Characterization of Framed Curves Arising from Local Isometric Immersions

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Abstract

We characterize the class of framed curves that are induced by local isometric immersions (bending deformations) defined in a neighbourhood of a curve in the reference configuration. This characterization is sharp; in particular, for every framed curve belonging to the class we construct a local isometric immersion from which it arises.

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1 Introduction

Asymptotic theories play an important role in modelling thin elastic sheets such as plates. We refer to the monograph [2] for an overview of the field. We also refer to [3] and e.g. [4] for applications of differential geometry in the context of elasticity; the present article is another such application.
In [8] (see also [19]) Kirchhoff's nonlinear plate theory [16] was derived from nonlinear three dimensional elasticity in terms of Γ-convergence. An essential feature of Kirchhoff's plate theory is that the class of deformations with finite energy consists precisely of the isometric immersions with square integrable second derivatives, i.e., they belong to the set

$$W^{2,2}_\delta(U) = \{u \in W^{2,2}(U, \mathbb{R}^3) : (\nabla u)^T \nabla u = I \text{ almost everywhere } \},$$

where $I \in \mathbb{R}^{2\times 2}$ is the identity matrix and $U \subset \mathbb{R}^2$ is the two-dimensional reference configuration of the plate.

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A further reduction of dimensions allows to derive theories for narrow ribbons [14, 10, 21, 22, 6, 7]. Such a ribbon may be defined near any embedded smooth curve in the reference configuration.
In order to establish a connection between asymptotically narrow ribbons (i.e., one-dimensional framed curves) and the (two-dimensional) plates they arise from, one needs to understand the constraints imposed on the framed curves arising from $W^{2,2}$ isometric immersions. Such a local understanding is also useful in the context of plates which are not infinitesimally narrow, because it allows to separate global constraints from purely local ones. Indeed, the purely local constraints obtained in the present article play an essential role in the context of folded isometric immersions.

In this article we consider a smooth embedded curve $b : (0, 1) \to \mathbb{R}^2$ and let $U \subset \mathbb{R}^2$ be a neighbourhood of its trace $b(0, 1)$. Let $u : U \to \mathbb{R}^3$ be a smooth isometric immersion. Then u determines a frame $r : (0, 1) \to SO(3)$, where $SO(3)$ is the set of rotation matrices, whose first row is given by $r_1 = D_{b'}u \circ b$ and whose third row is given by $r_3 = n \circ b$. Here $n = \partial_1 u \times \partial_2 u$ is the normal (the Gauss map) to u. This frame r is the (pulled back) Darboux frame of the curve $u \circ b$ on the surface u.
Conversely, given a frame r one can reconstruct u from it, at least locally near b. However, even leaving aside regularity considerations, this is clearly not possible for all frames r. In fact, the geodesic curvature of the curve $u \circ b$ is $a_{12} = r_1' \cdot r_2$. Since u is an isometric immersion and geodesic curvature is an intrinsic quantity, a necessary condition on r is that a_{12} must agree with the curvature of b, which we denote by κ.
There is, however, a second condition which is related to the developability [9, 20, 15, 18] of regular isometric immersions such as u. It arises as a sufficient condition when one reconstructs u from r, such as in the context of ribbons. Its failure is illustrated by the frame along the reference curve $b(t) = (t, 0)$ given by

$$r(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{pmatrix}. \qquad (1)$$

This frame satisfies $a_{12} = 0 = \kappa$. But it is not clear how to construct u giving rise to such a frame, and indeed our results show that there is no such isometric immersion u defined in a neighbourhood of b, no matter how narrow the neighbourhood.
Thirdly, the regularity of u clearly affects that of r and viceversa.
Our main result, Theorem 2.1 below, shows that these three constraints completely characterize frames r which arise from local isometric immersions

with finite bending energy [8], i.e., from u in the class $W^{2,2}_\delta(U)$ where U is a neighbourhood of b. More precisely, Theorem 2.1 asserts that a frame $r : (0,1) \to SO(3)$ arises from a $W^{2,2}$ isometric immersion defined in a neighbourhood of b if, and only if,

(a) $r \in W^{1,2}_{loc}(0,1)$.

(b) $r_1' \cdot r_2 = \kappa$.

(c) there exists a locally Lipschitz continuous function $F : (0,1) \to \mathbb{R}$ such that $r_2' \cdot r_3 = F r_1' \cdot r_3$.

As before, we have used the notation $r_i = r^T e_i$, where e_i are the standard basis vectors of $\mathbb{R}^3$.
Observe that condition (c) is violated by the frame (1) because it satisfies $r_2' \cdot r_3 \equiv 1$ and $r_1' \cdot r_3 \equiv 0$. More generally, the sufficiency of conditions (a) through (c) has proven useful in the contexts of ribbons [7, 6, 13] and of folded isometric immersions [1, 11] when the folding curve is neither a geodesic nor a line of curvature.
In Section 5 we will compare two other viewpoints to the one adopted in the present article. In Section 5.1 we will rephrase our main existence result Theorem 2.1 (ii) in terms of the setting in [13], where the normal to the deformed surface along the reference curve is the central variable. In Corollary 5.2 we will therefore obtain natural conditions on a curve $\beta : (0,1) \to \mathbb{S}^2$ which ensure that β can be realized as the normal of an isometric immersion defined in a neighbourhood of b. This result has been used in [13] to construct ribbons with finite width satisfying prescribed clamped boundary conditions.
In Section 5.2 we will link the viewpoint in [7], where the main variable is the second fundamental form along the reference curve b, to the one adopted here. As a result, in Corollary 5.6 we show that Theorem 2.1 (ii) allows us to recover (and in fact generalize) the existence result in [7], despite apparently different hypotheses.

2 Local Isometric Immersions and Framed Curves

2.1 Developability of Isometric Immersions

If u is an isometric immersion from a domain $U \subset \mathbb{R}^2$ into $\mathbb{R}^3$, i.e., u is Lipschitz and

$$\partial_i u \cdot \partial_j u = \delta_{ij} \text{ almost everywhere on } U, \tag{2}$$

then its Gauss map is the map $n : U \to \mathbb{S}^2$ defined by

$$n(x) = \partial_1 u(x) \times \partial_2 u(x).$$

If, moreover, $u \in W^{2,1}_{loc}$ then we can differentiate equation (2) to find that

$$\partial_i \partial_j u(x) = A_{ij}(x) n(x) \text{ for almost every } x \in U, \tag{3}$$

where $A_{ij} = n \cdot \partial_i \partial_j u$ denotes the second fundamental form of u.
Let us next recall some facts from [20, 9, 15, 18, 17] about maps $u \in W^{2,2}_\delta(U)$. Firstly, such u are continuously differentiable on U and thus their Gauss map is continuous. Secondly, such u are developable in the following sense: for every $x_0 \in U$ there exists $\delta > 0$ and a Lipschitz continuous map $q_u : B_\delta(x_0) \to \mathbb{S}^1$ such that, for all $x \in B_\delta(x_0)$,

$$\nabla u \text{ is constant on the segment } B_\delta(x_0) \cap (x + \mathbb{R} q_u(x))\,; \tag{4}$$

observe that this implies that n is constant on these segments, too.

