Data analysis: Statistical principals and computational methods

Energy Minimization: Search Techniques

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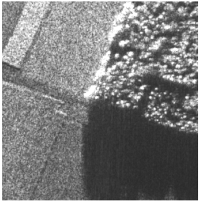
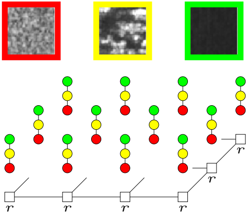
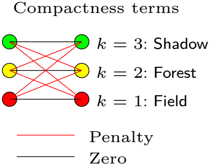
Energy Minimization (recap – segmentation)

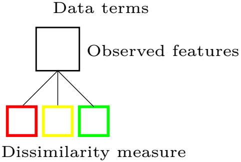


Original

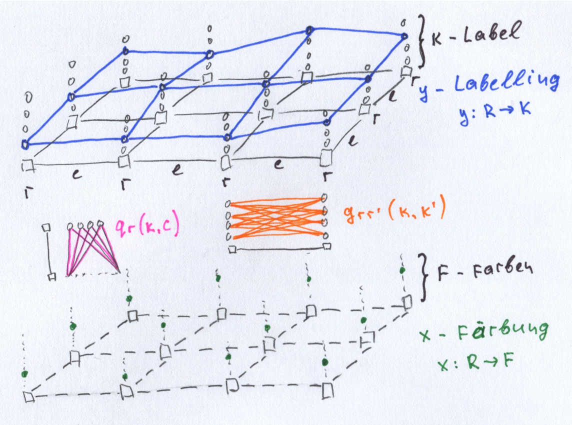
A possible segmentation







Energy Minimization (recap)



Today – Energy Minimization Problems:

$$y^* = \arg\min_y \left[\sum_i \psi_i(y_i) + \sum_{ij} \psi_{ij}(y_i, y_j) \right]$$

Outline

- Iterated Conditional Modes (+Variants)

- Equivalent transformations
- Binary MinSum problems – canonical forms
- Binary MinSum problems $\leftrightarrow$ MinCut
- MinCut, MaxFlow

- Search techniques – general idea
- α-expansion and $\alpha\beta$-swap

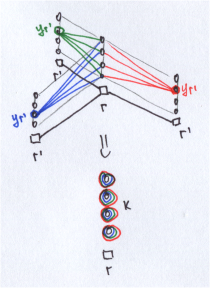
Iterated Conditional Modes

$$y^* = \arg\min_y \Big[\sum_i \psi_i(y_i) + \sum_{ij} \psi_{ij}(y_i, y_j) \Big]$$

Idea: choose (locally) the label that leads to the best energy given the fixed rest [Besag, 1986]

Repeat until convergence for all i:

$$y_i = \arg\min_k \Big[\psi_i(k) + \sum_{j:ij\in\mathcal{E}} \psi_{ij}(k, y_j) \Big]$$

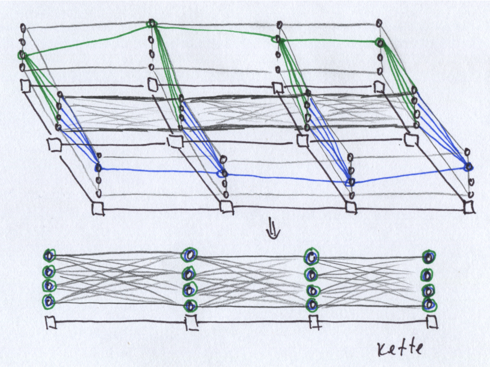


+ Extremely simple, easy to parallelize
− "Coordinate-wise" optimization → does not converge to the global minimum even for very simple energies

  Example: strong Ising model (Potts with $K=2$)

