

## 8 Bifurkationen

21.12.12

### Bifurkation:

Änderung des qualitativen Systemverhaltens

globale u. lokale Bif.

Hier lokale Bif. von Ruhelagen / Fixpunkten.

$\mu$ -Systemparameter, Bif. bei  $\mu = \mu_0 (= 0)$ :

$$\dot{x} = f(x, \mu) \text{ mit FP } \bar{x} : f(\bar{x}, \mu) = 0$$

bzw.

$$x(n+1) = \varphi(x, \mu) \text{ mit FP } \bar{x} : \dot{x} = f(\bar{x}, \mu)$$

Bifurkation wenn:

$$\text{analog: } \operatorname{Re}\lambda = 0, \quad \lambda = \underbrace{\operatorname{EW}(Df_x(\bar{x}, \mu_0))}_{\frac{df(x, \mu)}{dx} \mid_{x=\bar{x}}}$$

$$\frac{df(x, \mu)}{dx} \mid_{x=\bar{x}}$$

Fold-Bif.  
(Saddle-  
Node-Bif.)

$\operatorname{Im}(\lambda)$

$\operatorname{Im}(\lambda)$

$\operatorname{Re}\lambda$

$\operatorname{Re}\lambda$

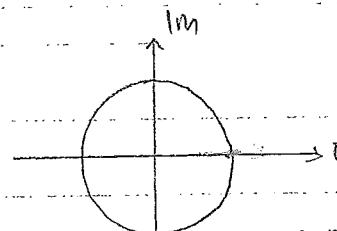
$\operatorname{Re}\lambda$

$\operatorname{Re}\lambda$

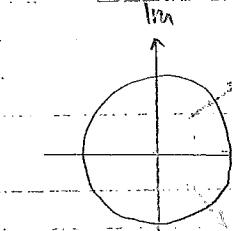
$\operatorname{Re}\lambda$

zeitdiskret:

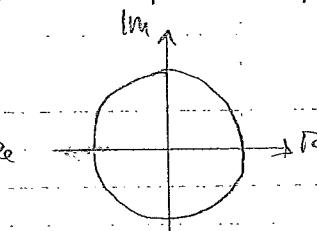
$$|\lambda| = 1$$



Fold-Bif.



Neimark-  
Sacker-Bif.



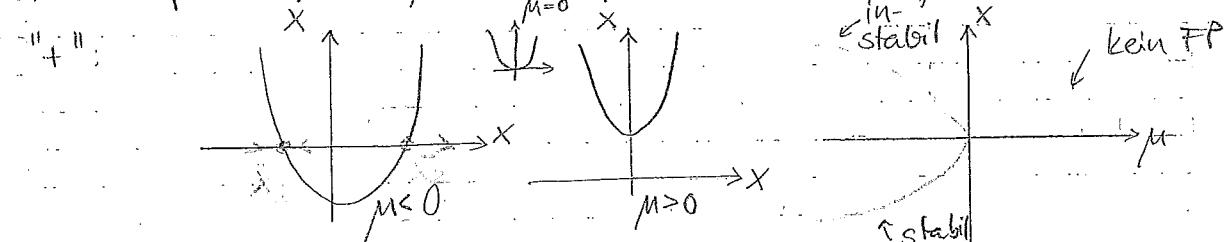
Flip-Bif.

### 8.1 Fold-Bif. (allgemein)

Ang.:  $\bar{x} = 0$  und Bif. bei  $\mu = \mu_0 = 0 \rightarrow Df_x(0, 0) = 0$  (analog)

(Taylorreihe + Normierung). Bzw.  $Df_x(0, 0) = 1$  (zd)

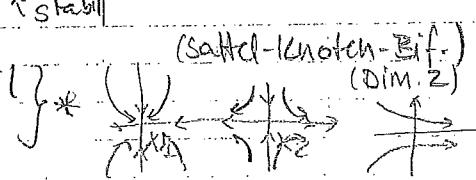
Normalform:  $\dot{x} = \mu \pm x^2$  bzw.  $x(n+1) = \mu + x(n) \pm x^2(n)$

