \mathrm{Inv}\ F = Q_i \subseteq E_i \cap \mathrm{LOC}[Q]$ (since $Q_i \subseteq E_i \cap Q$) and we are done. "$\Leftarrow$": Take $F = \mathrm{Pol}\ Q$. Then (by 4.2a) $Q_i = E_i \cap \mathrm{LOC}[Q] =$ $E_i \cap \mathrm{Inv\ Pol}\ Q = E_i \cap \mathrm{Inv}\ F$. Case (b) can be proved analogeously. ∎

In addition to 6.2 it would be very interesting to have a condition for the finiteness of F (open problem in [J6n72 (p.41)]). For finite A we obtain:

6.3 Proposition. A clone $Q \subseteq R_A$ is the set of all invariants of a finite algebra $\langle A;F \rangle$ of finite type iff there exists an $s \in \mathbb{N}$ such that $s\text{-LOC}\ Q = Q$. ∎ (cf. 5.6)

For infinite A, we do not have such a full answer. Clearly, $\exists s \in \mathbb{N}$: $s\text{-LOC}\ Q = Q$ is still a necessary condition for $Q \subseteq R_A$ to be equal $\mathrm{Inv}\ F$ for a finite set $F \subseteq O_A$ (but unfortunately no longer sufficient). In the following proposition we give a sufficient condition by means of a chain condition.

6.4 Proposition. Let $Q \subseteq R_A$ and consider the conditions

(i) $LOC[Q] = Q$ and, for every down-directed system $\{Q_i \mid i \in I\}$ of local closed clones of relations (i.e., $\forall i \in I$: $Q_i = LOC[Q_i]$, $\forall i,j \in I \; \exists k \in I$: $Q_i \cap Q_j \quad Q_k$) $\bigcap_{i \in I} Q_i = Q$ implies the existence of a finite subset $I' \subseteq I$ such that $\bigcap_{i \in I'} Q_i = Q$;

(ii) $Q = \mathrm{Inv}\, F$ for a finite set $F \subseteq O_A$;

(iii) $\exists \, s \in \mathbb{N}$: $Q = s\text{-}LOC[Q]$.

Then the following implications hold:

(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) .

For finite A, all conditions are equivalent.

Proof. It remains (cf. 6.2) to prove (i)$\Rightarrow$(ii). Let $I = \{F \mid F \subseteq \mathrm{Pol}\, Q$ and F finite$\}$ and $Q_F = \mathrm{Inv}\, F$ for $F \in I$. Then $\{Q_F \mid F \in I\}$ is a down-directed set of local closed clones and $\bigcap_{F \in I} Q_F = \bigcap_{F \in I} \mathrm{Inv}\, F = \mathrm{Inv} \bigcup_{F \in I} F = \mathrm{Inv}\, \mathrm{Pol}\, Q \overset{(4.2)}{=} LOC[Q] = Q$. Thus, by (i), there exists a finite set $I' \subseteq I$ such that $\bigcap_{F \in I'} Q_F = Q$, i.e., $Q = \mathrm{Inv} \bigcup_{F \in I'} F$ with $\left| \bigcup_{F \in I'} F \right| < \aleph_0$. ∎

6.5 Unfortunately the author was not able to find "inner" conditions for Q which are equivalent to 6.4(ii) in case $|A| \geq \aleph_0$. Condition 6.4(i) looks on Q only "from the outside" because one has to consider the lattice of all clones on A.

To get (from 6.4(i)) a necessary and sufficient condition
sideration to be less than or equal to $|A| \geq \aleph_0$. For $Q \subsetneq R_A$ we define

$$Q^{\infty} := \mathrm{Inv}^{\infty}\mathrm{Pol}\ Q\ .$$

To give some sense to the theorem below we mention (but will not go into details) that Q^{∞} can be characterized as $\mathrm{LOC}[Q]_{\infty}$ (analogeously to $\mathrm{Inv\ Pol}\ Q = \mathrm{LOC}[Q]$) where $[Q]_{\infty}$ is the clone of relations (of arity $\leq |A|$) generated by Q. The definition of the clone $[Q]_{\infty}$ is a quite natural generalization of 3.5 ($[Q]_{\infty}$ is the closure with respect to 3.2(R0), (R1), (R2) and (R3)).

<u>6.6</u> Lemma. <u>For</u> $F \subseteq O_A$, $\langle F \rangle = \mathrm{Pol\ Inv}^{\infty} F$ (cf. [Kr/Poi], [Ros72]).

<u>6.7</u> Theorem. <u>For</u> $Q \subseteq R_A$, <u>the following conditions are equivalent</u> (assume $|A| \geq \aleph_0$):

(*) $Q = \mathrm{LOC}[Q]_{R_A}$ <u>and</u> <u>for every down-directed system</u> $\{Q_i \mid i \in I\}$ <u>of local closed clones</u> (cf. 6.4(i)) $\bigcap_{i \in I} Q_i^{\infty} = Q^{\infty}$ <u>implies the existence of a finite set</u> $I' \subsetneq I$ <u>such that</u> $\bigcap_{i \in I'} Q_i = Q$.

(**) <u>There exists a finite set</u> $F \subseteq O_A$ <u>with</u> $Q = \mathrm{Inv}_A F$.

P r o o f. (*)$\Rightarrow$(**): Let $I=\{F \mid F \subseteq \mathrm{Pol}\, Q,\ F \text{ finite}\}$ and $Q_F = \mathrm{Inv}\, F$. Then $Q_F = \mathrm{LOC}[Q_F]$ and $\{Q_F \mid F \in I\}$ is a down-directed system. We have $\mathrm{Pol}\ Q = \bigcup\{F \mid F\in I\} \subseteq \bigcup\{\mathrm{Pol\ Inv}\ F \mid F\in I\} \subseteq$ $\mathrm{Pol\ Inv\ Pol}\ Q = \mathrm{Pol}\ Q$, i.e., $\mathrm{Pol}\ Q = \bigcup_{F\in I} \mathrm{Pol\ Inv}\ F$. Therefore $\bigcap_{F\in I} Q_F^{\infty} = \bigcap_{F\in I} \mathrm{Inv}^{\infty}\mathrm{Pol\ Inv}\ F = \mathrm{Inv}^{\infty} \bigcup_{F\in I} \mathrm{Pol\ Inv}\ F = \mathrm{Inv}^{\infty}\mathrm{Pol}\ Q = Q^{\infty}$. By (*), $\bigcap_{F\in I'} Q_F = Q$ for a finite $I' \subseteq I$, i.e., $Q = \bigcap_{F\in I'} \mathrm{Inv}\ F$ $= \mathrm{Inv} \bigcup_{F\in I'} F$ where $|\bigcup_{F\in I'} F| < \aleph_0$.

(**)$\Rightarrow$(*): Assume $Q = \mathrm{Inv}\ F$, F finite. Let $\bigcap_{i\in I} Q_i^{\infty} = Q^{\infty}$ and $Q_i = \mathrm{LOC}[Q_i]$. We divide the proof of the existence of a finite I' into six parts a)-f). We have:

a) $\mathrm{Pol}\ Q_i^{\infty} = \mathrm{Pol}\ Q_i$ since $\mathrm{Pol}\ Q_i^{\infty} = \mathrm{Pol\ Inv}^{\infty}\mathrm{Pol}\ Q_i \overset{(6.6)}{=} \langle \mathrm{Pol}\ Q_i\rangle$.

b) $\mathrm{Inv}^{\infty}\tilde{F} = Q^{\infty}$ where $\tilde{F} := \bigcup_{i\in I} \mathrm{Pol}\ Q_i^{\infty}$, in fact: $Q^{\infty} = \bigcap Q_i^{\infty} =$ $\bigcap \mathrm{Inv}^{\infty}\mathrm{Pol}\ Q_i^{\infty} = \mathrm{Inv}^{\infty}(\bigcup \mathrm{Pol}\ Q_i^{\infty}) = \mathrm{Inv}^{\infty}\tilde{F}$.

c) $\langle F\rangle \subseteq \langle \tilde{F}\rangle$, since $\mathrm{Inv}^{\infty}\tilde{F} \overset{b)}{=} Q^{\infty} = \mathrm{Inv}^{\infty}\mathrm{Pol}\ Q = \mathrm{Inv}^{\infty}\mathrm{Pol\ Inv}\ F \subseteq \mathrm{Inv}^{\infty}$ $\mathrm{Pol\ Inv}^{\infty} F \overset{(6.6)}{=} \mathrm{Inv}^{\infty}\langle F\rangle$, therefore $\mathrm{Pol\ Inv}^{\infty}\tilde{F} \supseteq \mathrm{Pol\ Inv}^{\infty}\langle F\rangle$.

d) $\exists I' \subseteq I$, I' finite: $\langle F\rangle \subseteq \langle \bigcup_{i\in I'} \mathrm{Pol}\ Q_i^{\infty}\rangle$, since: By c), $F \subseteq \langle\tilde{F}\rangle$ implies $\exists \tilde{F}' \subseteq \tilde{F}$, $\tilde{F}$ finite: $F \subseteq \langle \tilde{F}'\rangle$. By definition of $\tilde{F}$, there exists a finite $I' \subseteq I$ such that $\langle \bigcup_{i\in I'} \mathrm{Pol}\ Q_i^{\infty}\rangle \supseteq \langle \tilde{F}'\rangle \supseteq \langle F\rangle$.

e) $Q = \mathrm{Inv}(\bigcup_{i\in I'} \mathrm{Pol}\ Q_i^{\infty})$, since $Q = \mathrm{Inv}\ F = \mathrm{Inv}\langle F\rangle \overset{d)}{\supseteq} \mathrm{Inv}(\bigcup_{i\in I'}$ $\mathrm{Pol}\ Q_i^{\infty}) \supseteq \mathrm{Inv}\ \tilde{F} = \mathrm{Inv}^{\infty}\tilde{F} \cap R_A \overset{b)}{=} Q^{\infty} \cap R_A = \mathrm{Inv}^{\infty}\mathrm{Pol}\ Q \cap R_A = \mathrm{Inv\ Pol}\ Q$ $= \mathrm{LOC}[Q] = Q$.

f) $Q = \bigcap_{i\in I'} Q_i$, since: $Q \overset{e)}{=} \mathrm{Inv}(\bigcup_{i\in I'} \mathrm{Pol}\ Q_i^{\infty}) = \bigcap_{i\in I'} \mathrm{Inv\ Pol}\ Q_i^{\infty}$ $\overset{a)}{=} \bigcap_{i\in I'} \mathrm{Inv\ Pol}\ Q_i = \bigcap_{i\in I'} Q_i$. We are done. ∎

§7 Concrete characterizations III.

7.1 Characterization problem:

Let A be a set, $G \leq S_A$ a permutation group, $H \leq O_A^{(1)}$ a transformation semigroup, L a subset of 2^A(power set of A) and let C be a subset of $\mathcal{E}(A)$(set of all equivalence relations on A). Does there exist a universal algebra $\mathfrak{A} = \langle A; F \rangle$ where

(A) $F \subseteq O_A$ arbitrary ("general case")

(B) $F \subseteq O_A^{(s)}$ for some $s \in \mathbb{N}$ ("bounded case")

(C) F finite ("finite case"),

such that

$G = \mathrm{Aut}\,\mathfrak{A}$ (automorphism group) and/or

$H = \mathrm{End}\,\mathfrak{A}$ (endomorphism monoid) and/or

$L = \mathrm{Sub}\,\mathfrak{A}$ (subalgebra lattice) and/or

$C = \mathrm{Con}\,\mathfrak{A}$ (congruence lattice) ?

Remarks: (i) Because of 1.6d) we need not distinguish between w-Aut and Aut.

(ii) Without loss of generality one can assume at once L and C to be algebraic sublattices.

Theorem 6.2 provides an answer for case (A) and (B) while case (C) can be treated with 6.4 and 6.7.

Some of the above problems 7.1 have a well-known answer, some others were still open (in particular the simultaneous characterization of G,H,L,C). The known answers sometimes are better than those given by 6.2 because the use of clones of relations is avoided or reduced to simpler closure properties. In general however, we think there is little hope to find conditions for G, H, L and C which are not based on closure properties of the clone of relations generated by $G^{\bullet} \cup H^{\bullet} \cup L \cup C$. We have:

7.2 Theorem. Let G, H, L and C as in 7.1 and $Q = G^{\bullet} \cup H^{\bullet} \cup L \cup C$. Then there exists a universal algebra $\mathfrak{A} = \langle A; F \rangle$ with

(A) $F \subsetneq O_A$ or

(B) $F \subsetneq O_A^{(s)}$ $(s \in \mathbb{N})$ resp.,

such that

$$G = \operatorname{Aut} \mathfrak{A} \quad , \quad H = \operatorname{End} \mathfrak{A}$$
$$L = \operatorname{Sub} \mathfrak{A} \quad , \quad C = \operatorname{Con} \mathfrak{A}$$

if and only if

(A) $G = S_A \cap H$,
$H^{\bullet} = (O_A^{(1)})^{\bullet} \cap \mathrm{LOC}[Q]$,
$L = \mathrm{LOC}[Q]^{(1)}$,
$C = \mathcal{E}(A) \cap \mathrm{LOC}[Q]$ or

(B) $G = S_A \cap H$,
$H^{\bullet} = (O_A^{(1)})^{\bullet} \cap s\text{-}\mathrm{LOC}[Q]$,
$L = s\text{-}\mathrm{LOC}[Q]^{(1)}$
$C = \mathcal{E}(A) \cap s\text{-}\mathrm{LOC}[Q]$ resp. ∎ (6.2)

Another application of 6.2 provides the characterization of bicentralizers of universal algebras ($F = \mathrm{Pol\ Pol}\ F'$ is called the <u>bicentralizer</u> of $F' \subseteq O_A$):

<u>7.3</u> Proposition. <u>For</u> $F \subseteq O_A$, F <u>is a bicentralizer iff</u>

$$F^{\bullet} = O_A^{\bullet} \cap \mathrm{LOC}[F^{\bullet}].$$

<u>Proof</u>. by 6.2, note $(\mathrm{Pol\ Pol}\ F')^{\bullet} = O_A^{\bullet} \cap \mathrm{Inv\ Pol}\ F'$. ■

<u>Remarks</u>: 7.3 can be found also in [Sz78(Thm.13)]; for finite A this is a result of A.V. Kuznecov (cf.[Va],[Sz78]). Sufficient conditions for F to be a bicentralizer were also given in [Fa]. Bicentralizers in $O_{\underline{3}}$ are described in [Dani].