Throughout this article $b : \overline{I} \to \mathbb{R}^2$ denotes an arclength parametrized embedded $W^{2,\infty}$-curve. We denote its curvature by $\kappa = b'' \cdot (b')^\perp$.
If $u \in W^{2,2}_\delta(U)$ and $b(I) \subset U$ then (4) implies that there exists a neighbourhood $M \subset \mathbb{R}^2$ of $I \times \{0\}$ such that the map $N = q_u \circ b$ satisfies

$$\nabla u(b(t) + sN(t)) = \nabla u(b(t)) \text{ for all } (t, s) \in M. \tag{5}$$

This suggests the following definition: given a bounded domain $U \subset \mathbb{R}^2$ with $b(I) \subset U$ and $u \in W^{1,1}_{loc}(U, \mathbb{R}^3)$, we will say that b is noncharacteristic for u if there exists a locally Lipschitz continuous map $N : I \to \mathbb{S}^1$ with $b'(t) \cdot N^\perp(t) \neq 0$ for all $t \in I$ and there exists a neighbourhood $M \subset \mathbb{R}^2$ of $I \times \{0\}$ such that (5) is satisfied.

2.2 Main Result

For p_1, $p_2 \in L^1_{loc}(I)$ we will say that $\alpha : I \to \mathbb{R}$ is an argument for (p_1, p_2) if

$$p_1 \sin \alpha = p_2 \cos \alpha \text{ almost everywhere on } I.$$

Our main result is the following theorem. It characterizes framed curves that arise from local isometric immersions. In its statement and in what follows, for a given map $r : I \to SO(3)$ we denote its rows by r_i and we set $a_{ij} = r_i' \cdot r_j$; here i, $j = 1, 2, 3$.

Theorem 2.1. *Let $b : \overline{I} \to \mathbb{R}^2$ be an arclength parametrized embedded $W^{2,\infty}$-curve with curvature $\kappa = b'' \cdot (b')^\perp$. The following are true:*

(i) Let $U \subset \mathbb{R}^2$ be a neighbourhood of $b(I)$, let $u \in W^{2,2}_\delta(U)$ and assume that b is noncharacteristic for u. Define $r : I \to SO(3)$ by

$$r_3 = n \circ b \text{ and } r_1 = (D_{b'}u) \circ b. \tag{6}$$

Then $r \in W^{1,2}_{loc}(I)$ and we have $a_{12} = \kappa$. Moreover, (a_{13}, a_{23}) admits a locally Lipschitz continuous argument $I \to (-\frac{\pi}{2}, \frac{\pi}{2})$.

(ii) Let $r \in W^{1,2}_{loc}(I, SO(3))$ be such that $a_{12} = \kappa$ and assume that (a_{13}, a_{23}) admits a locally Lipschitz continuous argument $I \to (-\frac{\pi}{2}, \frac{\pi}{2})$. Then there exists a neighbourhood $U \subset \mathbb{R}^2$ of $b(I)$ and an isometric immersion $u \in W^{2,2}_\delta(U)$ such that (6) holds on I. Moreover, b is noncharacteristic for u.

Remarks.

(i) Theorem 2.1 shows that a framed curve arises from a local isometric immersion along a noncharacteristic curve b with curvature κ precisely if it satisfies the following two conditions:

(a) $a_{12} = \kappa$.

(b) (a_{13}, a_{23}) admits a locally Lipschitz continuous argument $I \to (-\frac{\pi}{2}, \frac{\pi}{2})$.

In view of Lemma 6.3, condition (ib) is equivalent to the condition

(b') There exists a locally Lipschitz continuous function $F : I \to \mathbb{R}$ such that $a_{23} = Fa_{13}$.

Condition (ia) is due to the fact that geodesic curvature is an intrinsic quantity and is therefore preserved under isometric immersions. Condition (ib) is more subtle. It is inherited from the rigidity related to the nonzero width of the local isometric immersion.

(ii) In Theorem 2.1 (ii) the condition $r \in W^{1,2}_{loc}$ can be replaced by the apparently weaker conditions $r \in W^{1,1}_{loc}$ and $r_3 \in W^{1,2}_{loc}$, see Proposition 3.3.

(iii) Theorem 2.1 (ii) generalizes earlier results in [7]; see Section 5.2. We also refer to the constructions in [18, 12] for the particular case when $b' = -N^\perp$.

(iv) We cannot expect r to be globally in $W^{1,2}(I)$ in Theorem 2.1 (i). For instance, we can take $b(t) = te_1$ and $N \equiv e_2$, and for given $\mu : (0,1) \to \mathbb{R}$ we can define the isometric immersion $u : (0,1) \times \mathbb{R} \to \mathbb{R}^3$ by

$$u(x_1, x_2) = \begin{pmatrix} \int_0^{x_1} \cos\left(\int_0^t \mu\right) dt \\ x_2 \\ \int_0^{x_1} \sin\left(\int_0^t \mu\right) dt \end{pmatrix}.$$

Then

$$|\partial_1\partial_1 u(x_1, x_2)| = |r_1'(x_1)| = |\mu(x_1)| \text{ for all } (x_1, x_2) \in (0,1) \times \mathbb{R}.$$

Let $\alpha > 0$ and

$$U = \{(x_1, x_2) : x_1 \in (0,1) \text{ and } |x_2| < x_1^{2\alpha}\}.$$

Then

$$\int_U |\partial_1\partial_1 u|^2 = 2\int_0^1 \mu^2(t) t^{2\alpha}\, dt.$$

Taking $\mu(t) = t^{-\frac{\alpha+1}{2}}$ we see that $u \in W^{2,2}(U)$, but $\int_0^1 |r_1'|^2 = \int_0^1 \mu^2 = \infty$.

Theorem 2.1 has proven useful in the context of ribbons with finite width [13] as well as in the context of folded paper [11].

3 From Framed Curve to Local Isometric Immersion

We will obtain Theorem 2.1 (ii) as a consequence of Proposition 3.3 below (defining $\widetilde{R}_1$ by (14)). Theorem 2.1 (i) will be proven in Section 4.

3.1 Local Coordinates

For the proof of both parts of Theorem 2.1 we will make use of the coordinates introduced in Lemma 3.1 below. These are natural coordinates, and they generalize those used, e.g., in [18, 12, 6, 7]. While the proof could be shortened somewhat by applying a degree argument, we prefer to give an elementary proof.

Lemma 3.1. *Let $\mathcal{I} \subset \mathbb{R}$ be an open bounded interval and let $b : \overline{\mathcal{I}} \to \mathbb{R}^2$ be an arclength parametrized embedded $W^{2,\infty}$-curve. Let $\widetilde{R}_1 : \mathcal{I} \to \mathbb{S}^2$ be locally Lipschitz and such that $b' \cdot \widetilde{R}_1 > 0$ everywhere on $\mathcal{I}$. Set $\widetilde{R}_2 = (\widetilde{R}_1)^\perp$ and define $\Phi : \mathcal{I} \times \mathbb{R} \to \mathbb{R}^2$ by*

$$\Phi(t, s) = b(t) + s\widetilde{R}_2(t).$$

Then there exists a neighbourhood $M \subset \mathbb{R}^2$ of $\mathcal{I} \times \{0\}$ such that $\Phi|_M$ is a homeomorphism of M onto the open set $U = \Phi(M)$. Moreover, $\Phi : M \to U$ as well as $\Phi^{-1} : U \to M$ are locally Lipschitz.

Remarks.