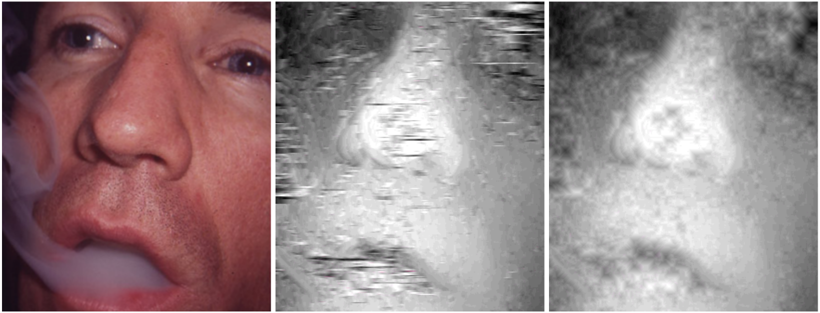
Iterated Conditional Modes

Extension: instead to fix all variables but one, fix a subset of variables so that the rest is easy to optimise (e.g. a chain or a tree). For images – e.g. row-wise/columl-wise optimization



→ can be solved exact and efficient by Dynamic Programming

Row-wise ICM:
the labels for a nodes of a chain can vary. The rest is fix.



Start from an y^0 (e.g. the result of *independent* row-wise Dynamic Programming), continue with the row-wise ICM

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Equivalent transformations (aka re-parameterization)

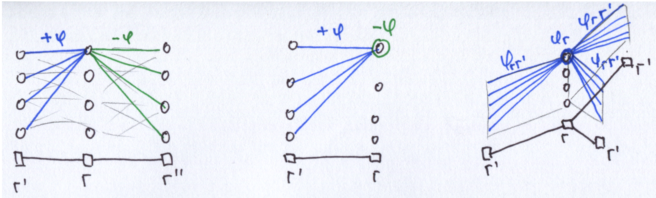
Two tasks $A = (\psi)$ and $A' = (\psi')$ are called **equivalent**, iff

$$\left[\sum_i \psi_i(y_i) + \sum_{ij} \psi_{ij}(y_i, y_j)\right] = \left[\sum_i \psi'_i(y_i) + \sum_{ij} \psi'_{ij}(y_i, y_j)\right]$$

holds for all labelings y

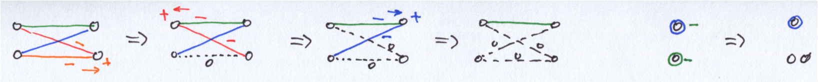
$\mathcal{A}(A)$ – Equivalence class (all thasks that are equivalent to A).

Equivalent transformations (re-parameterization):



Binary MinSum Problems – canonical forms

For **binary** Problems $K = \{0, 1\}$ the functions ψ can be re-parameterized as follows



$\Rightarrow$ ψ are not equal zero only for $k = 1$ and $(k, k') = (1, 1)$
$\Rightarrow$ the energy can be written as

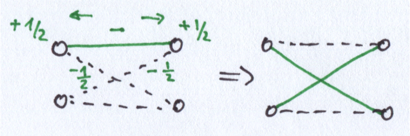
$$E(y) = \sum_i y_i \cdot \psi_i + \sum_{ij} y_i \cdot y_j \cdot \psi_{ij}$$

with nodes- and edge-specific **numbers** ψ_i and ψ_{ij} (not functions anymore) – a polynom of second order

– is used for **Quadratic Pseudo-Boolean Optimization**

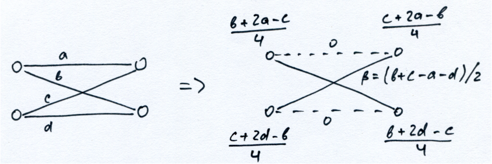
Binary MinSum Problems – canonical forms

Furthermore (in order to transform into a **MinCut** problem):



$$E(y) = (\dots) + \sum_{rr'} \beta_{ij} \cdot \delta(y_i \neq y_j)$$

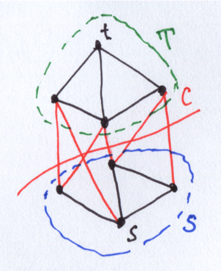
All-in-all:



with $\beta = (b + c - a - d)/2$ (keep in mind this expression !!!)

MinCut

Attention!!! Similar notations, different meaning



Let a graph $G = (V, \mathcal{E})$ be given

There are two "special" nodes
– s (source) and t (target).