In the next paragraphs we list (most of) the problems in 7.1 and (some of) their solutions adjoining some (but surely not all) references. The propositions are marked with

(<u>A</u>), (<u>B</u>) or (<u>C</u>)

whenever case 7.1(<u>A</u>),(<u>B</u>) or (<u>C</u>) is treated. The notations of this paragraph will also be used in the following ones.

It is worthy to note that all the characterization problems above have a non-tivial solution, i.e. not all structures under consideration are related to some universal algebra. We do not mention this fact explicitely every time but it follows directly from the corresponding characterization theorems which allow the construction of counterexamples.

§8 Concrete characterization of Aut $\mathfrak{A}$

$$\boxed{\exists\, \mathfrak{A} = \langle A; F\rangle : \quad G = \mathrm{Aut}\,\mathfrak{A} \quad ?}$$

The full answer to this problem firstly was given by B. Jónsson in [Jón68] (cf. [Jón72 (2.4.3)], namely:

(A)8.1 Theorem ([Jón68]). $\exists\, \mathfrak{A} : G = \mathrm{Aut}\,\mathfrak{A} \iff G = S_A \cap \mathrm{Loc}\langle G\rangle_{S_A}$;
i.e., a group G of permutations (on A) is the automorphism group of some algebra iff the following condition holds: For every $h \in S_A$, if for every finite subset B of A there exists a member of G that agrees with h on B, then $h \in G$.

Remark: $\mathfrak{A}$ can be chosen as a simple algebra (cf. 13.5).

The following answer follows from 6.2:

(A)8.2 Proposition. $\exists\, \mathfrak{A} : G = \mathrm{Aut}\,\mathfrak{A} \iff G^{\bullet} = S_A^{\bullet} \cap \mathrm{LOC}[G^{\bullet}]$ (and $G \leq S_A$). ■

Note that the result 8.1 is much better than 8.2 because the local closure of the clone of operations generated by G is much less complicated than the local closure of the clone of relations generated by $G^{\bullet}$. But this case shall serve us as an example to show how to get completely new proofs within the framework of our General Galois theory; we shall see how to work with clones of relations (because in other cases only this method works). Therefore we give the

P r o o f of 8.1 using 8.2:

"$\Rightarrow$": Clearly, $G=\mathrm{Aut}\,\mathfrak{A}$ implies $G=\langle G\rangle_{S_A}$ and $G=S_A\cap \mathrm{Loc}\,G$ (4.6).

"$\Leftarrow$": We show $S_A^{\bullet}\cap \mathrm{LOC}[G^{\bullet}]=(S_A\cap \mathrm{Loc}\langle G\rangle_{S_A})^{\bullet}$ because then 8.1 immediately follows from 8.2. Firstly,

$$(S_A\cap \mathrm{Loc}\langle G\rangle_{S_A})^{\bullet}\subseteq S_A^{\bullet}\cap \mathrm{LOC}[G^{\bullet}] \text{ by 4.7.}$$

Secondly we show the opposite inclusion. Since $\langle G\rangle_{S_A}{}^{\bullet}\subseteq[G^{\bullet}]$ we can assume $G=\langle G\rangle_{S_A}$. Let $f^{\bullet}\in S_A^{\bullet}\cap \mathrm{LOC}[G^{\bullet}]$. We must show $f\in \mathrm{Loc}\,G$.

By definition 1.9, for each finite subset $\overline{B}\subseteq f^{\bullet}$ there exists a general superposition $\sigma\in[G^{\bullet}]$ such that

$$\overline{B}\subseteq\sigma\subseteq f^{\bullet}.$$

Note that there is a 1-1 correspondence between $\overline{B}$ and finite $B\subseteq A$ by $B=\{x \mid (x,y)\in f^{\bullet}\}$ and we have $\overline{B}=(f|B)^{\bullet}$.

Since $\sigma\in[G^{\bullet}]$ there exist $g_i\in G$ ($i\in I$) such that

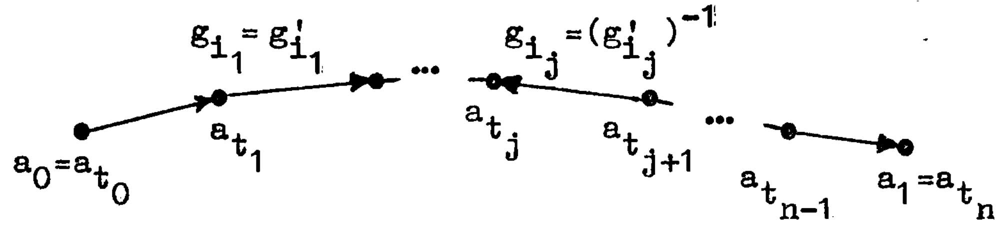
$$(a_0,a_1)\in\sigma \iff \exists (a_j)_{j\in\underline{\alpha}\setminus\underline{2}}: (a_{\pi_i(0)},a_{\pi_i(1)})\in g_i^{\bullet} \ (i\in I)$$

for suitable $\pi_i:\underline{2}\longrightarrow\underline{\alpha}$ (α ordinal), cf. 3.2(R4).

Consider (the quantifier free part of) this formula as a labled graph with vertex set $V=\{a_j \mid j\in\alpha\}$, and for $(\pi_i(0),\pi_i(1))=(t,t')$ we take an arrow from a_t to $a_{t'}$ with label g_i ($i\in I$). We distinguish two cases (Remark: the used arguments turn out to be here the same as discussed in [Sa/St77c(p.225)] in the language of equations in semigroups):

Case (i): The vertices a_0 and a_1 are connected, i.e. there exist distinct vertices $a_0=a_{t_0},a_{t_1},\dots,a_{t_{n-1}},a_{t_n}=a_1$ such that there is an edge from a_{t_j} to $a_{t_{j+1}}$ or from $a_{t_{j+1}}$ to a_{t_j} with label g_{i_j} ($j\in\underline{n}$). Put $g'_{i_j}:=g_{i_j}$ in the former case and $g'_{i_j}:=g_{i_j}^{-1}$ in the latter one.

E.g.:



Then we get for σ:

$$\sigma \subseteq \{(a_{t_0}, a_{t_n}) \mid \exists a_{t_1}, \dots, a_{t_{n-1}} : g'_{i_0}(a_{t_0}) = a_{t_1}, \\ g'_{i_1}(a_{t_1}) = a_{t_2}, \dots, g'_{i_{n-1}}(a_{t_{n-1}}) = a_{t_n}\}$$

i.e., $\sigma \subseteq g^\bullet$ where $g = g'_{i_0} g'_{i_1} \dots g'_{i_{n-1}} \in \langle G \rangle_{S_A} = G$.
Since $\overline{B} \subseteq \sigma$ we have $\overline{B} \subseteq g^\bullet$. But this implies $(g|B)^\bullet = \overline{B}$, i.e., $g|B = f|B$ (and $g \in G$).

Case (ii): The vertices a_0 and a_1 are not connected. Let $(x,y),(x',y')$ be distinct elements of $\overline{B} \subseteq \sigma$. By the disconnectedness of a_0 and a_1, the element (x,y') also satisfies the above formula for $\sigma \in [G^\bullet]$, i.e. $(x,y') \in \sigma$ in contradiction to $\sigma \subseteq f^\bullet$ ($f \in S_A$ therefore $f(x) = y \neq y' = f(x')$). Clearly, $\overline{B}$ must have at least two elements what we can assume without loss of generality.

Thus only case (i) occurs: For all finite $B \subseteq A$ there is a $g \in G$ with $f|B = g|B$, i.e. $f \in \mathrm{Loc}\, G$. ■

Remark: If one permits to use algebras with infinitary operations then every permutation group G is the automorphism-group of an algebra (cf. [Ar/Schm]).

Let us consider now the "bounded case" (B) (cf. 7.1). This case was treated by E. Płonka [Pł] and B. Jónsson [Jón72], too: Defining (for $G \leq S_A$)

<u>8.3</u> $C_G(B) := \{a \in A \mid \forall f, g \in G : f|B = g|B \Rightarrow f(a) = g(a)\}$ for $B \subseteq A$,

they obtained the following theorems:

(<u>B</u>)<u>8.4</u> Theorem([P1],[J6n72(2.4.1)]). <u>There exists an algebra</u> $\mathfrak{A} = \langle A; F\rangle$ <u>with</u> $F \subseteq O_A^{(s)}$ $(s \geq 2)$ <u>such that</u> $G = \mathrm{Aut}\,\mathfrak{A}$ <u>iff</u> G <u>is a permutation group and</u> $f \in S_A$ <u>belongs to</u> G, <u>whenever for all</u> $B \subseteq A$ <u>with at most</u> s <u>elements there exists a</u> $g \in G$ <u>that agrees with</u> f <u>on</u> $C_G(B)$.

(<u>B</u>)<u>8.5</u> Theorem([J6n72(2.4.4)]). $\exists\, F \subseteq O_A^{(1)} : G = \mathrm{Aut}\, F \Leftrightarrow$

$$G = \langle G\rangle_{S_A} \text{ and } \forall f \in S_A \Big(\forall a \in A: C_G(\{a\}) \neq \{a\} \text{ or } C_G(\{f(a)\}) \neq \{f(a)\} \Rightarrow \exists g \in G : g|_{C_G(\{a\})} = f|_{C_G(\{a\})} \Big) \Rightarrow f \in G.$$

Let us compare this with the result which follows directly from 6.2 for arbitrary $s \in \mathbb{N}$:

(<u>B</u>)<u>8.6</u> Proposition. <u>There exists an algebra</u> $\mathfrak{A} = \langle A; F\rangle$ <u>with</u> $F \subseteq O_A^{(s)}$ <u>such that</u> $G = \mathrm{Aut}\,\mathfrak{A}$ <u>iff</u> $G^{\bullet} = S_A^{\bullet} \cap s\text{-LOC}[G^{\bullet}]$. ∎

We see that it is possible (cf. 8.4 and 8.6) to describe the elements of s-LOC$[G^{\bullet}]$ which are permutations (i.e. elements of $S_A^{\bullet}$) in terms of permutations and very special relations $C_G(B)$. Of course, one can deduce 8.4 from 8.6 in the same manner as 8.1 from 8.2.

The structure of $C_G(B)$ can be described also as follows (cf. 8.3):

8.7 Proposition. For $B=\{b_1,\dots,b_n\}\subseteq A$ and $G\leqq S_A$ we have:

$$C_G(B)=\{h(b_1,\dots,b_n)\mid h\in Pol^{(n)}G\} = \Gamma_{Pol^{(n)}G}(B)\in[G^{\cdot}]^{(1)}.$$

Proof. The right part follows from 2.4 and 4.3. We show that $C_G(B)$ contains exactly all $h(b_1,\dots,b_n)$ $(h\in Pol^{(n)}G)$. Clearly, for $f,g\in G$ and $f|B=g|B$ we have $f(h(b_1,\dots,b_n)) = h(fb_1,\dots,fb_n) = h(gb_1,\dots,gb_n) = g(h(b_1,\dots,b_n))$, i.e., all $h(b_1,\dots,b_n)$ belong to $C_G(B)$. Let $a\in C_G(B)$. It remains to prove that there is an $h\in Pol^{(n)}G$ such that $a=h(b_1,\dots,b_n)$. Define $h\in O_A^{(n)}$ as follows:

$$h(fb_1,\dots,fb_n):= f(a) \quad \text{for} \quad f\in G \quad \text{and}$$
$$h(x_1,\dots,x_n):=x_1 \quad \text{for} \quad (x_1,\dots,x_n)\in \varrho := A^n\setminus \{(fb_1,\dots,fb_n)\mid f\in G\}.$$

Then, for $g\in G$, we get

$$g(h(fb_1,\dots,fb_n)) = g(f(a)) = h(g(fb_1),\dots,g(fb_n))$$

or $$g(h(x_1,\dots,x_n)) = g(x_1) = h(gx_1,\dots,gx_n);$$

(note that $(x_1,\dots,x_n)\in\varrho \Longleftrightarrow (gx_1,\dots,gx_n)\in\varrho$ because G is a group), consequently, $h\in Pol^{(n)}G$ and $a=h(b_1,\dots,b_n)$. ∎

A comparision of 8.1 and 8.2 with 8.4 and 8.6 leads to the question which role plays the condition $G = s\text{-}Loc\langle G\rangle_{S_A}$ for case (B). The result coincides (for $s\in\mathbb{N}$) with Lemma 2.4.2 in [J6n72]:

(B)8.8 Proposition. For $G\subseteq S_A$, $s\in\mathbb{N}$, consider the conditions

(i) $G = S_A \cap s\text{-}Loc\langle G\rangle_{S_A}$

(ii) $\exists\ F\subseteq O_A^{(s)}$: $G = Aut_A F$ (or equivalently, $G = Aut\ Pol^{(s)}G$)

(iii) $G = S_A \cap (s+1)\text{-Loc}\langle G\rangle_{S_A}$.