(i) In the proof we will also show the following: if $h \in L^1_{loc}(\mathcal{I})$ is nonnegative and $m : \mathcal{I} \to \mathbb{R}$ is measurable and locally bounded, then we can choose M such that, in addition to the conclusions of the lemma,

$$\int_M h(t)\, dtds < \infty$$

and

$$|s| \leq \frac{1}{1 + |m(t)|} \text{ for all } (t, s) \in M. \tag{7}$$

(ii) Applying Remark (i) with

$$m = \frac{1 + 2|\widetilde{R}_1'|}{\widetilde{R}_1 \cdot b'}$$

we have

$$\widetilde{R}_1 \cdot b' - s\widetilde{R}_1' \cdot \widetilde{R}_2 \geq \frac{\widetilde{R}_1 \cdot b'}{2} \text{ on } M. \tag{8}$$

Since

$$\det(\partial_t \Phi \mid \partial_s \Phi) = \widetilde{R}_1 \cdot b' - s\widetilde{R}_1' \cdot \widetilde{R}_2,$$

we therefore have

$$\det(\partial_t \Phi \mid \partial_s \Phi) \geq \frac{\widetilde{R}_1 \cdot b'}{2} \text{ on } M.$$

Hence the Jacobian of Φ is bounded from below by a positive constant on $(J \times \mathbb{R}) \cap M$, whenever J is a compact subset of $\mathcal{I}$.

(iii) We will also show that if $\widetilde{R}_1$ is Lipschitz on $\mathcal{I}$ and

$$\inf_{\mathcal{I}} b' \cdot \widetilde{R}_1 > 0,$$

then there is $\delta > 0$ such that we can choose $M = \mathcal{I} \times (-\delta, \delta)$.

In the proof of Lemma 3.1 we will use the following simple observation.

Lemma 3.2. *Let $\ell > 0$ and for all $n \in \mathbb{N}$ let $c_n > 0$ and define*

$$\tau_n = (1 - 2^{-n})\ell \tag{9}$$

and

$$\mathcal{I}_n = (-\tau_{n+1}, -\tau_n) \cup (\tau_n, \tau_{n+1}). \tag{10}$$

Then there exists $\xi \in C^\infty(-\ell, \ell)$ which is positive everywhere on $(-\ell, \ell)$ and which for all $n \in \mathbb{N}$ satisfies

$$\xi(t) \le c_n \text{ for all } t \in \overline{\mathcal{I}}_n. \tag{11}$$

Proof. By taking $\bar{c}_n = \min_{k \le n} c_k$ we may assume without loss of generality that $(c_n) \subset (0, \infty)$ is a nonincreasing sequence.
Let $\varphi \in C^\infty(\mathbb{R})$ be nonnegative with $\operatorname{spt} \varphi \subset \left(\frac{1}{4}, \frac{3}{4}\right)$ and $\int \varphi = 1$. Define $\eta : \mathbb{R} \to [0, 1]$ by

$$\eta(t) = \int_0^t \varphi.$$

We define $\widetilde{\xi} : \mathbb{R} \to \mathbb{R}$ by

$$\widetilde{\xi}(t) = \sum_{n=0}^{\infty} \left(c_n + (c_{n+1} - c_n)\eta\left(\frac{t - \tau_n}{\tau_{n+1} - \tau_n}\right)\right).$$

Then $\widetilde{\xi} \in C^\infty(0, \ell)$ because locally the series is a finite sum. Finally, we define $\xi : \mathbb{R} \to \mathbb{R}$ by setting

$$\xi(t) = \begin{cases} \widetilde{\xi}(t) & \text{if } t \ge 0 \\ \widetilde{\xi}(-t) & \text{if } t < 0. \end{cases}$$

Since $\widetilde{\xi} \equiv c_0$ in a neighbourhood of 0, we have $\xi \in C^\infty(-\ell, \ell)$. □

Proof of Lemma 3.1. Without loss of generality we may assume that $\mathcal{I} = (-\ell, \ell)$ for some $\ell > 0$.
Since $\widetilde{R}_2$ is locally Lipschitz, so is Φ. We compute $\partial_s \Phi = \widetilde{R}_2$ and

$$\partial_t \Phi = b' + s\widetilde{R}_2'. \tag{12}$$

Hence (ii) follows.

Claim 1. For all $0 < \tau < \ell$ there is $\rho_\tau > 0$ such that for each $t \in [-\tau, \tau]$ the restriction of Φ to the ball $B_{\rho_\tau}(t, 0)$ is injective.
To prove this, we define the generalized gradient $\partial\Phi(\zeta_0)$ to be the convex hull of the set consisting of all limits

$$\lim_{\zeta \to \zeta_0} \nabla\Phi(\zeta),$$

where ζ belongs to the set of full measure where $\nabla\Phi(\zeta)$ exists. Since

$$\nabla\Phi(t, s) = (b'(t) + s\widetilde{R}_2'(t) \mid \widetilde{R}_2(t))$$

and since $\widetilde{R}_2$ and b' are continuous and $\widetilde{R}_2'$ is locally bounded, we conclude that

$$\partial\Phi(t, 0) = \{(b'(t) \mid \widetilde{R}_2(t))\}$$

for every $t \in (-\ell, \ell)$. Since $(b'(t)|\widetilde{R}_2(t))$ is invertible because $b' \cdot \widetilde{R}_1 \neq 0$ on $(-\ell, \ell)$, Clarke's inverse function theorem [5, Theorem 1] implies that for all $t \in (-\ell, \ell)$ there exists $r > 0$ such that $\Phi(B_r(t, 0))$ is open and Φ is injective on $B_r(t, 0)$. Covering $[-\tau, \tau] \times \{0\}$ with a finite number of such open balls, the claim follows from the Lebesgue lemma.

Claim 2. There exist positive constants ε and η such that for all t, $t' \in [-\ell, \ell]$ we have

$$|b(t) - b(t')| \geq \begin{cases} \frac{|t-t'|}{2} & \text{if } |t - t'| \leq \varepsilon \\ \eta & \text{if } |t - t'| \geq \varepsilon. \end{cases} \tag{13}$$

In fact, denote by $\operatorname{Lip} b'$ the Lipschitz constant of b', set $\varepsilon = \frac{1}{2 \operatorname{Lip} b'}$ and let $|t - t'| \leq \varepsilon$ with $t < t'$. Then

$$\begin{aligned} |b(t') - b(t)| &= \left| b'(t)\,(t' - t) + \int_t^{t'} (b'(s) - b'(t))\ ds \right| \\ &\geq |t' - t| - (\operatorname{Lip} b') \int_t^{t'} |s - t|\ ds \\ &\geq (1 - \varepsilon \operatorname{Lip} b')\,|t' - t|. \end{aligned}$$

This proves the first estimate in (13). To prove the second one assume that there were no $\eta > 0$ as in the claim. Then for all $n \in \mathbb{N}$ there would exist t_n, $t_n' \in [-\ell, \ell]$ with $|t_n - t_n'| \geq \varepsilon$ and $|b(t_n) - b(t_n')| \leq 1/n$. Their accumulation points t, $t' \in [-\ell, \ell]$ would satisfy $|t - t'| \geq \varepsilon$ yet $b(t) = b(t')$, contradicting the injectivity of b. This concludes the proof of the claim.