Each edge $\{i, j\} \in \mathcal{E}$ has its **costs** c_{ij}.

A **cut** C is an edge subset so,
that there is no path from s to t

The cut has to be **minimal**: a removal of an edge from this subset leads to existence of a path

MinCut

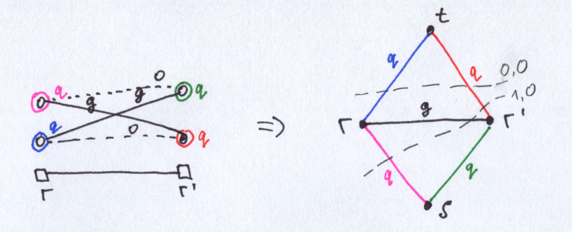
The quality of a cut is the summed costs of all involved edges

The task is to find the cut of the minimal quality:

$$C^* = \arg\min_C \sum_{ij \in C} c_{ij}$$

Alternatively: a cut corresponds to a **partition** of the node set into two subsets S and T with $s \in S$ and $t \in T$, $S \cup T = V$, $S \cap T = \emptyset$

$$(S,T)^* = \arg\min_{(S,T)} \sum_{ij \in \mathcal{E}, i \in S, j \in T} c_{ij}$$



Each node of the MinSum problem correspond to an "intrinsic" node in MinCut. There are also two additional nodes s and t

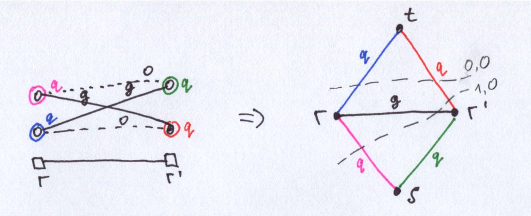
Each labeling $y : V \to \{0, 1\}$ correspont to a partition (S, T), with $y_i = 0 \Leftrightarrow i \in S$ and $y_i = 1 \Leftrightarrow i \in T$

The edge costs of the MinCut problem are:

$c_{ij} = \beta_{ij}$ for all edges connecting intrinsic nodes

$c_{si} = \psi_i(1)$ and $c_{it} = \psi_i(0)$.

Binary MinSum Problems ↔ MinCut



The energy of a labeling y is equal to the quality of the corresponding cut (partition)

- The relation MinSum ↔ MinCut works **always** (the problems are identical)
- MinCut is NP-complete in general
- MinCut is **polynomially** solvable if all edge costs are **non-negative**, i.e. $a + d \geq b + c$ holds for all edges (remember the "expression")
- Such problems are called **submodular**

Solvability, MinCut $\leftrightarrow$ MaxFlow

MinCut can be transformed into the corresponding **MaxFlow**:

There is a "Pipe network" (a Graph with nodes i and edges (i, j)). There are two special nodes s and t.

There is a **flow** x_{ij} through each pipe (i, j).

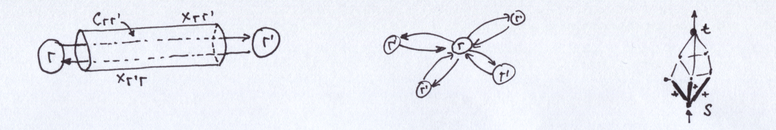
Each pipe has its **capacity** c_{ij}

The task is to find the **maximal** flow that can be sent from the source to the target

If the capacities of the pipe network are the same as the edge costs in a MinCut problem, these two problems are **dual** to each other (attention!!! only if the edge costs are non-negative).