<u>Then the following implications hold:</u>

(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) .

P r o o f. If $G = S_A \cap s\text{-Loc}\langle G\rangle_{S_A}$ (cf. 1.9) then the condition in 8.4 is fulfilled since $g|_{C_G(B)} = f|_{C_G(B)}$ implies $g|_B = f|_B$, consequently $G = \text{Aut } F$ for some $F \subseteq O_A^{(s)}$ by 8.4.
Now, if $G = \text{Aut } F = S_A \cap \text{Pol } F^{\bullet}$ (cf.1.6d), $F \subseteq O_A^{(s)}$, then, by 5.2, we have $G = S_A \cap (s+1)\text{-Loc } G$ (since $F^{\bullet} \subseteq R_A^{(s+1)}$). ∎

In 8.8, the inverse implications do not hold in general (as shown in [Jōn72(p.37)]). The next theorem will show the nice result of M. Gould that the solution of the "bounded case"(B) provides at the same time the solution of the "finite case"(C)(cf.7.1). This result also shows that the conditions in 8.8 become equivalent after quantifying s .

(C)8.9 Theorem([Go72a]). <u>For</u> $G \subseteq S_A$, <u>the following conditions are equivalent</u>:

(i) $\exists$ finite $F \subseteq O_A$: $G = \text{Aut } F$;
(ii) $\exists\, s \in \mathbb{N}\ \exists\, F \subseteq O_A^{(s)}$: $G = \text{Aut } F$;
(iii) $\exists\, f \in O_A$: $G = \text{Aut}\{f\}$;
(iv) $\exists\, s \in \mathbb{N}$: $G = S_A \cap s\text{-Loc}\langle G\rangle_{S_A}$.

P r o o f. Clearly (iii)$\Rightarrow$(i)$\Rightarrow$(ii)$\overset{(8.8)}{\Rightarrow}$(iv). For (i)$\Rightarrow$(iii) (easy to do) and (iv)$\Rightarrow$(i)(crucial point of the proof) we refer to [Go72a(pp. 1066,1067)]. (∎)

8.10 Remarks.

a) Loc G (or s-Loc G resp.) is the least group which contains a permutation group $G \leq S_A$ and which is at the same time the automorphism group of an algebra (or an algebra with operations of rank $\leq s$, resp., $s \in \mathbb{N}$). A similiar observation fails for algebras with finitely many operations: If

$$S_A \cap 1\text{-Loc}\, G \supset \dots \supset S_A \cap s\text{-Loc}\, G \supset S_A \cap (s+1)\text{-Loc}\, G \supset \dots$$

is an infinite chain than - by 8.9 - there does not exist a least automorphism group of an algebra $\mathfrak{A} = \langle A; F \rangle$ with finite F and $G \subseteq \text{Aut}\,\mathfrak{A}$.

Example:

Let $A = \bigcup \{A_n \mid n \in \mathbb{N}\}$ be the union of disjoint sets A_n with $|A_n| = n$. Let $G \leq S_A$ be the group consisting of all permutations $f \in S_A$ such that $(f|A_n) \in \mathfrak{A}_n$ (for $n \in \mathbb{N}$ where $\mathfrak{A}_n$ is the alternating group on A_n).

Because $(n-2)\text{-Loc}\, \mathfrak{A}_n = S_n > \mathfrak{A}_n$ but $(n-1)\text{-Loc}\, \mathfrak{A}_n = \mathfrak{A}_n$ one easily proves that

$$S_A \cap s\text{-Loc}\, G \gneqq S_A \cap (s+1)\text{-Loc}\, G \quad (\text{for } s \geq 2). \blacksquare$$

b) For the concrete characterization of the (lattice of all) automorphism groups of all subalgebras (of a given universal algebra) we refer to [Ko] (for the abstract version see [Fr/Si]). The semigroup of all local automorphisms is investigated and characterized in [Br], [Sz75].

§9 Concrete characterization of End $\mathfrak{A}$

$\exists\, \mathfrak{A} = \langle A; F\rangle$: $H = \mathrm{End}\, \mathfrak{A}$?

M. Armbrust and J. Schmidt showed in [Ar/Schm] that every (abstract) monoid is isomorphic to the endomorphism semigroup of some universal algebra. They observed also that the concrete characterization problem for transformation semigroups has a non-trivial solution (i.e. there are monoids which are not equal to the endomorphism semigroup of an algebra).

Obviously, the existence of an $\mathfrak{A}$ with $H = \mathrm{End}\,\mathfrak{A}$ is equivalent to the condition $H = \mathrm{End\ Pol}\ H$. In [Sa/St77c] the set End Pol H (for $H \subseteq O_A^{(1)}$) is called the *algebraic closure* of H.

As N. Sauer and M.G. Stone pointed out in [Sa/St77c], the determination of the algebraic closure of semigroups is related to broad questions posed by E.S. Ljapin [Lj(p.25)] and S. Ulam [Ul(p.32)] regarding the determination of algebraic structures from given endomorphisms.

By a result of J. Sichler [Si](for finite A, $|A| \geq 5$) the algebraic closure of H contains all maps in $O_A^{(1)}$ if H contains anything more than constant maps and all of the permutations $f \in S_A$. If H consists of locally invertible and constant maps only, a necessary and sufficient condition for H to be algebraic (i.e. $H = \mathrm{End\ Pol}\ H$) is to be found in [St75] (cf. [St69]), where results of W.A. Lampe and G. Grätzer ([La68], [Gr/La68]) had been generalized.

N. Sauer and M.G. Stone gave a characterization of the algebraic closure of H for $H = \{f\}$ $(f \in O_A^{(1)})$ in [Sa/St77b] and for arbitrary $H \subseteq O_A^{(1)}$ in [Sa/St77c] in terms of "equational conditions". The same question (which was posed explicitely as problem 3 in [Gr(p.77)]) is treated by L. Szabó in [Sz78(Thm.15)] who gave a characterization by means of formula schemes.

Our approach provides a characterization theorem in terms of clones of relations: From 6.2 we conclude directly:

(A)9.1 Theorem(cf.[Sz78(Thm.15)]. For $H \subseteq O_A^{(1)}$, there exists an algebra $\mathfrak{A} = \langle A; F \rangle$ with $H = \mathrm{End}\,\mathfrak{A}$ iff $H^\bullet = O_A^{(1)\bullet} \cap \mathrm{LOC}[H^\bullet]$. ∎

(B)9.2 Theorem. For $H \subseteq O_A^{(1)}$, $s \in \mathbb{N}$, there exists an algebra $\mathfrak{A} = \langle A; F \rangle$ with $F \subseteq O_A^{(s)}$ and $H = \mathrm{End}\,\mathfrak{A}$ iff $H^\bullet = O_A^{(1)\bullet} \cap s\text{-LOC}[H^\bullet]$. ∎

9.3 Remarks.

(i) The algebraic closure of $H \subseteq O_A^{(1)}$ is the least endomorphism monoid containing H. Thus we have (by 9.1, 9.2):

For $H \subseteq O_A^{(1)}$, the set of all $f \in O_A^{(1)}$ with $f^\bullet \in \mathrm{LOC}[H^\bullet]$ (or $f^\bullet \in s\text{-LOC}[H^\bullet]$, resp.) is the least semigroup which contains H and which is the endomorphism monoid of an algebra (or an algebra with operations of rank $\leq s$, resp.).

(ii) The operators Pol and End define a Galois connection between subsets of $O_A^{(1)}$ and subsets of O_A. The Galois closed sets with respect to this Galois connection were characterized in [Sa/St78](F=Pol End F) and [Sa/St77c](H = End Pol H, cf. 9.1). Because Pol-End is the restriction of the Galois connection Inv-End (i.e. Inv-Pol$^{(1)}$) to operations only, the result in [Sa/St78(Thm.1)] can be considered as a special case of theorem 4.2(b)(or 6.2) for $s = 1$. In fact, we have:

Proposition: $F = \mathrm{Pol}\ \mathrm{End}\ F \iff F^\bullet = O_A^\bullet \cap \mathrm{Inv}\ \mathrm{Pol}^{(1)} F$

$\iff F = O_A^\bullet \cap 1\text{-LOC}[F^\bullet]$. ∎

Note that $1\text{-LOC}[F^\bullet]$ is easy to describe: Take the clone of relations generated by $F^{\bullet}$ and then the closure with respect to arbitrary unions (cf. 1.13).

(iii) The condition for H in 9.1 cannot be replaced by $H = \mathrm{Loc}\langle H\rangle_{O_A^{(1)}}$ (Thm.9.6 (or Thm.1 in [St75]) provides counter-examples).

(C)9.4 For the finite case (C) of the characterization problem we do not have such a good solution as theorem 8.9 for groups. From 6.7 we get:

$\exists$ finite F : $H = \mathrm{End}_A F \iff H^{\bullet} = (O_A^{(1)})^{\bullet} \cap \mathrm{LOC}[H^{\bullet}]$ and $Q = \mathrm{LOC}[H^{\bullet}]$ satisfies 6.7(*).

But this condition is not very satisfactorily.

At the end of this paragraph we present the s-local version of a result of M.G. Stone [St75] (which is an extension of proposition 8.8 to certain monoids). A careful examination of the proof of theorem 1 in [St75] shows that, in fact, there was proven theorem 9.6 below, too. However we shall present another proof based on the general characterization theorem 9.2 (or 6.2).

9.5 Definitions. Call a monoid $M \leqq O_A^{(1)}$ s-algebraic ($s \in \mathbb{N}$) if $\exists \mathfrak{A} = \langle A; F\rangle$: $F \subseteq O_A^{(s)}$ and $M = \mathrm{End}\,\mathfrak{A}$. A monoid $E \leqq O_A^{(1)}$ is said to be s-locally invertible (or locally invertible) if for all $n \leqq s$ (or $n \in \mathbb{N}$, respectively), for all $(a_i)_{i\in \underline{n}}, (b_i)_{i\in\underline{n}} \in A^n$ and for all $f,g \in E$, $f(a_i) = g(b_i)$ $(i \in \underline{n})$ implies the existence of an $h \in E$ such that $h(a_i) = b_i$ $(i \in \underline{n})$.

In an s-locally invertible monoid ($s \geq 2$) each map is injectiv (but not necessarily surjectiv). Let c_a be the constant function $c_a: x \mapsto a$.

For $M \subseteq O_A^{(1)}$ we define:

$$\varkappa(M) := \{c_a \mid \forall b \in A,\ b \neq a,\ \exists\ f,g \in M : f(a)=g(a)\ \&\ f(b) \neq g(b)\}.$$

(B)9.6 Theorem(cf.8.8). Let $s \in \mathbb{N}$ and let $M = E \cup K$ be a monoid where $E \leq O_A^{(1)}$ is s-locally invertible and K is a set of constant maps on A. Then the implications

$$(i) \Rightarrow (ii) \Rightarrow (iii)$$

hold for the following conditions:

(i) $\varkappa(M) \subseteq M$ and M is s-locally closed (i.e. $M = s\text{-Loc } M$),

(ii) M is s-algebraic, i.e. $\exists\ F \subseteq O_A^{(s)}: M = \mathrm{End}_A F$,

(iii) $\varkappa(M) \subseteq M$ and M is (s+1)-locally closed (i.e. $M = (s+1)\text{-Loc } M$).

9.7 Remarks. a) From 9.6 we get immediately the following Proposition. If E is locally invertible then M is s-algebraic for some $s \in \mathbb{N}$ iff M is s'-locally closed for some $s' \in \mathbb{N}$ and $\varkappa(M) \subseteq M$. ∎

b) This proposition (and 9.6) show (in comparision with 9.2) that the restriction to certain monoids M (of a relatively small class) leads to an improvement of the conditions for characterizing s-algebraicity, namely the involved s-local closure of $[M^\bullet]$ can be replaced by the simpler s-local closure of $\langle M \rangle_{O_A^{(1)}}$ ($= M$ since M is supposed here to be a monoid).

Proof of 9.6 (using 9.2):

We have (ii)$\Rightarrow$(iii) since $M=\mathrm{End}_A F$ implies $M=\mathrm{End}_A F^{\bullet}$ and therefore $M=(s+1)$-Loc M by 5.2; moreover, for $c_a \in \varkappa(M)$ and $h \in F^{(n)}$ we have $h(a,\dots,a)=a$, i.e. $c_a \in M$, because $h(a,\dots,a)=b \neq a$ would imply $f(b)=f(h(a,\dots,a))=h(fa,\dots,fa)=h(ga,\dots,ga) =g(h(a,\dots,a))=g(b)$ in contradiction to $f(b)\neq g(b)$ (where f,g are as in the definition of $\varkappa(M)$, cf. 9.5).