Claim 3. Let ε as in Claim 2, let $\tau \in (0, \ell)$ and denote by L_τ the Lipschitz constant of $\widetilde{R}_2$ on $[-\tau, \tau]$. Assume that t, $t' \in [-\tau, \tau]$ and s, $s' \in \mathbb{R}$ are such that $\Phi(t, s) = \Phi(t', s')$. If

$$|t - t'| \leq \varepsilon \text{ and } |s| \leq \frac{1}{4L_\tau}$$

then $|t - t'| \leq 4|s - s'|$.
In fact, $\Phi(t, s) = \Phi(t', s')$ implies that

$$|b(t) - b(t')| = \left| s\widetilde{R}_2(t) - s'\widetilde{R}_2(t') \right| \leq |s||\widetilde{R}_2(t) - \widetilde{R}_2(t')| + |(s - s')\widetilde{R}_2(t')|.$$

Since $|\widetilde{R}_2(t) - \widetilde{R}_2(t')| \leq L_\tau |t - t'|$ and $|s| \leq (4L_\tau)^{-1}$, using (13) we deduce that

$$\frac{|t - t'|}{2} \leq \frac{|t - t'|}{4} + |s - s'|.$$

Absorbing the first term on the right-hand side into the left-hand side the claim follows.

For $\tau \in [0, \ell)$ let ρ_τ as in Claim 1. For $n \in \mathbb{N}$ define τ_n as in (9) and $\mathcal{I}_n$ as in (10). In order to include the proof of Remark (i), let $h \in L^1_{loc}(-\ell, \ell)$ be nonnegative and let $m : (-\ell, \ell) \to \mathbb{R}$ be measurable and locally bounded. Define $h_n = \int_{\mathcal{I}_n} h$ and

$$m_n = \|m\|_{L^\infty(\mathcal{I}_n)},$$

and set

$$c_n = \min_{k \leq n} \left\{ \frac{2^{-k}}{1 + h_k}, \frac{\eta}{3}, \frac{\rho_{\tau_{k+1}}}{20}, \frac{1}{1 + 4L_{\tau_{k+1}}}, \frac{1}{1 + m_k} \right\}.$$

Clearly $(c_n) \subset (0, \infty)$ is a nonincreasing sequence. By Lemma 3.2 there exists $\xi \in C^\infty(-\ell, \ell)$ such that (11) is satisfied. Define

$$M = \{(t, s) \in (-\ell, \ell) \times \mathbb{R} : |s| < \xi(t)\}.$$

Then M is a neighbourhood of $(-\ell, \ell) \times \{0\}$. Moreover, by (11), the choice of c_n and since $(-\ell, \ell)$ agrees with $\bigcup_{n=0}^\infty \mathcal{I}_n$ up to a countable set, we can estimate

$$\begin{aligned}
\int_M h(t)\ dtds &= 2 \int_{-\ell}^{\ell} h(t)\xi(t)\ dt \\
&= 2 \sum_{n=0}^{\infty} \int_{\mathcal{I}_n} h\xi \\
&\leq 2 \sum_{n=0}^{\infty} h_n \cdot \frac{2^{-n}}{1 + h_n} < \infty.
\end{aligned}$$

Moreover, since $c_n \le \frac{1}{1+m_k}$ for all $k \le n$, we have

$$\xi(t) \le \frac{1}{|m(t)|+1} \text{ for almost every } t \in \mathcal{I}.$$

Hence (7) is satisfied as well. It remains to show that Φ is injective on M. Then the invariance of domain theorem implies that $\Phi(M)$ is open and the proof is complete.
In order to prove that Φ is injective on M, let ζ, $\zeta' \in M$ be such that $\Phi(\zeta) = \Phi(\zeta')$. We write $(t,s) = \zeta$ and $(t',s') = \zeta'$ and choose the labels such that $|t| \le |t'|$. There exist k, $k' \in \mathbb{N}$ with $k \le k'$ such that $t \in \overline{\mathcal{I}}_k$ and $t' \in \overline{\mathcal{I}}_{k'}$. By definition of M and by (11) we have $|s| \le c_k$ and $|s'| \le c_{k'} \le c_k$ because (c_n) is nonincreasing.
Since $\Phi(\zeta) = \Phi(\zeta')$, we have

$$|b(t') - b(t)| = |s\widetilde{R}_2(t) - s'\widetilde{R}_2(t')| \le |s| + |s'| \le \frac{2\eta}{3}.$$

Hence (13) implies that $|t - t'| \le \varepsilon$. Since

$$|s'| \le c_{k'} \le \frac{1}{1 + 4L_{\tau_{k'+1}}} \le \frac{1}{1 + 4L_{|t'|}},$$

and $|t| \le |t'|$, Claim 3 implies that $|t - t'| \le 4|s - s'|$. Hence

$$|\zeta - \zeta'| \le 5|s - s'| \le 10c_k \le \frac{\rho_{\tau_{k+1}}}{2}.$$

Since

$$|\zeta - (t,0)| = |s| \le c_k \le \frac{\rho_{\tau_{k+1}}}{20},$$

we conclude that

$$|\zeta' - (t,0)| \le |\zeta' - \zeta| + |\zeta - (t,0)| < \rho_{\tau_{k+1}}.$$

Therefore ζ', $\zeta \in B_{\rho_{\tau_{k+1}}}(t,0)$. Since $t \in [-\tau_{k+1}, \tau_{k+1}]$, by Claim 1 the map Φ is injective on this ball. Hence $\zeta = \zeta'$.

Finally, in order to prove Remark (iii) let us assume that $\widetilde{R}_2$ is Lipschitz on $(-\ell,\ell)$ and $\widetilde{R}_1 \cdot b'$ is bounded from below on $(-\ell,\ell)$ by a positive constant. In this case there is $r > 0$ such that we can extend both $\widetilde{R}_2$ and b to $(-\ell-r, \ell+r)$; we define σ and M as before, but with $(-\ell-r, \ell+r)$ replacing $(-\ell,\ell)$. The infimum δ of σ over $[0,\ell]$ is positive. Clearly $(-\ell,\ell) \times (-\delta,\delta) \subset M$. $\square$

3.2 Construction of Local Isometric Immersion

We recall that $b : \overline{I} \to \mathbb{R}^2$ always denotes an embedded $W^{2,\infty}$-curve that is parametrized by arclength; its curvature is denoted by κ. The following is our main result regarding the existence of local isometric immersions.

Proposition 3.3. *Let $\widetilde{R}_1 : I \to \mathbb{S}^1$ be locally Lipschitz and assume that $\widetilde{R}_1 \cdot b' > 0$ on I. Set $\widetilde{R}_2 = \widetilde{R}_1^\perp$ and define*

$$\begin{aligned}\Phi : I \times \mathbb{R} &\to \mathbb{R}^2\\ (t,s) &\mapsto b(t) + s\widetilde{R}_2(t).\end{aligned}$$

Then there exists a neighbourhood $M \subset \mathbb{R}^2$ of $I \times \{0\}$ such that $U = \Phi(M)$ is open and $\Phi : M \to U$ is an orientation preserving locally bi-Lipschitz homeomorphism.
Let $\alpha \in W^{1,\infty}_{loc}(I)$ be the unique function $I \to (-\frac{\pi}{2}, \frac{\pi}{2})$ such that

$$\widetilde{R}_1 = b' \cos\alpha + (b')^\perp \sin\alpha. \tag{14}$$

Let $r \in W^{1,1}_{loc}(I, SO(3))$ satisfy

$$a_{12} = \kappa \tag{15}$$
$$a_{13} \sin\alpha = a_{23} \cos\alpha \tag{16}$$

almost everywhere on I.
Define $\widetilde{r} : I \to SO(3)$ by setting $\widetilde{r}_3 = r_3$ and

$$\widetilde{r}_1 = r_1 \cos\alpha + r_2 \sin\alpha.$$

Then the map $u : U \to \mathbb{R}^3$ defined by setting

$$u\left(\Phi(t,s)\right) = s\widetilde{r}_2(t) + \int_0^t r_1 \text{ for all } (t,s) \in M \tag{17}$$

belongs to $C^1_\delta(U)$ and we have

$$(\nabla u)(\Phi) = \widetilde{r}_1 \otimes \widetilde{R}_1 + \widetilde{r}_2 \otimes \widetilde{R}_2 \text{ on } M. \tag{18}$$