The values of the maximal flow and of the optimal cut are equal. The latter can be obtained given the former

MaxFlow



Flows are **directed**: x_{ij} means "from i to j"

Flows are restricted by the corresponding capacities:

$$0 \leq x_{ij} \leq c_{ij}$$

Nothing appears or disappears on the way:

$$\sum_{j:ij\in\mathcal{E}} x_{ji} = \sum_{j:ij\in\mathcal{E}} x_{ij} \quad \forall i \neq s, t$$

The total flow to be maximized is:

$$\sum_i x_{si} \to \max_x$$

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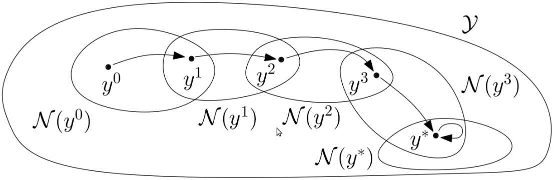
Search techniques – general idea

There is a neighbourhoodfor each labelling
– a subset of labelling so that

a) it can be described constructively"

b) the current labelling belongs to this subset

c) the optimal labelling in the subset is easy to find

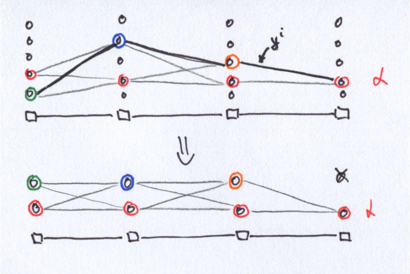


The algorithm is an iterative search for the best labelling in the neighbourhood of the actual one
– converges to a local optimum

α-expansion

The neighbourhood of a labelling – for **all** nodes restrict the **label set** [Boykov et al., 2001]

α-**expansion**: consider a label α, for each node consider two labels (at most) – the actual one and α



the auxiliary task is a **binary MinSum problem** – can be solved by MinCut under circumstances

This is repeated for all α-s until convergence

α-expansion

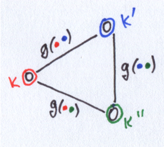
In which cases the auxiliary tasks can be solved exactly?

Sufficient: if the pairwise functions ψ_{ij} are **Metrices**, i.e.

a) $\psi(k,k) = 0$

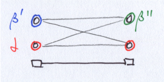
b) $\psi(k,k') = \psi(k',k) \geq 0$

c) $\psi(k,k') \leq \psi(k,k'') + \psi(k'',k')$



Then the auxiliary tasks are **submodular**:

$$\psi(\alpha,\alpha) + \psi(\beta',\beta'') =$$
$$= 0 + \psi(\beta',\beta'') \leq \psi(\beta',\alpha) + \psi(\alpha,\beta'')$$



Examples:

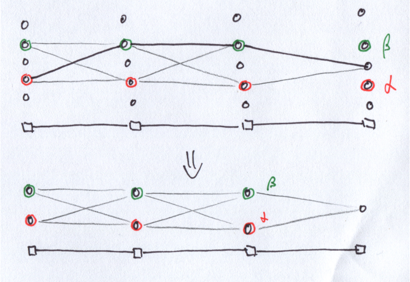
- the Potts Model $\psi(k,k') = \delta(k \neq k')$ – segmentation
- linear metric $\psi(k,k') = |k - k'|$ – stereo
- truncatedmetrices e.g. $\psi(k,k') = \min(|k - k'|, C)$

$\alpha\beta$-swap

Consider a label pair α, β, in each node

- if the current label is α or β, only α and β are allowed,
- otherwise, only the current label is allowed.

$\rightarrow$ each node can swap from α to β and back



the auxiliary task is a binary MinSum problem – solvable by MinCut, if e.g. $\psi(k,k) = 0$ and $\psi(k, k' \neq k) \geq 0$ (**Semimetric**)

This is repeated for all pairs α and β until convergence

A comparison

For an $n \times n$ grig as the graph, K labels, random labeling

$\#l$ – the number of labelings in the neighbourhood
$\#n$ – the number of neighbourhoods

	ICM	ICM+	α-exp.	$\alpha\beta$-swap
$\#l$	K	K^n	$2^{\frac{n^2\cdot(K-1)}{K}}$	$2^{\frac{n^2\cdot 2}{K}}$
$\#n$	n^2	$2 \cdot n$	K	$\frac{K(K-1)}{2}$
applicable ψ	arbitrary	arbitrary	metric	semimetric
exact for	never	chain	$K=2$ (?)	$K=2$

– very easy to parallelise

– can be freely combined with each other