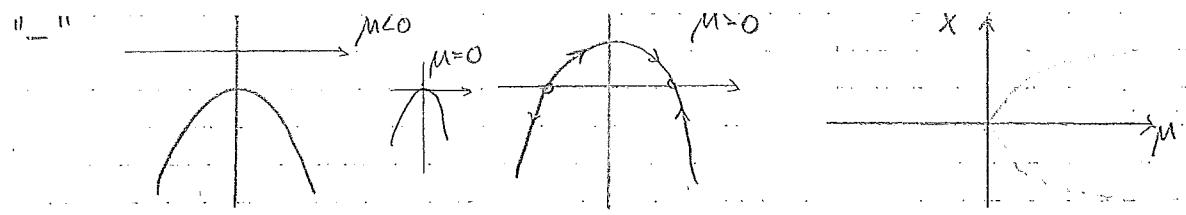


$\mu < \mu_0: 2\text{FP} \rightarrow 1 \text{ as. stabil, 1 instabil}$

$\mu > \mu_0: \text{keine FP}$

\* oder andernam



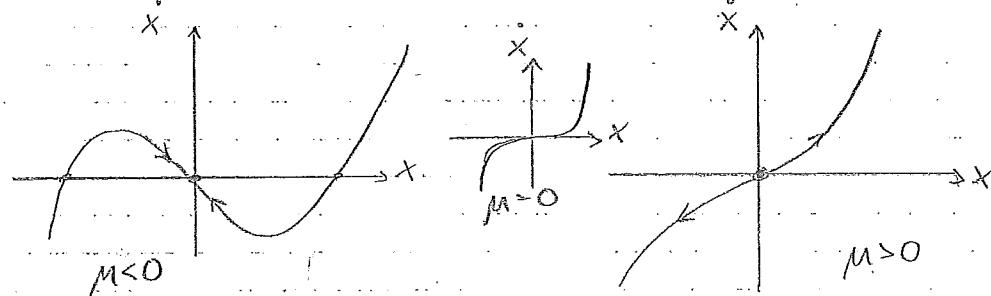


### 8.2 Pitchfork-Bif. (Spezialfall $\lambda=0$ )

f ungerade Fkt. in x

Normalfkt.  $\dot{x} = \mu x \pm x^3$  bzw.  $x(n+1) = (1+\mu)x \pm x^3$

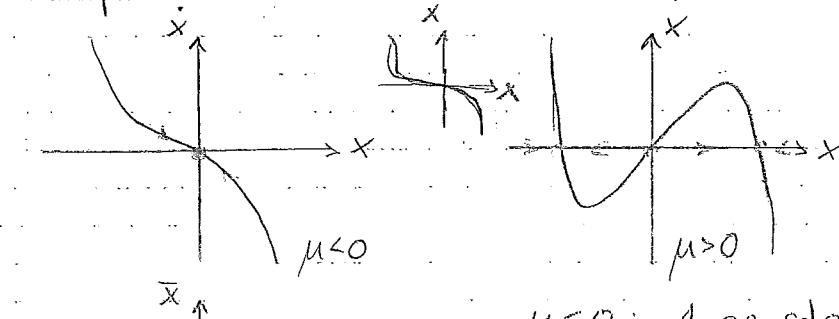
"+" Subkritisch:  $\bar{x}=0: \mu<0 \quad \bar{x}=\pm\sqrt{-\mu}$



$\mu < 0: 1$  as. stabil. FP  
 $2$  instabile FP

$\mu > 0: 1$  instabiler FP

"-" Superkritisch:



$\mu < 0: 1$  as. stabiller FP

$\mu > 0: 1$  instabiler FP  
 $2$  as. stabile FP

### 8.3 Hopf-Bif.

Normalform:  $\dot{x}_1 = \mu x_1 - x_2 \pm x_1(x_1^2 + x_2^2)$

$$\dot{x}_2 = x_1 + \mu x_2 \pm x_2(x_1^2 + x_2^2)$$

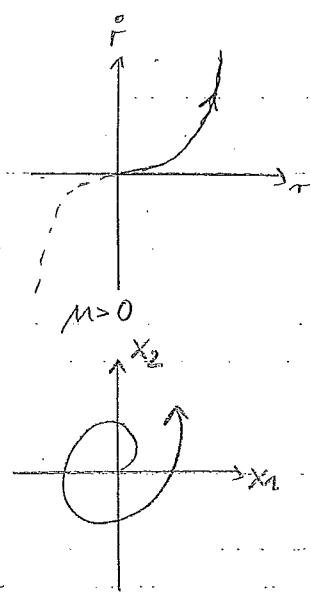
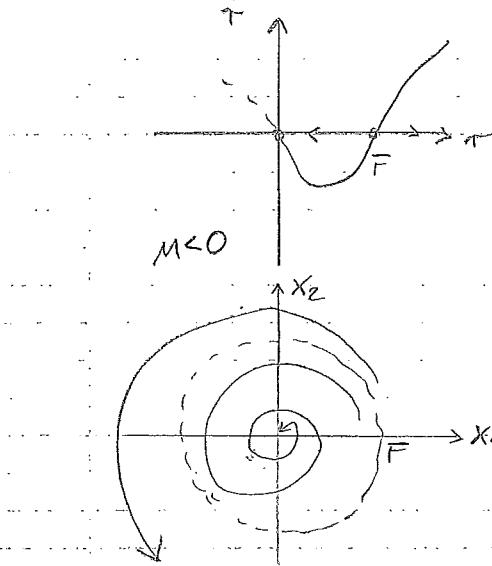
$$Df_x(0,0,\mu) = \begin{pmatrix} \mu, -1 \\ 1, \mu \end{pmatrix} \rightarrow (\mu - \lambda)^2 + 1 = 0 \rightarrow \lambda = \mu \pm i$$

$$r = \sqrt{x_1^2 + x_2^2}, \quad \psi = \arctan \frac{x_2}{x_1}$$

$$\dot{r} = \mu r \pm r^3$$

$$\dot{\psi} = 1$$

## "+" subkritisch

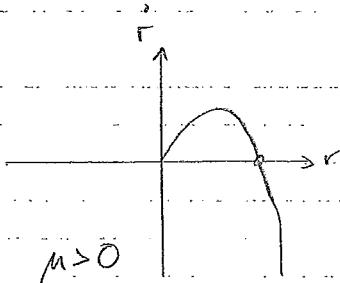
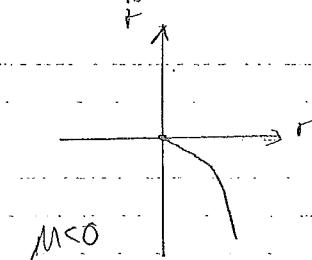


period. Lsg. nur stabil,  
nie asympt. stabil

$\mu < 0$ : 1 as. stabiler FP  
1 instabile period. Lsg.

$\mu > 0$ : 1 instabiler FP

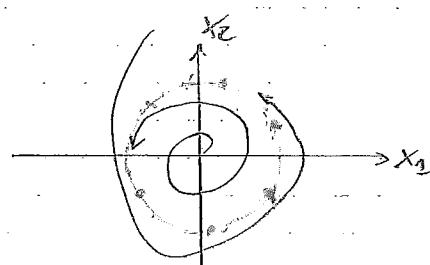
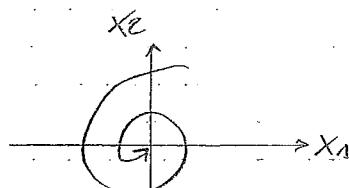
## "-" Superkritisch



$\mu < 0$ : 1 as. stab. FP

$\mu > 0$ : 1 instab. FP

1 stab. period. Lsg.