Moreover, $u \in W^{2,1}_{loc}(U, \mathbb{R}^3)$ and its second fundamental form satisfies

$$A(\Phi) = \frac{\widetilde{a}_{13}}{\cos\alpha - s\widetilde{R}_1' \cdot \widetilde{R}_2}\ \widetilde{R}_1 \otimes \widetilde{R}_1 \text{ on } M. \tag{19}$$

If, in addition, $r_3 \in W^{1,2}_{loc}(I)$, then M can be chosen such that $u \in W^{2,2}(U, \mathbb{R}^3)$.

Remarks.

(i) In (18), (19) and similar equations, we tacitly identify a function $f \in L^1_{loc}(I)$ with its trivial extension $f \circ P \in L^1_{loc}(I \times \mathbb{R})$, where P is the canonical projection $I \times \mathbb{R} \to I$.

(ii) In view of (18), we see that u satisfies (5) with $N = \widetilde{R}_2$ and

$$(D_{b'}u) \circ b = r_1 \text{ and } n \circ b = r_3 \text{ on } I.$$

(iii) If $J \subset I$ is an interval and $r_3 \in W^{1,2}(J)$, then clearly a_{13}, $a_{23} \in L^2(J)$. Since moreover $a_{12} \in L^\infty(I)$ by (15), we see that $r_3 \in W^{1,2}(J)$ implies that $r \in W^{1,2}(J)$.

(iv) If $\widetilde{R}_1 \in W^{1,\infty}(I)$ and

$$\inf_I \widetilde{R}_1 \cdot b' > 0, \tag{20}$$

then M can be chosen to be of the form $M = I \times (-\delta, \delta)$ for some $\delta > 0$. This follows from Lemma 3.1.

In order to prove Proposition 3.3, we begin with a simple lemma.

Lemma 3.4. *Let r, $\widetilde{r} \in W^{1,1}_{loc}(I, SO(3))$. Then $\widetilde{r}_3 = r_3$ if and only if there is $\alpha \in W^{1,1}_{loc}(I)$ (unique up to addition of an integer multiple of 2π) such that*

$$\begin{aligned} \widetilde{r}_1 &= r_1 \cos\alpha + r_2 \sin\alpha \\ \widetilde{r}_2 &= r_2 \cos\alpha - r_1 \sin\alpha. \end{aligned} \tag{21}$$

Moreover, if (21) is satisfied, then

$$\begin{aligned} \widetilde{a}_{12} &= a_{12} + \alpha' \\ \widetilde{a}_{13} &= a_{13} \cos\alpha + a_{23} \sin\alpha \\ \widetilde{a}_{23} &= a_{23} \cos\alpha - a_{13} \sin\alpha. \end{aligned}$$

In particular, α is an argument for (a_{13}, a_{23}) if and only if $\widetilde{a}_{23} = 0$.

Proof. The existence of α follows from Lemma 6.2. The formulae for the $\widetilde{a}_{ij}$ follow from a standard computation. □

In what follows $O(2,3) \subset \mathbb{R}^{3\times 2}$ denotes the set of matrices with orthonormal columns. The next lemma shows that most conclusions of Proposition 3.3 follow from (18).

Lemma 3.5. *Let $\widetilde{R}_1 : I \to \mathbb{S}^2$ be locally Lipschitz and such that $\widetilde{R}_1 \cdot b' > 0$ on I, and let $Q \in W^{1,1}_{loc}(I, O(2,3))$. Set $\widetilde{R}_2 = \widetilde{R}_1^{\perp}$ and define r, $\widetilde{r} : I \to SO(3)$ by setting $\widetilde{r}_3 = r_3 = Qe_1 \times Qe_2$ and $r_1 = Qb'$ as well as $\widetilde{r}_1 = Q\widetilde{R}_1$. Define Φ, M and U as in Lemma 3.1, let $u \in W^{1,1}_{loc}(U, \mathbb{R}^3)$ and assume that*

$$(\nabla u)(\Phi(t,s)) = Q(t) \text{ for almost every } (t,s) \in M. \tag{22}$$

Then Φ has the properties asserted in Proposition 3.3, we have

$$u \in C^1_\delta(U) \cap W^{2,1}_{loc}(U, \mathbb{R}^3)$$

and u is given, up to a translation, by (17). Moreover, we have

$$n(\Phi(t,s)) = r_3(t) \text{ for all } (t,s) \in M \tag{23}$$

and (18) as well as

$$A(\Phi(t,s)) = \frac{\widetilde{a}_{13}(t)}{b'(t) \cdot \widetilde{R}_1(t) - s\widetilde{R}_1'(t) \cdot \widetilde{R}_2(t)}\ \widetilde{R}_1(t) \otimes \widetilde{R}_1(t) \tag{24}$$

for almost every $(t,s) \in M$. Moreover, for almost every $t \in I$ the point $b(t)$ is a Lebesgue point for the representative of A defined by (24), and with this representative we have

$$\begin{aligned} (n \circ b)' &= (D_{b'} n) \circ b \text{ almost everywhere on } I \\ ((\nabla u) \circ b)' &= (D_{b'} \nabla u) \circ b \text{ almost everywhere on } I. \end{aligned} \tag{25}$$

In addition, we have

$$a_{12} = \kappa \tag{26}$$

$$\widetilde{a}_{12} = \widetilde{R}_1' \cdot \widetilde{R}_2 \tag{27}$$

$$\widetilde{a}_{23} = 0, \tag{28}$$

and the unique function $\alpha \in W^{1,\infty}_{loc}(I)$ satisfying $|\alpha| < \frac{\pi}{2}$ and (14) is an argument for (a_{13}, a_{23}).

Proof. Set $\kappa_g = \widetilde{R}_1' \cdot \widetilde{R}_2$. By Lemma 3.4 there exists a unique locally Lipschitz function $\alpha : I \to (-\pi/2, \pi/2)$ satisfying (14). Observe that (14) implies

$$r_1 = Qb' = \widetilde{r}_1 \cos\alpha - \widetilde{r}_2 \sin\alpha. \tag{29}$$

The existence of M and the properties of Φ follow from Lemma 3.1 and the remarks following it. In particular, by Remark (ii) following Lemma 3.1, we have

$$\cos\alpha - s\kappa_g > 0 \text{ on } M. \tag{30}$$

By (22) and the continuity of $Q : I \to O(2,3)$ and of $\Phi^{-1} : U \to M$, we have $u \in C^1_\delta(U)$. Formula (23) follows from (22). Since $Q \in W^{1,1}_{loc}(I)$, we have $\nabla u(\Phi) \in W^{1,1}_{loc}(M)$. Since $\Phi : M \to U$ is locally bi-Lipschitz, we deduce that $\nabla u \in W^{1,1}_{loc}(U)$, so $u \in W^{2,1}_{loc}(U)$, and that the chain rule applies. In particular,

$$\partial_t \left(\nabla u(\Phi)\right) = (D_{\partial_t \Phi} \nabla u)(\Phi) \text{ on } M, \tag{31}$$

and a similar equation for n.