For the proof of (i)$\Rightarrow$(ii) we proceed as in the proof of 8.1: We will show $M^{\bullet} = O_A^{(1)\,\bullet} \cap s\text{-LOC}[M^{\bullet}]$, then we are done by 9.2. Obviously, $M^{\bullet} \subseteq (O_A^{(1)})^{\bullet} \cap s\text{-LOC}[M^{\bullet}]$.

To show the opposite inclusion, let $f \in O_A^{(1)}$ and $f^{\bullet} \in s\text{-LOC}[M^{\bullet}]$. Then, for $B \subseteq A, |B| \leq s$, $\overline{B} = \{(x,f(x)) \mid x \in B\} = (f|_B)^{\bullet}$, there exists an $\sigma \in [M^{\bullet}]$ such that

$$\overline{B} \subseteq \sigma \subseteq f^{\bullet} \quad \text{(cf. 1.9).}$$

We will find an $g \in M$ such that $g^{\bullet} \supseteq \overline{B}$, obviously this will imply $f|_B = g|_B$, i.e. (since B was chosen arbitrarily) $f \in s$-Loc $M = M$ (by (i)), consequently $M^{\bullet} \supseteq O_A^{(1)\,\bullet} \cap s\text{-LOC}\, M^{\bullet}$ and the proof will be finished.

How to find now the $g \in M$? Since $\sigma \in [M^{\bullet}]$ there exist $g_i \in M$ ($i \in I$) such that

$$(a_0,a_1) \in \sigma \iff \exists (a_j)_{j \in \underline{\alpha} \setminus \underline{2}}\colon (a_{\pi_i(0)}, a_{\pi_i(1)}) \in g_i^{\bullet} \ (i \in I)$$

for suitable $\pi_i\colon \underline{2} \longrightarrow \underline{\alpha}$ (α ordinal, cf. 3.2(R4)).

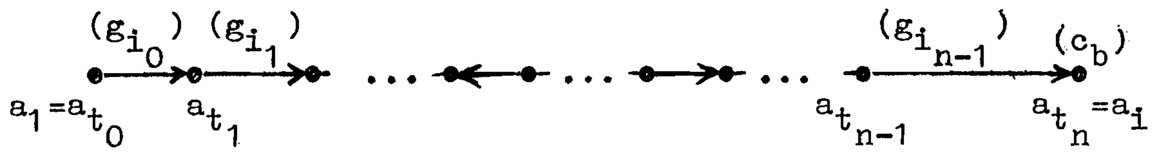
We consider this formula as a labled graph with vertex set $V = \{a_j \mid j \in \underline{\alpha}\}$ such that $a_j \in V$ will get the label c_b if, for some $i \in I$, $a_{\pi_i(1)} = a_j$ and $g_i \in K$ is the constant function c_b (we can assume $\sigma \neq \emptyset$, therefore this labeling is consistent); moreover, for $(a_{\pi_i(0)}, a_{\pi_i(1)}) = (a_t, a_{t'})$ we take an arrow (edge) from a_t to $a_{t'}$ with label g_i whenever $g_i \in E$

(more than one edge between two points is not excluded).

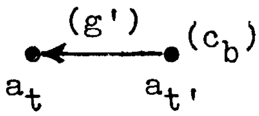
We distinguis two cases.

Case 1: The vertices a_0 and a_1 are not connected. Then there exists a constant $d \in A$ such that $(x,y) \in \sigma \Rightarrow y = d$, i.e., $\sigma \subseteq c_d^{\bullet}$, since $(x,y),(x',y') \in \sigma$ imply $(x,y') \in \sigma$ by disconnectedness, therefore $y=y'$ because of $\sigma \subseteq f^{\bullet}$ $(f \in O_A^{(1)})$.
We are going to show $c_d \in M$.

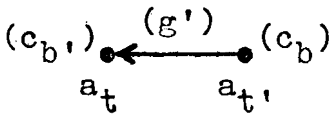
Subcase 1a: There is a vertex a_i which is connected with a_1 and labeled with a constant c_b, i.e. we have the following situation (the labels are put in parenthesis):



Now, we "move" the constant label from a_i to the vertex a_1 by the following induction steps (which do not change the property of (a_0,a_1) belonging to σ, i.e. we change the above formula for σ without changing the relation σ defined by this formula):

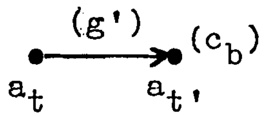


is equivalent to

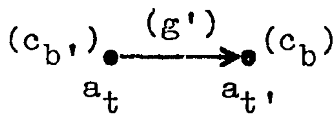


with $b' := g'(b)$.

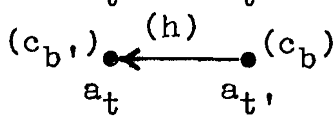
Since E is s-locally invertible, $g'(a_t) = a_{t'} = e(a_{t'})$ (e identity, $g' \in E$) implies the existence of an $h \in E$ such that $a_t = h(a_{t'})$ and we have:



is equivalent to



or to

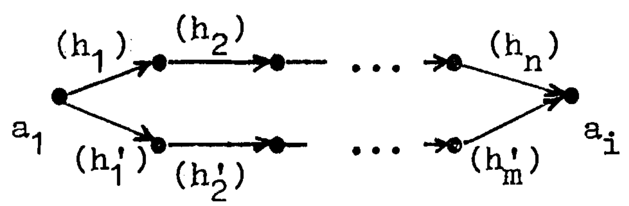


with $b' = h(b)$.

In both situations, $c_{b'} \in M$ (since $g', h, c_b \in M$). After n steps we get a label $c_{d'} \in M$ for the vertex a_1, i.e. $(a_0,a_1)\in\sigma \Rightarrow a_1 = d'$. We have $d=d'$ since $(a_0,a_1)\in\sigma \Rightarrow a_1 = d$. Thus $\sigma \subseteq c_d^{\bullet} \in M^{\bullet}$.

Subcase 1b: There is no vertex which is connected with a_1 and labeled with a constant. We prove $c_d \in \varkappa(M)$. We have $(a_0,d)\in\sigma$ but $(a_0,d')\notin\sigma$ for $d\neq d'$ ($a_0\in B$). We interpret this statement in the graph V which represents our formula σ, and get:

If we label a_1 with (c_d) or $(c_{d'})$, respectively, and if we label (in all possible ways) all vertices connected with a_1 consecutively by the above induction steps then all the obtained labels (for vertices) will be consistent (compatible) or inconsistent, respectively. That is, there exist two paths from a_1 to a vertex, say a_i,

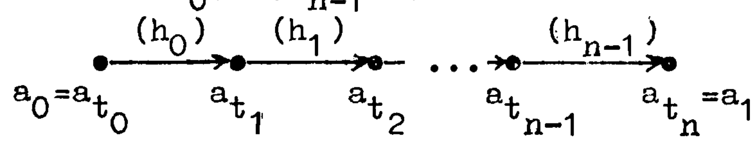


(by the above equivalent transformations we can assume that all arrows have the same direction from a_1 to a_i)

such that $h(d) = h'(d)$ but $h(d') \neq h'(d')$ where $h = h_1 h_2 \ldots h_n \in M$ and $h' = h_1' h_2' \ldots h_m' \in M$. This is exactly the condition for c_d to be an element of $\varkappa(M)$, i.e. $\sigma \subseteq c_d^{\bullet}$ with $c_d \in \varkappa(M) \subseteq M$ (by (i)).

Summarizing case 1 we get $\exists\, g\in M$: $\overline{B} \subseteq g^{\bullet}$ (the wanted result).

Case 2: The vertices a_0 and a_1 are connected. If $a_0\in B$, then we can assume that there is a path from a_0 to a_1 as follows (with $h_0,\ldots,h_{n-1}\in E$):



This is correct because – by the s-locally invertibility of E –

$$a_t \xleftarrow{(g')} a_{t'} \quad \text{is equivalent to} \quad a_t \xrightarrow{(h)} a_{t'}$$

for some $h \in E$ if at most s values will be assigned to a_0 (and therefore to each of the a_t), cf. 9.5. If we delete now in the formula (i.e. in the graph) for σ all members (i.e. all vertices, edges and labels) except the above string $h_0, h_1, \ldots, h_{n-1}$, then we get the relation $g^{\bullet}$ with $g = h_0 h_1 \cdots$ $\cdots h_{n-1} \in \langle E \rangle_{0}(1) \underset{A}{} \subseteq M$ which obviously contains $\bar{B} \cap \sigma = \bar{B}$, i.e., $\bar{B} \subseteq g^{\bullet} \in M^{\bullet}$. As discussed before the proof is finished. ■

§10 Concrete characterization of Sub $\mathfrak{A}$

$$\boxed{\exists\, \mathfrak{A} = \langle A;F\rangle \;:\; L = \mathrm{Sub}\,\mathfrak{A} \quad ?}$$

10.1 It is easy to see that $Q \subseteq R_A^{(1)}$ coincides with $[Q]^{(1)} = [Q] \cap R_A^{(1)}$ iff Q is an intersection structure (i.e., the intersection of every subfamily of Q belongs to Q). Moreover, this implies (by definition) that $Q = \mathrm{LOC}[Q]^{(1)}$ iff Q is an algebraic (cf. e.g. [J6n72(3.6.1)]) intersection structure (i.e. closed under unions of directed subfamilies of Q, too; cf. 1.13).

Thus we conclude from 6.2 the well-known result of G. Birkhoff and O. Frink([Bi/Fr], cf. [J6n72(3.6.4)]):

(A)10.2 Theorem. $L \subseteq 2^A$ is the subalgebra lattice of a universal algebra iff L is an algebraic intersection structure (or, equivalently, iff $L = \mathrm{LOC}[L]^{(1)}$). ■

Again by 6.2, we have:

(B)10.3 Theorem. $L \subseteq 2^A$ is the subalgebra lattice of an algebra with operations of rank at most s iff L is an intersection structure which is closed under unions of s-directed systems (cf. 1.12), i.e., iff $L = s\text{-}\mathrm{LOC}[L]^{(1)}$; $s \in \mathbb{N}$. ■

The equivalent condition 1.14(b) (put $Q = L$) was given by G. Fuhrken and rediscovered by M. Gould[Go68] (cf. [J6n72(p.94)]).

For unary algebras see also [J6n72(3.6.7)],[Joh/Sei].

The subalgebra systems of algebras of finite type can be characterized nicely by nearly the same condition (adjoining a condition on cardinalities only):

(C)<u>10.4</u> Theorem(cf.[Go72b]). <u>The following conditions are equivalent</u>:

(i) $\exists\, \mathfrak{A} = \langle A;F\rangle$ & F <u>finite</u> : $L = \mathrm{Sub}\,\mathfrak{A}$,

(ii) $\exists\, f \in O_A : L = \mathrm{Inv}_A^{(1)} f$ ($= \mathrm{Sub}\langle A;f\rangle$),

(iii) <u>There exists an</u> $s \in \mathbb{N}$ <u>such that</u>

a) $L = s\text{-LOC}[L]^{(1)}$ (cf. 10.3), or equivalently,
$\forall B \subseteq A \left(\forall X \subseteq B, |X| \leq s : \Gamma^L(X) \subseteq B\right) \Rightarrow B \in L$ (cf. 1.14)

b) $|\Gamma^L(X)| \leq \aleph_0$ <u>for all</u> $X \subseteq A$ <u>with</u> $|X| \leq s$.

(For 1-unary algebras see [J6n72(3.6.8)].)

Proof. The proof follows from a more general result of M. Gould [Go72b(p.370)](cf. 12.7). Nevertheless we give the prove for this simpler case: (ii)$\Rightarrow$(i)$\overset{10.3}{\Rightarrow}$(iii) is obvious, thus we have to prove (iii)$\Rightarrow$(ii):

For $X = \{x_0, \dots, x_{s-1}\}$ we can enumerate (by (iii)b)) the elements of $\Gamma^L(X)$:

$$\Gamma^L(X) = \{a_0^X, a_1^X, \dots, a_n^X, \dots\}$$

(if this set is finite with n elements take $a_i^X = a_j^X$ for $i \equiv j$ mod n) such that $a_0^X = a_0$. Define the (s+1)-ary function

$$f(x_0,\dots,x_{s-1},y) := \begin{cases} a_{t+1}^X & \text{if } y = a_t^X \text{ for } X = \{x_0,\dots,x_{s-1}\} \\ x_0 & \text{otherwise,} \end{cases}$$

and let $\mathfrak{A} = \langle A;f\rangle$. Clearly, $L \subseteq \mathrm{Sub}\,\mathfrak{A}$. To prove the inverse, let $B \in \mathrm{Sub}\,\mathfrak{A}$ and $X = \{x_0, \dots, x_{s-1}\} \subseteq B$. Because of $f(x_0, \dots, x_{s-1}, a_t^X) = a_{t+1}^X$ we get by induction on t: $\Gamma^L(X) \subseteq B$, hence $B \in L$ by (iii)a). Thus $L = \mathrm{Sub}\,\mathfrak{A}$. ■

<u>10.5</u> <u>Remarks</u>. a) Since $\mathrm{Sub}\langle A;F\rangle = \mathrm{Inv}_A^{(1)}F$, there naturally arises the question how to characterize the subalgebra systems of cartesian powers of $\langle A;F\rangle$, i.e., the set

$$\mathrm{Sub}\langle A;F\rangle^s = \mathrm{Inv}_A^{(s)}F .$$

The answer was given in [Ros78] by I.G. Rosenberg (in terms of subdirect closure systems; cf. also [Sz78(Thm.9)]). Clearly, from 6.2 we have at once:

$$\exists\, \mathfrak{A} = \langle A;F\rangle : L = \mathrm{Sub}\,\mathfrak{A}^s \iff L = \mathrm{LOC}[L]^{(s)} \;;\; s \in \mathbb{N}. \blacksquare$$

b) Theorem 10.2 also provides the abstract characterization (i.e. up to isomorphisms) of subalgebra lattices (cf.[Bi/Fr]). For more special results we refer to [Joh/Sei],[Jón72(section 3.8)],[Ha],[Wh].