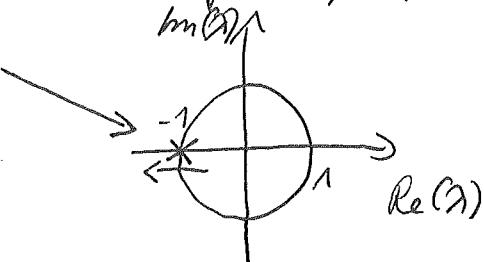


## 8.4 Flip-Bifurkation

Zeitdiskretes System  $x(n+1) = f(x, \mu)$

Ang: mit FP  $\bar{x} = 0$  und Bif. bei  $\mu = \mu_0 = 0$

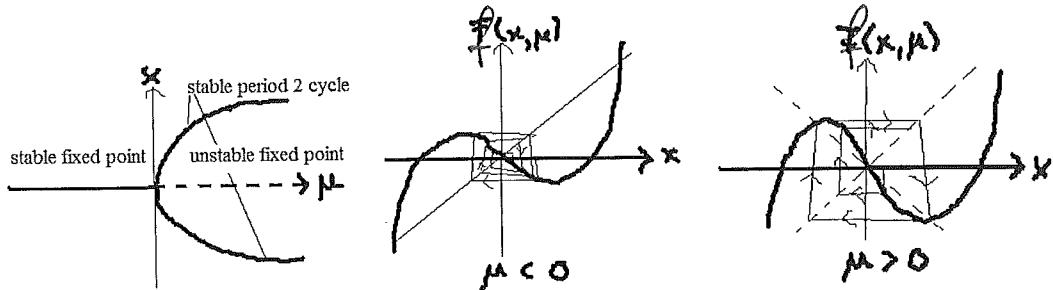
$$\text{Df}(0, 0) = -1$$



Normalform:  $x(n+1) = -(1+\mu)x \pm x^3$

2 Fälle:

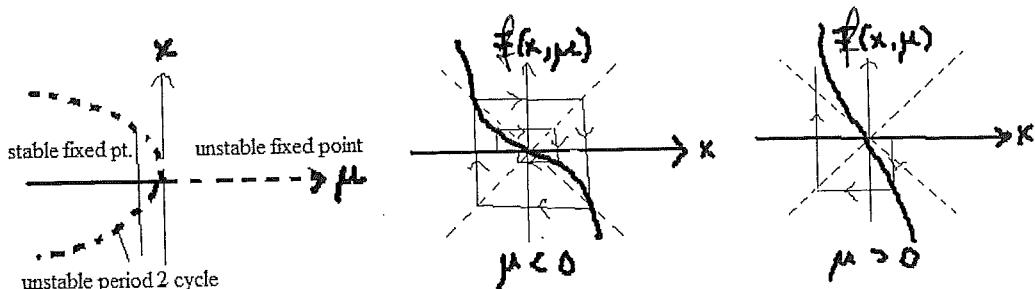
Superkritisch "+":



$\mu < 0$ : 1 as. stab. FP  
 $\mu > 0$ : 1 instab. FP +  
 as. stab. P2-Lsg.  
 Period 2

} oder andernum

Subkritisch "-":



$\mu < 0$ : 1 as. stab. FP  
 + instab. P2-Lsg.  
 $\mu > 0$ : 1 instab. FP

} oder andernum

Bemerkung: P2-Lsg. sind Fixpunkte der 2-fach iterierten Abb.  $f^2(\cdot) = f \circ f$ .  
 Diese hat im Bif. Pkt.  $\lambda = \text{Df}^2(0, 0) = (-1) \cdot (-1) = 1 \rightarrow$  Pitchfork-Bif.  
 → Flip-Bif. von  $f \leq$  Pitchfork-Bif von  $f^2$  mit den 2 Fällen "Super" + "Sub"-Kritikus