We claim that the map $\widetilde{u} : U \to \mathbb{R}^3$ defined by the right-hand side of (17) satisfies (18) (with $\widetilde{u}$ instead of u on the left-hand side).

In fact, since Φ is injective, $\widetilde{u}$ is well-defined. Taking the derivative with respect to s we see that $(D_{\widetilde{R}_2}\widetilde{u})(\Phi) = \widetilde{r}_2$. Using this and (12) as well as (29), we find

$$\begin{aligned}
(\cos\alpha - s\kappa_g)(D_{\widetilde{R}_1}\widetilde{u})(\Phi) - \widetilde{r}_2 \sin\alpha &= \partial_t(\widetilde{u}\circ\Phi) \\
&= r_1 - s\kappa_g \widetilde{r}_1 \\
&= (\cos\alpha - s\kappa_g)\ \widetilde{r}_1 - \widetilde{r}_2 \sin\alpha.
\end{aligned}$$

In view of (30) we deduce that $(D_{\widetilde{R}_1}\widetilde{u})(\Phi) = \widetilde{r}_1$. This concludes the proof of (18).

On the other hand the definition of $\widetilde{r}_1$ and $\widetilde{r}_3$ means that

$$Q = \widetilde{r}_1 \otimes \widetilde{R}_1 + \widetilde{r}_2 \otimes \widetilde{R}_2.$$

Hence the differentials of u and $\widetilde{u}$ agree, so after adding a constant vector to u, formula (17) is satisfied.

Next we observe that (18) implies that

$$(D_{\widetilde{R}_2}\nabla u)(\Phi) = \partial_s\left((\nabla u)(\Phi)\right) = 0 \text{ on } M. \tag{32}$$

Since $\nabla^2 u$ is symmetric, we deduce from (3) and (23) that there is a measurable function $\lambda : M \to \mathbb{R}$ such that

$$(\nabla^2 u)(\Phi) = \lambda r_3 \otimes \widetilde{R}_1 \otimes \widetilde{R}_1 \text{ on } M. \tag{33}$$

In particular, $(\nabla^2 u)(\Phi)(\widetilde{R}_2, \cdot) = 0$. Hence

$$\widetilde{r}_2' = \partial_t\left((D_{\widetilde{R}_2}u)(\Phi)\right) = (D_{\widetilde{R}_2'}u)(\Phi) = -\kappa_g \widetilde{r}_1$$

almost everywhere on M. Hence $\widetilde{a}_{23} = 0$ and $\widetilde{a}_{12} = \kappa_g$ almost everywhere on I. Using these and taking the derivative with respect to t in (18), we find

$$\left(b' \cdot \widetilde{R}_1 - s\kappa_g\right)(D_{\widetilde{R}_1}\nabla u)(\Phi) = \widetilde{a}_{13} r_3 \otimes \widetilde{R}_1 \text{ on } M. \tag{34}$$

Hence we deduce from (30) that in (33) we have

$$\lambda = \frac{\widetilde{a}_{13}}{b' \cdot \widetilde{R}_1 - s\kappa_g} \text{ on } M.$$

Hence (24) follows from (33).
Let us denote by A the representative defined pointwise by (24). Let $t_0 \in I$ be a Lebesgue point of $\widetilde{a}_{13}$ and let $r > 0$ be such that the closure of $B := B_r(b(t_0))$ is contained in U. Since Φ^{-1} is Lipschitz on B, there is a constant k such that

$$\Phi^{-1}(B) \subset B_{kr}(t_0, 0). \tag{35}$$

In view of (24) and of (ii), setting $m = \widetilde{a}_{13}\widetilde{R}_1 \otimes \widetilde{R}_1$ and $a = b' \cdot \widetilde{R}_1$, we have

$$\begin{aligned} \int_B |A - A(b(t_0))| &= \int_{\Phi^{-1}(B)} \left| m(t) - m(t_0)\frac{a(t) - s\kappa_g(t)}{a(t_0)} \right| \, dtds \\ &\leq \frac{1}{a(t_0)} \int_{\Phi^{-1}(B)} |m(t)a(t_0) - m(t_0)a(t)| \, dtds + \frac{C}{a(t_0)} \int_{\Phi^{-1}(B)} |s| \, dtds. \end{aligned}$$

In the last step we used that κ_g is locally bounded because $\widetilde{R}_1$ is locally Lipschitz. Since by the continuity of $\widetilde{R}_1$ the point t_0 is a Lebesgue point of $m(t)$, in view of (35) as $r \to 0$ the first term on the right-hand side converges to zero much faster than $|B_{kr}|$, hence much faster than r^2. The same is true for the second term, as it is dominated by $r|B_{kr}|$. Hence $b(t_0)$ is a Lebesgue point of A and (25) follows from (31).
Finally, we compute using (3) and (22):

$$\begin{aligned} a_{12} &= ((D_{b'}u)(b))' \cdot (D_{(b')^\perp}u)(b) \\ &= (D_{b''}u)(b) \cdot (D_{(b')^\perp}u)(b). \\ &= Qb'' \cdot Q(b')^\perp = \kappa \end{aligned}$$

because Q takes values in $O(2,3)$. □

Proof of Proposition 3.3. The existence and uniqueness of α follows from Lemma 3.4. Set $\kappa = b'' \cdot (b')^\perp$ and $\kappa_g = \alpha' + \kappa$. A short computation shows that $\widetilde{R}_1' = \kappa_g \widetilde{R}_2$. Lemma 3.4 shows that (15) implies $\widetilde{a}_{12} = \kappa_g$ and that (16) implies $\widetilde{a}_{23} = 0$. Hence

$$\widetilde{r}_2' = -\kappa_g \widetilde{r}_1. \tag{36}$$

As before, the existence of M and the properties of Φ follow from Lemma 3.1.

According to Remark (ii) following Lemma 3.1, we may assume that (8) is satisfied. If, moreover, $r_3 \in W^{1,2}_{loc}(I)$, then by Remark (i) following Lemma 3.1 we may also assume that

$$\int_M \frac{|r_3'|^2}{\cos\alpha}\, dtds < \infty. \tag{37}$$

As noted in the proof of Lemma 3.5, the map $u : U \to \mathbb{R}^3$ given by (17) is well-defined and satisfies (18). Hence Lemma 3.5 shows that $u \in C^1_\delta(U) \cap W^{2,1}_{loc}(U)$, that $n(\Phi) = r_3$ on M and that (19) is satisfied.
Finally, assume that $r_3 \in W^{1,2}_{loc}(I)$. Since $|\widetilde{a}_{13}| = |r_3'|$, equation (19) implies that

$$\int_U |A|^2 = \int_M |A(\Phi)|^2 |\det \nabla\Phi| = \int_M \frac{|r_3'|^2}{\cos\alpha - s\kappa_g}. \tag{38}$$

By (8) we have

$$\cos\alpha - s\kappa_g \geq \frac{\cos\alpha}{2} \text{ on } M.$$

Hence

$$\int_M \frac{|r_3'|^2}{\cos\alpha - s\kappa_g} \leq 2 \int_M \frac{|r_3'|^2}{\cos\alpha}. \tag{39}$$

In view of (37) the right-hand side of (39) is finite. Hence (38) implies that $A \in L^2(U)$. In view of (3) this implies that $\nabla^2 u \in L^2(U)$, too, and therefore $u \in W^{2,2}(U)$. □

4 From Isometric Immersion to Framed Curve

The converse to Proposition 3.3 will be addressed now, thus leading to a proof of Theorem 2.1 (i).