§11 Concrete characterization of Con $\mathfrak{A}$

$$\boxed{\exists\, \mathfrak{A} = \langle A;F\rangle : \quad C = \mathrm{Con}\,\mathfrak{A} \quad ?}$$

Only few results concerning the concrete characterization problem for Con $\mathfrak{A}$ can be found in the literature. A partial solution has been given by M. Armbrust [Ar]. In [Qu/Wo] R. Quackenbush and B. Wolk proved that any finite distributive sublattice of $\mathcal{E}(A)$ (containing the least and the greatest element) is a congruence lattice.

This result can be extended to arbitrary complete distributive sublattices of $\mathcal{E}(A)$ as it is done by H. Draškovičova in [Dr] and (independently) by S. Burrs, H. Crapo, A.Day, D. Higgs and W. Nickols (during 1970)(cf. [Jón72(p.174)]).

In [Jón72(Thm.4.4.1)] B. Jónsson gave a solution of the characterization ~~theorem~~ problem for Con $\mathfrak{A}$. H. Werner gave in [We74] the substantially same (but nevertheless better) result by using so-called graphical compositions.

We can interpret these results as modified versions of the following theorem, namely as the description of the closure operator LOC[Q] especially for sets Q of equivalence relations.

(A)11.1 Theorem. $C \subseteq \mathcal{E}(A)$ <u>is the congruence lattice of a universal algebra iff</u> $C = \mathcal{E}(A) \cap \mathrm{LOC}[C]$ (or equivalently $C = \mathcal{E}(A) \cap 1\text{-}\mathrm{LOC}[C]$, cf. 11.2). ∎ (by 6.2).

(B)11.2 Remark. It is well-known that $\mathrm{Con}\langle A;F\rangle = \mathrm{Con}\langle A;F'\rangle$ where F' is the set of all unary polynomial functions of $\langle A;F\rangle$ (i.e. $F' = \langle F \cup \{c_a \mid a \in A\}\rangle^{(1)}$). Therefore the algebra in 11.1 can be chosen always as a unary algebra and, by 6.2, we can replace LOC[C] by 1-LOC[C].

If $\Theta \in LOC[C]^{(2)}$ is reflexive then the equivalence relation $\overline{\Theta}$ generated by Θ belongs to (1-)LOC[C] (cf. 1.13), since $\overline{\Theta}$ can be expressed as a union of compositions of Θ and Θ^{-1}.
This gives an "algorithm" to prove $C = \mathcal{E}(A) \cap LOC[C]$:
Take $\Theta \in LOC[C]^{(2)}$ (then Θ is reflexive) and prove $\overline{\Theta} \in C$.
This must be valid for all such Θ.
If C has the property that the equivalence relation generated by $\bigcup\{\Theta_i \mid i \in I\}$ also belongs to C whenever all $\Theta_i \in C$, then the following condition does the job as well:

$$\forall r \in \Theta \in [C]^{(2)} : \overline{\{r\}} \in C.$$

(C)11.3 Proposition. $C \subseteq \mathcal{E}(A)$ is the congruence lattice of an algebra $\mathfrak{A} = \langle A; F\rangle$ of finite typ (i.e. F is finite) iff $Q = LOC[C]$ satisfies 6.7(*) and $C = \mathcal{E}(A) \cap Q$. ∎ (by 6.7)

Remark. In connection with congruence relations of universal algebras there arises the problem how to characterize (concrete!) the lattice of all congruence classes of an algebra. We refer to [Wil] for such a concrete characterization.

§12 Concrete characterization of $\mathrm{Aut}\,\mathfrak{A}$ and $\mathrm{Sub}\,\mathfrak{A}$

$$\boxed{\exists\ \mathfrak{A} = \langle A;F\rangle\ :\quad G = \mathrm{Aut}\,\mathfrak{A}\quad \&\quad L = \mathrm{Sub}\,\mathfrak{A}\ \ ?}$$

From 6.2 we get:

(A)12.1 Theorem. For $L \subseteq 2^A$ and $G \leq S_A$, there exists an algebra $\mathfrak{A} = \langle A;F\rangle$ such that $L = \mathrm{Sub}\,\mathfrak{A}$ and $G = \mathrm{Aut}\,\mathfrak{A}$ iff $L = Q^{(1)}$ and $G^{\bullet} = Q \cap S_A^{\bullet}$ where $Q = \mathrm{LOC}[L \cup G^{\bullet}]$. ∎ (6.2)

Because clones of relations are rather complicated we seak for better conditions combining the results of §8 and §10.

(A)12.2 Theorem([St72]). For $L \subseteq 2^A$ and $G \leq S_A$, there is an algebra $\mathfrak{A} = \langle A;F\rangle$ such that $L = \mathrm{Sub}\,\mathfrak{A}$ and $G = \mathrm{Aut}\,\mathfrak{A}$ iff the following conditions are satisfied:

(i) L is an algebraic intersection structure (or equivalently, $L = \mathrm{LOC}[L]^{(1)}$, cf. 10.2);

(ii) G is a locally closed permutation group (i.e. $G = S_A \cap \mathrm{Loc}\langle G\rangle_{S_A}$, (8.1))

(iii) $g(B) := \{g(b) \mid b \in B\}$ belongs to L for all $g \in G$ and $B \in L$;

(iv) $C_G(B) \in L$ for all finite $B \subseteq A$ (notation cf. 8.3).

12.3 Remarks. We mention here some equivalent conditions. Assume 12.2(i) and (ii), then 12.2(iii) is equivalent to each of the following conditions:

$(iii)_1$ $\Gamma^L(g(B)) \subseteq g(\Gamma^L(B))$ *for* $g \in G$ *and all finite* $B \subseteq A$;

$(iii)_2$ $g(\Gamma^L(B)) \subseteq \Gamma^L(g(B))$ *for* $g \in G$ *and all finite* $B \subseteq A$;

$(iii)_3$ $\Gamma^L(g(B)) = g(\Gamma^L(B))$ *for* $g \in G$ *and all finite* $B \subseteq A$;

$(iii)_4$ $g(\Gamma^L(B)) \in L$ *for* $g \in G$ *and all finite* $B \subseteq A$.

Moreover, 12.2(iv) *is equivalent to*

$(iv)_1$ $\Gamma^L(B) \subseteq C_G(B)$ *for all finite* $B \subseteq A$ (i.e., if two permutations of G agree on B then they agree on $\Gamma^L(B)$ as well).

Theorem 12.2 was given by M.G. Stone in [St72(Thm.4, p.46)] with condition $(iv)_1$ instead of (iv) and independently by the present author (unpublished, with condition $(iii)_2$ instead of (iii)).

Proof of 12.3: $(iii) \Rightarrow (iii)_4$ obvious.

$(iii)_4 \Rightarrow (iii)_1$: $gB \subseteq g(\Gamma^L(B)) \Rightarrow \Gamma^L(gB) \subseteq \Gamma^L(g(\Gamma^L(B))) \overset{(iii)_4}{=} g(\Gamma^L(B))$.

$(iii)_1 \Rightarrow (iii)_2$: $B \subseteq g^{-1}(\Gamma^L(gB)) \Rightarrow \Gamma^L(B) \subseteq \Gamma^L(g^{-1}(\Gamma^L(gB))) \subseteq g^{-1}(\Gamma^L\Gamma^L(gB))$ (by $(iii)_1$) $\Rightarrow g(\Gamma^L(B)) \subseteq \Gamma^L(gB)$.

$(iii)_2 \Rightarrow (iii)$: Let $B \in L$. We show $\Gamma^L(Y) \subseteq g(B)$ for all finite $Y \subseteq g(B)$ (since this implies $g(B) \in L$, cf. 1.8, 1.14). From $g^{-1}Y \subseteq B$ we have $g^{-1}(\Gamma^L(Y)) \overset{(iii)_2}{\subseteq} \Gamma^L(g^{-1}(Y)) \subseteq \Gamma^L(B) = B$, thus $\Gamma^L(Y) \subseteq g(B)$.

$(iii)_3 \Leftrightarrow (iii)_1$ & $(iii)_2$ obviously.

$(iv) \Rightarrow (iv)_1$: $B \subseteq C_G(B) \Rightarrow \Gamma^L(B) \subseteq \Gamma^L(C_G(B)) \overset{(iv)}{=} C_G(B)$.

$(iv)_1 \Rightarrow (iv)$: For finite $D \subseteq C_G(B)$ we get $D \subseteq \Gamma^L(D) \overset{(iv)_1}{\subseteq} C_G(D) \subseteq C_G C_G(B) = C_G(B)$ (C_G is a closure operator!), consequently $C_G(B) = \bigcup\{\Gamma^L(D) \mid D \subseteq C_G(B) \ \& \ D \text{ finite}\}$ belongs to L by (i) (cf. 1.13). ∎

P r o o f of 12.2: The ("if"-part of the) proof can be done by a straightforward construction of the algebra $\mathfrak{A}$ (cf.[St72]). We choose another proof (using 12.1) in order to show again how to work with clones of relations and their properties (because in other cases - e.g. 7.2 - only this method works).

Part I. The conditions (i)-(iv) are necessary: If $L = \mathrm{Sub}\,\mathfrak{A}$, $G = \mathrm{Aut}\,\mathfrak{A}$, $\mathfrak{A} = \langle A;F\rangle$, then (i),(ii),(iii) are obvious (cf. 10.2, 8.1), moreover $\Gamma^{L}(B) \overset{(1.8)}{=} \Gamma_F(B) \overset{(cf.2.4)}{\subseteq} \Gamma_{\mathrm{Pol}\,G}(B) \overset{(8.7)}{=} C_G(B)$, thus (iv)$_1$ (and therefore (iv), cf. 12.3) holds.

Part II. The conditions (i)-(iv) are sufficient.

Step 1: We prove $G^{\bullet} = S_A^{\bullet} \cap \mathrm{LOC}[G^{\bullet} \cup L]$ (similiar to the proof of 8.1). Let $g \in S_A$ and $g^{\bullet} \in \mathrm{LOC}[G^{\bullet} \cup L]$. Then for all finite $B \subseteq A$ there is a binary relation $\varrho_B \in [G^{\bullet} \cup L]$ such that $(g|_B)^{\bullet} \subseteq \varrho_B \subseteq g^{\bullet}$. Then ϱ_B must be defined by a formula of the following type (cf. 3.2(R4)):

$$(a_0,a_1) \in \varrho_B \iff$$

$$\exists (a_i)_{i\in I}:\ a_j \in B_j\ (j\in J)\ \&\ g_k(a_{k(0)}) = a_{k(1)}\ (k\in K)$$

where $J \subseteq I \cup \{0,1\}$, $B_j \in L$, $k(0),k(1) \in I \cup \{0,1\}$, $g_k \in G$ (J,K,I index sets). By the same arguments as in the proof of 8.1 (p. 53), a_0 and a_1 must be "connected", i.e., there are $a_0 = a_{i_0}, a_{i_1}, \dots, a_{i_n} = a_1$ such that (i_t,i_{t+1}) or (i_{t+1},i_t) are equal to $(k_t(0),k_t(1))$ for some $k_t \in K$ $(0 \leq t \leq n-1)$ (because otherwise (provided $|B| \geq 2$) ϱ_B could not be a partial 1-1-function in contradiction to $\varrho_B \subseteq g^{\bullet}$). Hence $(g|_B)^{\bullet} \subseteq (f|_B)^{\bullet}$ where

$$f = g'_{k_0} g'_{k_1} \dots g'_{k_{n-1}} \in \langle G\rangle_{S_A} \underset{(ii)}{=} G \quad \text{with}$$

$$g'_{k_t} = \begin{cases} g_{k_t} & \text{if } (i_t,i_{t+1}) = (k_t(0),k_t(1)) \\ (g_{k_t})^{-1} & \text{if } (i_{t+1},i_t) = (k_t(0),k_t(1)) \end{cases}.$$

Thus $g|_B = f|_B$ and we get $g \in \mathrm{Loc}\langle G\rangle_{S_A} \overset{(ii)}{=} G$. (Q.E.D)

Step 2: We prove $L = \mathrm{LOC}[G^{\bullet} \cup L]^{(1)}$.