Proof of Theorem 2.1 (i). Since b is noncharacteristic, after possibly replacing q_u by $-q_u$, the map $\widetilde{R}_2 : I \to \mathbb{S}^2$ defined by $\widetilde{R}_2 = q_u \circ b$ is locally Lipschitz and $\widetilde{R}_1 = -(\widetilde{R}_2)^\perp$ satisfies $b' \cdot \widetilde{R}_1 > 0$ on I.
Define Φ and M as in Lemma 3.1. After possibly shrinking M or U, we may assume that $U = \Phi(M)$ and that

$$(\nabla u)(\Phi(t,s)) = (\nabla u)(b(t)) \text{ for all } (t,s) \in M. \tag{40}$$

Since $\nabla u \in W^{1,2}_{loc}(U)$ and Φ is locally bi-Lipschitz, we have $(\nabla u)(\Phi) \in W^{1,2}_{loc}(M)$. Therefore, $(\nabla u)(\Phi)$ is in $W^{1,2}_{loc}$ along almost every line $M \cap \{s \equiv const.\}$. Since it does not depend on s, we conclude that $(\nabla u)(b) \in W^{1,2}_{loc}(I)$. Hence $r \in W^{1,2}_{loc}(I)$.

In view of (40) the hypotheses of Lemma 3.5 are satisfied with $Q = (\nabla u)(b)$. Define $\widetilde{r}$ as in that lemma and notice that the definition of r used here agrees with that in the hypotheses of Lemma 3.5. We conclude that $a_{12} = \kappa$ and that the locally Lipschitz continuous function $\alpha : I \to (-\frac{\pi}{2}, \frac{\pi}{2})$ determined by (14) is an argument for (a_{13}, a_{23}). □

5 Relation to Other Viewpoints

In this section we link the setting used in the present article to the viewpoints adopted in [13] and in [7], and we restate some of our results in those settings.

5.1 Gauss Map along Reference Curve

In this section we interpret Theorem 2.1 in terms of the spherical image viewpoint adopted in [13]. In that setting, the key variable is the Gauss map of u along the reference curve b. The main result of this section, Corollary 5.2, identifies conditions for a spherical curve $\beta : I \to \mathbb{S}^2$ to arise as the Gauss map of an isometric immersion defined in a neighbourhood of b. We begin by recalling from [13] the following definition.

Definition 5.1. *An adapted frame for a curve $\beta \in W^{1,1}_{loc}(I, \mathbb{S}^2)$ is a map $\widetilde{r} \in L^\infty(I, SO(3))$ satisfying*

$$\widetilde{r}_3 = \beta \text{ almost everywhere on } I \tag{41}$$

$$\beta' \times \widetilde{r}_1 = 0 \text{ almost everywhere on } I. \tag{42}$$

The curve β is said to have geodesic curvature $\kappa_g \in L^1_{loc}(I)$ if there exists an adapted frame $\widetilde{r} \in W^{1,1}_{loc}(I, SO(3))$ for β satisfying

$$\widetilde{r}_1' \cdot \widetilde{r}_2 = \kappa_g \text{ almost everywhere on } I. \tag{43}$$

Remarks.

(i) If $\widetilde{r}$ is adapted to the spherical curve $r_3 : I \to \mathbb{S}^2$, then

$$\widetilde{a}_{23} = 0. \tag{44}$$

In view of (44) we have

$$r_3' = -\widetilde{a}_{13}\widetilde{r}_1 \tag{45}$$

and in particular $|\widetilde{a}_{13}| = |r_3'|$. Moreover, $\widetilde{a}_{12}$ is the geodesic curvature of r_3.

(ii) If r and $\widetilde{r}$ are as in Lemma 3.4, then that lemma shows that α is an argument for (a_{13}, a_{23}) if and only if $\widetilde{r}$ is adapted to the spherical curve $r_3 : I \to \mathbb{S}^2$.

The following corollary to Proposition 3.3 is the main result of this section. It has been used in [13] for the construction of ribbons with finite width satisfying prescribed clamped boundary conditions.
Corollary 5.2 essentially asserts that, given a curve $\beta : I \to \mathbb{S}^2$ and the usual arclength parametrized embedded curve b in the reference domain, one can construct a local isometric immersion whose Gauss map satisfies $n \circ b = \beta$, provided that the following two conditions are satisfied:

(a) β has locally bounded geodesic curvature κ_g.

(b) The oscillation of the relative phase $\int_0^t (\kappa_g - \kappa)$ is smaller than π.

Corollary 5.2. *Let $b : \overline{I} \to \mathbb{R}^2$ be an embedded arclength parametrized $W^{2,\infty}$ curve with curvature κ and let $\beta \in W^{1,2}_{loc}(I, \mathbb{S}^2)$ have geodesic curvature $\kappa_g \in L^\infty_{loc}(I)$. Assume that there is a primitive $\alpha : I \to \mathbb{R}$ of $\kappa_g - \kappa$ satisfying*

$$|\alpha(t)| < \frac{\pi}{2} \text{ for all } t \in I. \tag{46}$$

Then there exists a neighbourhood $U \subset \mathbb{R}^2$ of $b(I)$ and an isometric immersion $u \in W^{2,2}_\delta(U)$ for which b is noncharacteristic and whose Gauss map satisfies $n \circ b = \beta$ on I.

Remark. Note that whenever $r \in W^{1,1}_{loc}(I, SO(3))$ is such that $r_3 : I \to \mathbb{S}^2$ has geodesic curvature κ_g, then by definition there exists a frame $\widetilde{r} \in W^{1,1}_{loc}(I, SO(3))$ adapted to r_3 satisfying $\widetilde{a}_{12} = \kappa_g$. Lemma 3.4 then shows that (21) is satisfied by a primitive α of $\kappa_g - a_{12}$.

Proof of Corollary 5.2. Clearly α is locally Lipschitz. Define $\widetilde{R}_1$ by (14). Let $\widetilde{r}$ be an adapted frame for β with $\widetilde{r}_1' \cdot \widetilde{r}_2 = \kappa_g$; so in particular $\widetilde{a}_{23} = 0$.
Define $r : I \to SO(3)$ by setting $r_3 = \beta$ and

$$r_1 = \widetilde{r}_1 \cos\alpha - \widetilde{r}_2 \sin\alpha. \tag{47}$$

Then (21) is satisfied. Hence Lemma 3.4 implies that $a_{12} = \widetilde{a}_{12} - \alpha' = \kappa$ and that α is an argument for (a_{13}, a_{23}). Therefore all hypotheses of Proposition 3.3 are satisfied and the assertion follows from that proposition. $\square$

The following corollary is a converse to Corollary 5.2; it could be expanded using all of Theorem 2.1 (i).