Let $\varrho \in \mathrm{LOC}[G^{\bullet} \cup L]^{(1)}$ and B be a finite subset of ϱ. We show that there will be a $\sigma_B \in [L]^{(1)} = L$ such that $B \subsetneq \sigma_B \subsetneq \varrho$, because this implies $\varrho \in \mathrm{LOC}[L]^{(1)}$ and we are done by (i).

By our assumption, there is a $\varrho_B \in [G^{\bullet} \cup L]^{(1)}$ such that $B \subsetneq \varrho_B \subsetneq \varrho$, thus there is a defining formula for ϱ_B of the following form:

$$a_0 \in \varrho_B \iff$$

$$\exists (a_i)_{i\in I}:\ a_j \in B_j\,(j\in J)\ \&\ g_k(a_{k(0)}) = a_{k(1)}\,(k\in K),$$

where $J \subsetneq I \cup \{0\}$, $k(0), k(1) \in I \cup \{0\}$.

Consider the lab̲led graph with the vertices $a_i\,(i\in I)$ (a_j labled with B_j for $j\in J$) and edges $(a_{k(0)}, a_{k(1)})$ labled with g_k for $k\in K$. Clearly one can assume that all $a_i\,(i\in I)$ are connected with a_0 (unconnected components do not change ϱ_B (provided they are consistent)). By condition (iii) one can move all vertex-labels B_j to a_0 ($g(a_0)=a_j \in B_j \iff g(a_0)= a_j\ \&\ a_0 \in g^{-1}(B_j)$), and – since G is a group and L is closed under intersections – the above formula for ϱ_B can be transformed to the following form:

$$a_0 \in \varrho_B \iff$$

$$\exists (a_i)_{i\in I}:\ a_0 \in D\ \&\ f_t(a_0) = a_{i_t}\ (t\in T)$$

where $D \in L$, $f_t \in G$ ($t\in T$ index set). Define

$$\sigma_B := D \cap C_G(B).$$

Then $B \subsetneq \sigma_B$ since $B \subsetneq \varrho_B \subsetneq D$. Moreover, since $B \subsetneq \varrho_B$ we have $f_t(a_0) = f_{t'}(a_0)$ if $a_0 \in B$ and $i_t = i_{t'}$ $(t, t' \in T)$. This property holds not only for $a_0 \in B$ but – by 8.3 – also for all $a_0 \in C_G(B)$. Therfore $\sigma_B \subseteq \varsigma_B$, since $a_0' \in D \cap C_G(B)$ implies $f_t(a_0') = f_{t'}(a_0')$ if $i_t = i_{t'}$, i.e. $a_0' \in \varsigma_B$. Thus $B \subseteq \sigma_B \subseteq \varsigma_B \subseteq \varsigma$. Moreover, $\sigma_B \in L$ by (i) and (iv). Q.E.D.

Finally, step 1 and step 2 together finish the proof because of 12.1. ∎

Analogeously to 12.1 and 12.2 we get characterization theorems for the "bounded case" (B):

(B)12.4 Theorem. There exists an algebra with operations of rank at most $s \in \mathbb{N}$ such that $L = \mathrm{Sub}\,\mathfrak{A}$ and $G = \mathrm{Aut}\,\mathfrak{A}$ iff $L = Q^{(1)}$ and $G^\bullet = S_A^\bullet \cap Q$ for $Q = s\text{-LOC}[L \cup G^\bullet]$. ∎ (6.2(b))

(B)12.5 Theorem. For $L \subseteq 2^A$, $G \subseteq S_A$, there exists a universal algebra $\mathfrak{A} = \langle A; F\rangle$ with $F \subseteq O_A^{(s)}$ (for given $s \in \mathbb{N}$) such that

$$L = \mathrm{Sub}\,\mathfrak{A} \quad \text{and} \quad G = \mathrm{Aut}\,\mathfrak{A}$$

$\left\{\begin{matrix}\text{if}\\ \text{only if}\end{matrix}\right\}$ (or if and only if, respectively) the following conditions are fulfilled:

(i) $L = s\text{-LOC}[L]^{(1)}$ (cf. 10.3);

(ii) $\left\{\begin{matrix} G = S_A \cap s\text{-Loc}\langle G\rangle_{S_A} \\ G = S_A \cap (s+1)\text{-Loc}\langle G\rangle_{S_A}\end{matrix}\right\}$ (or the condition for G given in 8.4 (in case $s \geq 2$) and 8.5 (in case $s = 1$), respectively);

(iii) $g(\Gamma^L(B)) \subseteq \Gamma^L(g(B))$ for $g \in G$ and all $B \subseteq A$ with $|B| \leq s$;

(iv) $C_G(B) \in L$ for all $B \subseteq A$ with $|B| \leq s$.

12.6 Remark. Assume 12.5(i) and (ii). Then condition 12.5(iii) is equivalent to 12.2(iii)(cf. proof of 12.3, use 12.5(i) and 1.14). Moreover, condition 12.5(iv) is equivalent to 12.2(iv)(since $C_G(B) = \bigcup\{C_G(B') \mid B' \subseteq C_G(B)\ \&\ |B'| \leq s\} \in L$ by (i)).

Proof of 12.5: The conditions are necessary (this follows from 10.2 for (i); 8.4, 8.5, 8.8 for (ii), and from 12.2, 12.6 for (iii),(iv)).

The conditions are sufficient, too. In fact,

Step 1: $G^\bullet = S_A^\bullet \cap$ s-LOC$[G^\bullet \cup L]$ can be proved as in step 1 of the proof of 12.2 replacing "finite B" by "$B \subseteq A$ with at most s elements".

Step 2: $L =$ s-LOC$[G^\bullet \cup L]^{(1)}$ can be proved analogeously (cf. proof of 12.2) or shorter as follows: (i)-(iv) imply 12.2(i)-(iv) by 12.6, thus $L = \mathrm{LOC}[G^\bullet \cup L]^{(1)}$ (by 12.1) $\Rightarrow L = [G^\bullet \cup L]^{(1)} \Rightarrow$ s-LOC$[G^\bullet \cup L]^{(1)} =$ s-LOC $L \overset{(i)}{=} L$, and 12.4 finishes the proof.■

Finally we consider the "finite case" (C). The full answer was given by M. Gould in [Go72b(p. 370)]:

(C)12.7 Theorem. For an intersection structure $L \subseteq 2^A$ and a permutation group $G \leq S_A$ the following conditions are equivalent:

(I) $L = \mathrm{Sub}\,\mathfrak{A}$ and $G = \mathrm{Aut}\,\mathfrak{A}$ for some algebra $\mathfrak{A} = \langle A; F\rangle$ with finite F.

(II) $L = \mathrm{Sub}\,\mathfrak{A}$ and $G = \mathrm{Aut}\,\mathfrak{A}$ for some algebra $\mathfrak{A} = \langle A; f\rangle$ with one operation $f \in O_A$.

(III) *There exist* $s, n \in \mathbb{N}$ *such that*

(III.i) *For* $B \subseteq A$, B *belongs to* L *if* $\Gamma^L(X) \subseteq B$ *for all* $X \subseteq B$ *with* $|X| \leq s$ (*i.e.* $L = s\text{-LOC}\,L$, cf. 1.14);

(III.ii) G *is* *n-locally closed*, *i.e.* $G = S_A \cap n\text{-Loc}\,G$;

(III.iii) $g(\Gamma^L(B)) \in L$ *and* $|\Gamma^L(B)| \leq \aleph_0$ *for* $g \in G$ *and all* $B \subseteq A$ *with* $|B| \leq s$;

(III.iv) $C_G(B) \in L$ (*or equivalently* $\Gamma^L(B) \subseteq C_G(B)$, *cf.* 12.3) *for all* $B \subseteq A$ *with* $|B| \leq s$.

P r o o f. (II)$\Rightarrow$(I) trivial. (I)$\Rightarrow$(III) by 12.5 (take $n = s+1$). (III)$\Rightarrow$(II) follows from M.Gould's proof given in [Go72b(p. 370)] by remarking that M.Gould's conditions (IV.i), (IV.ii) and (IV.iii) follow from (III.ii), (III.i) and (III.iii&iv) above, resp. Nevertheless we sketch the proof of (III)$\Rightarrow$(II): Proceed as in the proof of 10.4 but choose the enumeration of $\Gamma^L(X)$ compatible with G in the sense that $g(a_i^X) = a_i^{gX}$ (cf. p. 67) for all $g \in G$. This is possible because of $\Gamma^L(gX) = g(\Gamma^L(X))$ by (III.iii) (cf. 12.3(iii)$_3$). Then all $g \in G$ commute with the f defined on p. 67, hence $L = \text{Sub}\langle A; f\rangle$, $G \subseteq \text{Aut}\langle A; f\rangle$. Further, one can choose (as proved in [Go72a]) an operation $f' \in O_A$ such that $2^A = R_A^{(1)} = \text{Sub}\langle A; f'\rangle$ and $G = \text{Aut}\langle A; f'\rangle$. Then $\mathfrak{A} = \langle A; \{f, f'\}\rangle$ fulfilles condition (I). One can assume f and f' to be of the same arity $m \geq 1$. Take $\mathfrak{A}' = \langle A; h\rangle$ with

$$h(x_0, \dots, x_m) := \begin{cases} f(x_0, \dots, x_{m-1}) & \text{if } x_m = x_0 \\ f'(x_0, \dots, x_{m-1}) & \text{if } x_m \neq x_0 \end{cases}.$$

Then $\mathfrak{A}'$ is the algebra required in (II) (cf. [Go72b(p. 372)]).

∎

§13 Concrete characterization of Aut $\mathfrak{A}$ and Con $\mathfrak{A}$

$$\boxed{\exists\ \mathfrak{A} = \langle A;F\rangle : \quad G = \mathrm{Aut}\,\mathfrak{A} \quad \& \quad C = \mathrm{Con}\,\mathfrak{A} \quad ?}$$

(A)
(B) 13.1 Theorem. For $G \subseteq S_A$ and $C \subseteq \mathcal{E}(A)$, there exists an algebra $\mathfrak{A} = \langle A;F\rangle$ with $F \subseteq O_A$ or $F \subseteq O_A^{(s)}$, resp., ($s \in \mathbb{N}$), such that $G = \mathrm{Aut}\,\mathfrak{A}$ and $C = \mathrm{Con}\,\mathfrak{A}$ iff

$$G^{\bullet} = S_A^{\bullet} \cap Q \quad \text{and} \quad C = \mathcal{E}(A) \cap Q$$

where $Q = \mathrm{LOC}[G^{\bullet} \vee C]$ or $Q = s\text{-}\mathrm{LOC}[G^{\bullet} \vee C]$, resp. ■ (6.2)

No other results concerning this case are known to the author except the following conjecture of H. Werner given in [We74] (here the characterization of Aut $\mathfrak{A}$ & Con $\mathfrak{A}$ is stated as problem 4):

13.2 Werner's Conjecture ([We74 (p. 452)]). Let C be a complete sublattice of $\mathcal{E}(A)$ (= equivalence relations on A) and G a permutation group on A. There is an algebra $\mathfrak{A} = \langle A;F\rangle$ such that $C = \mathrm{Con}\,\mathfrak{A}$ and $G = \mathrm{Aut}\,\mathfrak{A}$ iff

(a) C is closed under $P_{A^*,x,y}$ (we will not formulate this condition explicitely but we note, that it is equivalent to the existence of an algebra $\mathfrak{A}'$ with $C = \mathrm{Con}\,\mathfrak{A}'$ (cf. §11)).

(b) G is locally closed (cf. 8.1, this is equivalent to the existence of an algebra $\mathfrak{A}''$ with $G = \mathrm{Aut}\,\mathfrak{A}''$).

(c) If $g \in G$ and $\theta \in C$ then $\theta^g \in C$ where $\theta^g = \{(g(x),g(y)) \mid (x,y) \in \theta\}$.

Clearly the conditions are necessary but it turns out that they are not sufficient.

In the preceding paragraphs we applied the General Galois theory (in particular 6.2) to special cases in order to obtain "good" general characterization results. Now we become acquainted with another application of our theory: the construction of counterexamples.

13.3 Counterexample to 13.2. Let $A = \underline{4} = \{0,1,2,3\}$ and

$\Theta = \{(0,0),(0,1),(1,0),(1,1),(2,2),(2,3),(3,2),(3,3)\}$

$\hat{=}$ partition $\{\{0,1\},\{2,3\}\}$,

$\Theta' = \{(0,0),(0,2),(2,0),(2,2),(1,1),(1,3),(3,1),(3,3)\}$

$\hat{=}$ partition $\{\{0,2\},\{1,3\}\}$,

$\Theta_0 = \{(x,x) \mid x \in A\}$,

$\Theta_1 = A \times A$,

$g = (03)(12) \in S_A$, e = identity of S_A,

$C = \{\Theta_0, \Theta, \Theta', \Theta_1\}$,

$G = \{e, g\}$.

Then 13.2(a),(b),(c) are fulfilled for C and G (e.g., $C = \mathrm{Con}\langle A;\ g,h\rangle$ with $h(0)=h(1)=2$, $h(2)=h(3)=0$; and note that for finite A, every $G \leqq S_A$ is locally closed), but there is no algebra $\mathfrak{A}$ with $C = \mathrm{Con}\,\mathfrak{A}$ and $G = \mathrm{Aut}\,\mathfrak{A}$.