Corollary 5.3. *Let $U \subset \mathbb{R}^2$ be a bounded domain, let $u \in W^{2,2}_\delta(U)$ and let $b : I \to U$ be noncharacteristic for u. Then $n \circ b : I \to \mathbb{S}^2$ has locally bounded geodesic curvature.*

Proof. With the notation as in the previous proof and as in Lemma 3.5, the frame $\widetilde{r}$ is adapted to $r_3 = n \circ b$ and the geodesic curvature of r_3 agrees with $\widetilde{a}_{12}$. Since $\widetilde{R}_1$ is locally Lipschitz, we deduce from (27) that $\widetilde{a}_{12} \in L^\infty_{loc}(I)$. □

5.2 Second Fundamental Form along Reference Curve

The viewpoint adopted in the variational context in [7, 6] differs from the one in the earlier sections, in that the relevant variable is the second fundamental form Ξ of the immersion u along the reference curve b. The main result of this section, Corollary 5.6, is [7, Lemma 12]. While the hypotheses of Corollary 5.6 appear to be quite different, we will in fact obtain it as a consequence of Proposition 3.3 once we have linked, via Lemma 5.5 below, the viewpoint from [7] to the one adopted in the earlier sections.
We begin by recalling the following definition from [7].

Definition 5.4. *For $\Xi : I \to \mathbb{R}^{2\times 2}_{\text{sym}}$, a map $r \in W^{1,1}(I, SO(3))$ is said to be adapted to (b, Ξ) if the following equations are satisfied:*

$$\begin{aligned} a_{12} &= \kappa \\ a_{23} &= (b')^\perp \cdot \Xi b' \\ a_{13} &= b' \cdot \Xi b'. \end{aligned}$$

Let us compare this notion of adaptedness to the hypotheses of Proposition 3.3:

Lemma 5.5. *Let K, $K_g \in W^{1,\infty}(I)$ and assume that $b' = e^{iK}$. Let $r \in W^{1,1}(I, SO(3))$. Then the following are equivalent:*

(i) There exists $\lambda : I \to \mathbb{R}$ such that r is adapted to $(b, \lambda e^{iK_g} \otimes e^{iK_g})$ in the sense of Definition 5.4.

(ii) $K_g - K$ is an argument for (a_{13}, a_{23}) and $a_{12} = \kappa$.

Proof. When r is adapted to $(b, \lambda p \otimes p)$ for some $\lambda : I \to \mathbb{R}$, then

$$a_{13} = \lambda(p \cdot b')^2 \text{ and } a_{23} = \lambda(p \cdot b')(p \cdot (b')^\perp)$$

Absorbing $p \cdot b'$ into λ, this is equivalent to the existence of λ such that

$$a_{13} = \lambda p \cdot b' \text{ and } a_{23} = \lambda p \cdot (b')^\perp.$$

If $p = e^{iK_g}$ and $b' = e^{iK}$, then

$$p \cdot b' = \cos(K - K_g) \text{ and } p \cdot (b')^\perp = -\sin(K - K_g).$$

Hence we conclude that r is adapted to $(b, \lambda e^{iK_g} \otimes e^{iK_g})$ for some $\lambda : I \to \mathbb{R}$ precisely if $a_{12} = \kappa$ and

$$a_{13} = \lambda \cos(K_g - K) \text{ and } a_{23} = \lambda \sin(K_g - K). \tag{48}$$

In view of Remark 6.1 equations (48) are equivalent to $K_g - K$ being an argument for (a_{13}, a_{23}). □

Using Lemma 5.5 we can see that [7, Lemma 12] follows from Proposition 3.3:

Corollary 5.6. *Let $p \in C^1(\overline{I}, \mathbb{S}^2)$ be such that $p \cdot b' \neq 0$ on $\overline{I}$ and let $\lambda \in L^2(I)$. Let $r \in W^{1,2}(I, SO(3))$ be a frame adapted to the pair $(b, \lambda p \otimes p)$ in the sense of Definition 5.4. Then there exists a neighborhood $U \subset \mathbb{R}^2$ of $b(I)$ and $u \in W^{2,2}_\delta(U, \mathbb{R}^3)$ such that $u \circ b = \int_0^t r_1$ and $A \circ b = \lambda p \otimes p$.*

Proof. By continuity we may assume that $p \cdot b' > 0$ in $\overline{I}$, after possibly replacing p by $-p$. Hence by Lemma 6.2 there exist K_g, $K \in W^{1,\infty}(I)$ with $|K - K_g| < \frac{\pi}{2}$ on $\overline{I}$ such that $p = e^{iK_g}$ and $b' = e^{iK}$. So Lemma 5.5 (i) is satisfied. Hence $a_{12} = \kappa$ and $K_g - K$ is a Lipschitz continuous argument for (a_{13}, a_{23}).
Hence all hypotheses of Proposition 3.3 are satisfied with $\widetilde{R}_1 = p$. The assertions follow from that proposition. □

6 Appendix

In this appendix we collect some simple observations about the 'argument' α. The following observation follows immediately from the fact that (p_1, p_2) is parallel to $e^{i\alpha}$ if and only if it is orthogonal to $ie^{i\alpha}$.

Remark 6.1. *Let p_1, $p_2 \in L^1_{loc}(I)$ and $\alpha : I \to \mathbb{R}$. Then the following are equivalent:*

(i) α is an argument for (p_1, p_2).

(ii) There is a function $\lambda : I \to \mathbb{R}$ such that $(p_1, p_2) = \lambda e^{i\alpha}$ almost everywhere.

The proof of the following lemma is left to the reader.

Lemma 6.2. *Let $p \in [1, \infty]$ and let a, $\overline{a} \in W^{1,p}(I, \mathbb{S}^1)$. Then there exists a unique $\alpha \in W^{1,p}(I)$ with $\alpha(0) \in [0, 2\pi)$ such that $a = e^{i\alpha}\overline{a}$ almost everywhere on I.*

In the following Lemma we set $W^{0,\infty} = L^\infty$.

Lemma 6.3. *Let m_1, $m_2 \in L^1_{loc}(I)$ and let $k \geq 0$. Then the following are equivalent:*

(i) (m_1, m_2) *admits an argument* $\alpha \in W^{k,\infty}(I)$ *satisfying* $\|\alpha\|_{L^\infty(I)} < \frac{\pi}{2}$.

(ii) There is a function $F \in W^{k,\infty}(I)$ *such that* $m_2 = m_1 F$.

Similarly, (m_1, m_2) admits an argument $\overline{I} \to (0, \pi)$ in $W^{k,\infty}(I)$ if and only if $m_1 = m_2 F$ for some $F \in W^{k,\infty}(I)$. This follows by swapping the roles of m_1 and m_2 in Lemma 6.3.
By applying Lemma 6.3 on compact subintervals of I we obtain its local version: (m_1, m_2) admits an argument $I \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ in $W^{k,\infty}_{loc}(I)$ if and only if $m_2 = m_1 F$ for some $F \in W^{k,\infty}_{loc}(I)$.

Proof of Lemma 6.3. Assertion (i) means that there exists a function $\alpha \in W^{k,\infty}(I)$ with $|\alpha| < \frac{\pi}{2}$ on $\overline{I}$ such that $m_1 \sin\alpha = m_2 \cos\alpha$, i.e., $m_2 = m_1 \tan\alpha$. Hence (ii) is satisfied with $F = \tan\alpha$. Indeed, all derivatives of tan are bounded on compact subintervals of $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, hence $F \in W^{k,\infty}(I)$ by the chain rule.
Conversely, if (ii) is satisfied, then we set $\alpha = \arctan F$. Since arctan and its derivatives are uniformly bounded, $\alpha \in W^{k,\infty}(I)$ by the chain rule. Moreover, α takes values in a compact subinterval of $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ because F is bounded. □

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