To see this, consider the relation ϱ defined by

$$(a_0,a_1) \in \varrho \;:\Leftrightarrow\; \exists a \in A:\ (a_0,a_1) \in \Theta \ \&\ (a_0,a) \in \Theta' \ \&\ g(a_0) = a .$$

It is easy to check that $\varrho = f^{\bullet}$ where $f = (01)(23) \in S_A$. Therefore $f^{\bullet} \in S_A^{\bullet} \cap [G^{\bullet} \cup C]$ (cf.3.5). If the algebra $\mathfrak{A}$ would exist then $f \in G$ by 13.1, contradiction! ∎

13.4 Possibly Werner's conjecture becomes true if condition (b) will be replaced by a stronger one; e.g. we formulate the following conjecture:

Let $C \subseteq \mathcal{E}(A)$ and $G \leq S_A$. There is an algebra $\mathfrak{A} = \langle A; F\rangle$ such that $C = \mathrm{Con}\,\mathfrak{A}$ and $G = \mathrm{Aut}\,\mathfrak{A}$ iff

(a) $\exists\, \mathfrak{A}' = \langle A; F'\rangle :\ C = \mathrm{Con}\,\mathfrak{A}'$ (cf. 13.2(a), 11.1);

(b') $G^{\bullet} = S_A^{\bullet} \cap \mathrm{LOC}[G^{\bullet} \cup C]$ (this implies 13.2(b));

(c) $\forall g \in G\ \forall \theta \in C :\ \theta^{g} \in C$ (cf. 13.2(c)).

Comparing this conjecture with 13.1, the advantage of 13.4 consists in the following: The only influence of G on C is given by condition (c).

Clearly, after proving (if possible) this conjecture one should look for a simpler condition (b').

If C is the trivial congruence lattice, i.e. if the algebra $\mathfrak{A}$ is required to be simple, then 13.1 provides a full answer:

13.5 Proposition (cf. [Schm.E.T.64]). Let $G \leq S_A$. There exists a simple algebra $\mathfrak{A} = \langle A; F\rangle$ with $G = \mathrm{Aut}\,\mathfrak{A}$ if and only if

$$G = \mathrm{Loc}\langle G\rangle_{S_A} \cap S_A .$$

Proof.
We have $\mathrm{LOC}[G^{\bullet} \cup C] = \mathrm{LOC}[G^{\bullet}]$ if $C = \{\delta_2^0, \delta_2^1\}$, i.e., if C consists of trivial congruence relations only (note, $(x,y) \in \delta_2^1$ can be replaced by $x=y$, and $(x,y) \in \delta_2^0$ can be deleted in each formula which defines a relation of $[G^{\bullet} \cup C]$).

Thus, by 13.1 and 8.2, 8.1, we are done if $C = \mathcal{E}(A) \cap \mathrm{LOC}[G^{\bullet}]$.
This can be shown without difficulties (using e.g. 1.13 and 3.5 or proceed analogously as in proof of 8.1).
But there is also a simple direct proof. Clearly, the condition $G = S_A \cap \mathrm{Loc}\langle G\rangle_{S_A}$ is necessary (cf. 8.1). Now let $G = S_A \cap \mathrm{Loc}\langle G\rangle_{S_A}$. By 8.1 there is an $\mathfrak{A}' = \langle A; F'\rangle$ with $\mathrm{Aut}\,\mathfrak{A}' = G$.
Take $\mathfrak{A} = \langle A; F \cup \{t\}\rangle$ where t is the ternary discriminator

$$t(x,y,z) = \begin{cases} x & \text{if } x \neq y \\ z & \text{if } x = y \end{cases}.$$

Because $\mathrm{Aut}\langle A; t\rangle = S_A$ and $\langle A; t\rangle$ is simple (cf. e.g. [We78(Lemma 1.10)]), $\mathfrak{A}$ is also simple and $\mathrm{Aut}\,\mathfrak{A} = G$. ∎

§14 Concrete characterization of End $\mathfrak{A}$ and Sub $\mathfrak{A}$

$$\boxed{\exists\ \mathfrak{A} = \langle A;F\rangle :\quad H = \operatorname{End}\mathfrak{A} \quad \& \quad L = \operatorname{Sub}\mathfrak{A} \quad ?}$$

This concrete characterization problem was solved in [Sa/St77a] in terms of systems of equations which reflect - in our terminology - the properties of $LOC[H^{\bullet} \cup L]$.

From 6.2 we get the following theorem:

(A)
(B) 14.1 Theorem. For $H \subseteq O_A^{(1)}$ and $L \subseteq 2^A$, there exists an algebra $\mathfrak{A} = \langle A;F\rangle$ with

(A) $F \subseteq O_A$ or

(B) $F \subseteq O_A^{(s)}$ $(s \in \mathbb{N})$, resp.,

such that $H = \operatorname{End}\mathfrak{A}$ and $L = \operatorname{Sub}\mathfrak{A}$ iff (for $Q = H^{\bullet} \cup L$)

(A) $H = (O_A^{(1)})^{\bullet} \cap LOC[Q]$, $L = LOC[Q]^{(1)}$ or

(B) $H = (O_A^{(1)})^{\bullet} \cap s\text{-}LOC[Q]$, $L = s\text{-}LOC[Q]^{(1)}$, resp. ∎

The "finite case" (C) (cf. 7.1) might be treated with 6.7 (or with 14.1 (B) for finite A), but no criterion (for (C)) is known to the author which uses properties of H and L only.

How theorem 14.1 might be improved (e.g. analogously to 12.2 in comparision with 12.1)? We mention here some neces= sary condition:

14.2 Proposition. If $H = \operatorname{End}\mathfrak{A}$ and $L = \operatorname{Sub}\mathfrak{A}$ for some universal algebra $\mathfrak{A} = \langle A;F\rangle$ then the following conditions are satisfied:

(a) L <u>is an algebraic intersection structure</u> (cf. 10.2);

(b) $H^{\bullet}=(O_A^{(1)})^{\bullet} \cap \mathrm{LOC}[H^{\bullet}]$ (cf. 9.1);

(c) $h(B)=\{h(b) \mid b\in B\} \in L$ <u>for all</u> $h\in H$ <u>and</u> $B\in L$ (<u>in particular the image</u> $h(A)$ <u>of</u> h <u>belongs to</u> L <u>for all</u> $h\in H$);

(d) $h^{-1}(B)=\{a\in A \mid h(a)\in B\} \in L$ <u>for all</u> $h\in H$ <u>and</u> $B\in L$;

(e) $\{a\}\in L \iff c_a\in H$ (<u>where</u> $c_a: A\longrightarrow A: x\longmapsto a$);

(f) $\Gamma_{\mathrm{Pol}\,H}(B)\in L$ <u>for all finite</u> $B\subseteq A$;

(g) $C_H(B)=\{a\in A \mid \forall h,h'\in H:\ h|_B = h'|_B \Rightarrow h(a)=h'(a)\} \in L$ <u>for all finite</u> $B\subseteq A$;

(h) $\Gamma^L(B)\subseteq C_H(B)$ <u>for all</u> $B\subseteq A$.

<u>Proof</u>. (a),(b) are obvious(cf. 10.2, 9.1). (c) and (d) can be shown by a straightforeward proof(cf. 12.2(iii)). (e) is obvious. (f) and (g) are analogeously to 12.2(iv)(cf.8.7): We have $\Gamma_{\mathrm{Pol}\,H}(B)\in[H^{\bullet}]^{(1)}\subseteq \mathrm{LOC}[H^{\bullet}\cup L]^{(1)}=L$ by 4.3a and 14.1<u>A</u>. Further, for $a_1,\dots,a_n\in C_H(B)$, $f\in F^{(n)}$, we get $a=f(a_1,\dots,a_n)\in C_H(B)$ since $h|_B = h'|_B \Rightarrow f(ha_1,\dots,ha_n)=f(h'a_1,\dots,h'a_n) \Rightarrow h(f(a_1,\dots,a_n))=h'(f(a_1,\dots,a_n))$, i.e. $h(a)=h'(a)$. Condition (h) is equivalent to (f)(for the proof see 12.3(iv)). ∎

<u>Remark</u>. It is unknown to the author whether conditions 14.2(a)-(h) are sufficient, too (possibly under certain restrictions to H and/or L).

It is easy to see that $\Gamma_{\mathrm{Pol}\,H}(B)\subseteq C_H(B)$. It is not clear in which cases equality holds (cf. 8.7 for groups). Probably (g) follows from (f) and (a).

§15 Concrete characterizations IV.
(Survey on related Galois-connections)

With the preceding paragraph we close our considerations of special concrete characterization problems. Of course, there are some more problems than treated in §§8-14 (e.g., $\mathrm{End}\,\mathfrak{A}$ & $\mathrm{Con}\,\mathfrak{A}$, $\mathrm{Sub}\,\mathfrak{A}$ & $\mathrm{Con}\,\mathfrak{A}$). The simultaneous characterization of $\mathrm{Aut}\,\mathfrak{A}$, $\mathrm{End}\,\mathfrak{A}$, $\mathrm{Sub}\,\mathfrak{A}$ and $\mathrm{Con}\,\mathfrak{A}$ was given in §7.

The common background of all these results was the description of Galois closed sets of operations or relations with respect to the Galois connection Pol - Inv. In the most cases this Galois connection was restricted to special kinds of operations or relations, resp. Let us sketch once more this treatment in general:

<u>15.1</u> Let $E \subseteq O_A$ and $E' \subseteq R_A$ be sets of operations and relations with given "properties", resp.. Then the operators

$$F \longmapsto K'(F) := E' \cap \mathrm{Inv}_A F \quad \text{and}$$
$$Q \longmapsto K(Q) := E \cap \mathrm{Pol}_A Q \quad (F \subseteq E,\ Q \subseteq E')$$

define a Galois connection between subsets of E and E'. Suppose we have characterized the Galois closed sets

$$(^\circ) \quad F = K(K'(F)) \ (= E \cap \mathrm{Pol}(E' \cap \mathrm{Inv}\,F)) \quad \text{and}$$
$$(8) \quad Q = K'(K(Q)) \ (= E' \cap \mathrm{Inv}(E \cap \mathrm{Pol}\,Q)) .$$

Then we have (obviously) the following characterization theorems:

(*) <u>For $F \subseteq E$, there exists a relational algebra $\langle A;Q\rangle$ with $Q \subseteq E'$ such that $F = E \cap \mathrm{Pol}\,Q$ iff F satisfies $(^\circ)$.</u>

(**) <u>For</u> $Q \subseteq E'$, <u>there is a universal algebra</u> $\langle A;F\rangle$ <u>with</u> $F \subseteq E$ <u>such that</u> $Q = E' \cap \mathrm{Inv}\,F$ <u>iff</u> Q <u>satisfies</u> (8), (<u>i.e.</u> Q <u>is Galois closed</u>).

Proof. (*): Clearly, $F=K(Q) \Rightarrow K(K'(F))=K(K'(K(Q)))=K(Q)=F$. Conversely, if $F=K(K'(F))$ then define $Q=K'(F)$.
(**) can be proved analogeously. ∎

Conversely, every characterization theorem of kind (*) or (**) is (more or less implicitely) a characterization of Galois closed sets of operations or relations, respectively, where the Galois connection under consideration is given by the operators Pol - Inv modified as above. Note for example, that Inv F, Sub F ($=\mathrm{Inv}^{(1)}F$), Con F, Pol F and Pol Q, End Q ($=\mathrm{Pol}^{(1)}Q$), w-Aut Q can be expressed in the form $E' \cap \mathrm{Inv}\,F$ and $E \cap \mathrm{Pol}\,Q$ respectively.

<u>15.2</u> In the following table we summarize almost all results given in previous paragraphs under the point of view of Galois connections (cf. 15.1). We refer to the remark after 4.2 (and to 4.1, 4.2, §§8-14) for some references concerning Galois closed sets of relations (sometimes restricted to operations; we add here [Isk] ($\mathrm{Pol}_A \mathrm{Con}_A F$ for p-rings $\langle A;F\rangle$ and [Wie] (Aut $\mathrm{Inv}^{(n)}G$ for $G \leq S_A$)).

In the table, results on w-Aut F and Aut F considered in the next paragraph are mentioned, too. Note that w-Aut F = Aut F for $F \subseteq O_A$ (cf. 1.6d), therefore no distinction is needed in some cases.

The table gives the number of the theorem in which the Galois closed sets $K_1(K_2(Q))$ or $K_1(K_2(F))$ $(Q \subseteq R_A,\ F \subseteq O_A)$ were characterized (or from which this characterization immediately follows; have 15.1 in mind!).

⊡ K_1 = \ K_2 =	Inv	$Inv^{(s)}$	Pol	$Pol^{(s)}$ (& End)	$Pol^{(1-1)}$ w-Aut Aut
Inv	-	-	4.2(a) 4.5	4.2(b) 6.2(b) 16.2	16.6
$Inv^{(s)}$	-	-	6.2(a) 10.5		
Sub	-	-	6.2(a) 10.2	6.2(b) 10.3	
Con	-	-	6.2(a) 11.1	11.1 11.2	
Pol	4.1(a) 4.4	4.1(b) 5.2 β)	(4.2) 7.3	6.2(b) 9.3(ii)	
$Pol^{(n)}$	4.1(a) 5.2 α) 5.5 α)	4.1(b) 5.2 β) 5.5 β)	4.2 7.3	6.2(b)	
End			9.1	9.2	
w-Aut			6.2(a) 8.1	6.2(b) 8.4 8.5 8.6 8.8	
Aut	4.6 5.5 α) 5.2(remark)	5.5 β)			

<u>Table</u>: The Galois closure $K_1(K_2(Q))$(for $Q \subseteq R_A$ or O_A) was characterized in ⊡

Rem. Aut – Inv is <u>not</u> a Galois connection in general (since $G \subseteq \mathrm{Aut}\,\mathrm{Inv}\,G$ does not hold in general for arbitrary $G \subseteq S_A$ and infinite A)

§16 Krasner-clones of relations

16.1 The sets $Q = \mathrm{Inv}\, H$ of invariant relations of a set $H \subseteq O_A^{(1)}$ of unary operations are characterized by $1\text{-LOC}[Q] = Q$ (cf. 4.2(b)). The Galois closed sets $Q = 1\text{-LOC}[Q] = \mathrm{Inv\, End}\, Q$ (with respect to the Galois connection Inv - End) sometimes are called Krasner-algebras of 1st kind ([Bo/Kal],[Pö/Kal]).

Here we will use the name *Krasner - clone of 1st kind* for clones of relations (3.5) which additionally are closed under arbitrary unions (cf. 1.13).

16.2 Proposition(cf.[Pö/Kal(1.3.1, 1.3.4)]). *For* $Q \subseteq R_A$, *the following conditions are equivalent*:

(i) Q *is a Krasner-clone of 1st kind* ;

(ii) $Q = \mathrm{Inv\, End}\, Q$;

(iii) $\exists\, H \subseteq O_A^{(1)}: Q = \mathrm{Inv}\, H$. ■ (4.2(b) for s=1).

Now, let us consider the Galois connection Inv - Aut (or w-Aut), in particular the Galois closed sets Inv Aut Q (or Inv w-Aut Q) for $Q \subseteq R_A$ (for Aut Inv G see 4.6). Clearly these sets must be Krasner-clones of 1st kind, but they have to satisfy some more conditions, too. We get as a first observation:

16.3 Lemma.

(i) *For* $f \in S_A$ *and* $\varrho \in R_A^{(m)}$ *we have*:

f *preserves* $\varrho \iff f^{-1}$ *preserves* $\neg\varrho$ $(:= A^m \setminus \varrho)$

(ii) For a permutation group $G \leq S_A$, $Q = \mathrm{Inv}_A G$ is closed under $\neg$ (i.e., $\varrho \in Q \Rightarrow \neg\varrho \in Q$).

(iii) If $Q \subseteq R_A$ is closed under $\neg$, then $\mathrm{Aut}\ Q = \text{w-Aut}\ Q$.

Proof. (i) follows from the definitions; (ii),(iii) directly from (i) (e.g. (iii): $f \in \text{w-Aut}\ Q \Rightarrow \forall \varrho \in Q: \varrho \in \mathrm{Inv}\ f \overset{(i)}{\Rightarrow}$ $\forall \varrho \in Q: \neg\varrho \in \mathrm{Inv}\ f^{-1} \Rightarrow \forall \sigma \in Q: \sigma \in \mathrm{Inv}\ f^{-1} \Rightarrow f^{-1} \in \text{w-Aut}\ Q$, i.e. $f \in \mathrm{Aut}\ Q$). ■

For finite A, the property to be (a clone of relations and) closed under $\neg$ is strong enough to characterize the invariant relations of permutation groups. One could expect that this is true in general. Before discussing this conjecture we introduce the following notions:

16.4 Definition. A set $Q \subseteq R_A$ is called a Krasner-clone of 2^{nd} kind if

a) Q is a Krasner-clone of 1^{st} kind (i.e. $Q = 1\text{-LOC}[Q]$) and

b) Q is closed with respect to strong superposition, i.e. (for notation cf. 3.2(R4), 3.4): For $\varrho_i \in Q^{(m_i)}$, $\pi_i: \underline{m_i} \longrightarrow A$, $i \in I$ (index set) and $\pi: \underline{m} \longrightarrow A$, the relation

$$\bigwedge\nolimits^{\pi}_{(\pi_i)_{i\in I}} (\varrho_i)_{i\in I} := \{\pi a \mid a \in A^A \text{ \& } a \text{ is bijectiv \& } \forall i \in I: \pi_i a \in \varrho_i\} \quad ^{*)}$$

also belongs to Q, and

c) Q is closed under $\neg$, i.e. $\varrho \in Q \Rightarrow \neg\varrho \in Q$.

---

*) Note, $a \in A^A$ can be considered as a mapping $a: A \longrightarrow A: i \longmapsto a(i)$

16.5 Proposition. For $Q \subseteq R_A$, consider the following conditions:

(KCl 2) Q is a Krasner-clone of 2^{nd} kind;

(KCl 1) Q is a Krasner-clone of 1^{st} kind;

(Cl) Q is a clone of relations;

($\neg$) Q is closed under $\neg$;

(ν) Q contains the inequality relation $\nu = \{(x,y) \mid x \neq y\}$;

(sSup) Q is closed with respect to strong superposition.

Then the following equivalences and implications hold:

$$(\mathrm{KCl}\ 2) \overset{(+)}{\Longleftrightarrow} (\mathrm{Cl})\ \&\ (\neg)\ \&\ (\mathrm{sSup})$$

$$(\mathrm{KCl}\ 1)\ \&\ (\neg) \overset{(++)}{\Longleftrightarrow} (\mathrm{Cl})\ \&\ (\neg) \overset{(+++)}{\Longrightarrow} (\mathrm{Cl})\ \&\ (\nu).$$

$$(\mathrm{Cl})\ \&\ (\mathrm{sSup}) \Longrightarrow (\mathrm{Cl})\ \&\ (\nu).$$

P r o o f. (+) by definition. (++) holds because of $\bigcup_{i\in I} \varrho_i = \neg(\bigcap_{i\in I}(\neg\varrho_i))$. (+++) is clear since $\nu = \neg\delta_2^{1}$ ($\delta_2^{1} = \{(x,x) \mid x\in A\} \in Q \subseteq R_A$). ∎

Remarks. The converse of (+++) is not true in general. Example: $H = \mathrm{Pol}_{\mathbb{N}}^{(1-1)}\mathbb{N}'$ preserves ν and $\mathbb{N}' = \mathbb{N}\setminus\{1\}$, but $H \ni f : x \longmapsto x+1$ does not preserve $\neg\mathbb{N}' = \{1\}$. Take $Q = \mathrm{Inv}_{\mathbb{N}}H$. ∎

For finite A ($|A| \geq 3$) we have (cf. [Pö/Kal(1.3.5)])

$$(\mathrm{KCl}\ 2) \Longleftrightarrow (\mathrm{Cl})\ \&\ (\neg) \Longleftrightarrow (\mathrm{Cl})\ \&\ (\nu).$$

Krasner-clones Q of 2^{nd} kind satisfy (Cl),(sSup),($\neg$) and (ν). The next theorem clarifies which conditions characterize Galois closed sets of which Galois connection:

16.6 Theorem. For $Q \subseteq R_A$, the conditions given in (α),(β) or (γ), respectively, are equivalent:

(α) (i) (KCl 1) & (ν), i.e. Q is a Krasner-clone of 1^{st} kind and $\nu \in Q$;

(ii) $Q = \mathrm{Inv}\ \mathrm{Pol}^{(1-1)}Q$ (notation cf. 1.4);

(iii) There is a set H of unary injective mappings such that $Q = \mathrm{Inv}_A H$ ($H \subseteq O_A^{(1)}$).

(β) (i)' (KCl 1) & (sSup), i.e. Q is a Krasner-clone of 1^{st} kind and closed under strong superposition;

(ii)' $Q = \mathrm{Inv}$ w-Aut Q ;

(iii)' $\exists\, H \subseteq S_A :\ Q = \mathrm{Inv}\ H$.

(γ) (i)" (KCl 2)(or (KCl 1) & (sSup) & ($\neg$)), i.e. Q is a Krasner-clone of 2^{nd} kind (cf. 16.5);

(ii)" $Q = \mathrm{Inv}\ \mathrm{Aut}\ Q$;

(iii)" $\exists\, G \leq S_A :\ Q = \mathrm{Inv}\ G$.

Proof.

(α): (ii)⇒(iii) trivial. (iii)⇒(i): $Q = \mathrm{Inv}\ H$ is a Krasner-clone of 1^{st} kind by 16.2. Moreover, ν is invariant for every injective mapping. (i)⇒(iii): $Q = \mathrm{Inv}\ \mathrm{End}\ Q$ by 16.2. Because of $\nu \in Q$, every $f \in \mathrm{End}\ Q$ must be injective, i.e. $\mathrm{End}\ Q = \mathrm{Pol}^{(1-1)}Q$.

(β): (ii)'⇒(iii)' trivial. (iii)'⇒(i)': Q satisfies (KCl 1) by (α). Moreover, for $\varrho_i \in \mathrm{Inv}\ H$, also $\varrho := \bigwedge^{\pi}_{!} \pi_i(\varrho_i)$ (cf. 16.4b)) belongs to Inv H because $\pi a \in \varrho \Rightarrow \pi_i a \in \varrho_i$ and a is bijective $\Rightarrow \pi_i af \in \varrho_i$ for $f \in H$ (since f preserves ϱ_i); af is bijective (since f bijective) hence $\pi af \in \varrho$, i.e. f preserves ϱ; consequently $\varrho \in \mathrm{Inv}\ H$.

(i)'$\Rightarrow$(ii)': Let $\varrho \in \mathrm{Inv}^{(m)}$w-Aut Q. We show $\varrho \in Q$. Clearly, $\varrho = \bigcup_{b \in \varrho} \Gamma_H(b)$ for $H = $ w-Aut Q (cf. 1.8, 2.4). Since Q = 1-LOC Q is closed under arbitrary unions, we are done if $\Gamma_H(b)$ Q for all $b \in A^m$. The proof goes analogously to that of 4.3: $\Gamma_H(b) \overset{(2.4)}{=} \{f(b) \in A^m \mid f \in \text{w-Aut } Q\}$

$= \{f(b) \mid f \text{ bijective } \& \ f \in \mathrm{Pol}^{(1)}Q\}$

$= \{ba \in A^m \mid a: A \longrightarrow A \text{ bijective } \& \ a \in \mathrm{Pol}\ Q\}$

$= \{ba \mid a \in A^A \ \& \ a \text{ bijective } \& \ \forall r \in \varrho \in Q: ra \in \varrho\}$

belongs to Q by 16.4b) (note, $ra \in \varrho$ means $f(r) \in \varrho$ for $(r: \underline{m} \longrightarrow A) \in \varrho$ and $a = f \in S_A$, cf. 3.4).

(γ): (ii)"$\Rightarrow$(iii)" trivial. (iii)"$\Rightarrow$(i)" by (β) and 16.3(ii). (i)"$\Rightarrow$(ii)" by (β) and 16.3(iii). ■

Remarks. We state here the following open problem: *

$$\boxed{(KCl\ 2) \Longleftrightarrow (Cl)\ \&\ (\neg)\quad ?}$$

The author was unable to prove this equivalence (cf. 16.5), which holds for finite A, or to give a counterexample that (Cl) & ($\neg$) do not imply (KCl 2). The crucial point consists in proving whether

$$\mathrm{Pol}^{(1-1)}Q \subseteq \text{~~LOC~~ Loc}(\text{Aut } Q)$$

holds for all Q with (Cl)&($\neg$)(because this would imply $Q \subseteq \mathrm{Inv}$ Aut Q = Inv ~~LOC~~ Loc(Aut Q) $\subseteq$ Inv $\mathrm{Pol}^{(1-1)}Q \overset{(\alpha)}{=} Q$).

For investigations of Krasner-clones we also refer to [Kr50], [Kr66], [Kr68], [Kr76a], [Le76], [Le77], [Kr/Poi], [Poi71], [Poi75], [Poi80] (for finite A see also [Bo/Kal], [Pö/Kal]).

* This is solved in the meanwhile
(it is equivalent to the question if
$H \subseteq \mathrm{Loc}(S_A \cap \mathrm{Loc}\,H)$ for every locally invertible monoid $H \leq O_A^{(1)}$. ANSWER: NO, in general.

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SUBJECT INDEX

INDEX OF NOTATIONS

The cardinality of a set A is denoted by $|A|$; $\aleph_0$ is the least infinite cardinal number ($\aleph_0 = |\mathbb{N}|$); 2^A stands for the set (lattice) of all subsets of A; for $f: A^n \longrightarrow A$ and $B \subseteq A$, $f|B$ denotes the restiction of f to B ($f|_B : B^n \longrightarrow A$, $(f|B)^\bullet = f^\bullet \cap (B^n \times A)$); A^n is the n^{th} cartesian power of A; $A \times B$ denotes the cartesian product; the logical signs $\exists, \forall, \&$ are used in the usual sense.

Author's address

Dr. R. Pöschel
Akademie der Wissenschaften der DDR
Zentralinstitut für
Mathematik und Mechanik

DDR - 108 Berlin
Mohrenstr. 39